

MATH 475; FINAL EXAM, 150 points, December 17, 2004, 12:25-2:25 PM
(R.A.Brualdi)

TOTAL SCORE (8 problems; 150 points possible):

Do NOT multiply out binomial coefficients, factorials, Catalan numbers etc.

Name: Brief Solutions

1. [20 points] There are 20 identical pennies lined up in a row. Eight of the pennies are to be chosen in such a way that no two consecutive pennies are chosen. **How many possible choices are there?**

One way: This is the number of solutions in integers of $x_1 + x_2 + \dots + x_9 = 12$ where $x_1, x_9 \geq 0$ and $x_2, \dots, x_8 \geq 1$ (the x_i are number of spaces between chosen pennies) and so equals $\binom{13}{5}$.

2. [10 points] Determine the permutation of $\{1, 2, \dots, 9\}$ whose **inversion sequence** is

$$6, 3, 5, 5, 2, 3, 1, 1, 0.$$

Following either of the two algorithms we get 975286134.

3. [15 points] Give a **combinatorial proof** of the identity

$$n2^{n-1} = \binom{n}{1} \cdot 1 + \binom{n}{2} \cdot 2 + \binom{n}{3} \cdot 3 + \dots + \binom{n}{n} \cdot n$$

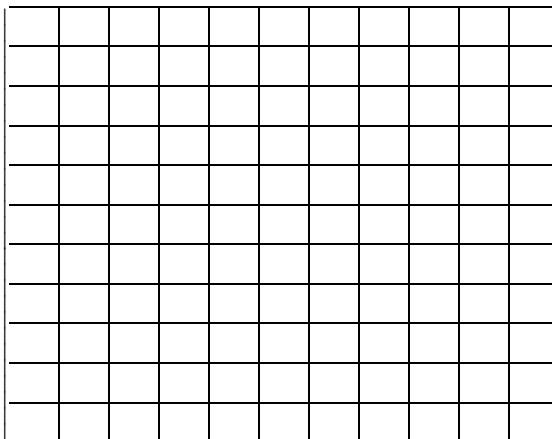
by showing that each side counts the same thing.

The LHS counts the number of nonempty committees with a leader chosen (choose the leader first, and then choose the rest (possibly empty) of the committee. The RHS counts the same thing according to the size of the committee (choose $k \geq 1$ members of the committee and then name one of them as leader).

4. [20 points] At closing a bagel store has 6 Sesame bagels, 6 Poppy bagels, 4 Egg bagels, and 2 Bran bagels. You want to buy a dozen (12) bagels. **How many possible dozens could you choose?**

This equals the number of solutions in nonnegative integers of $x_1 + x_2 + x_3 + x_4 = 12$ where $x_1, x_2 \leq 6$, $x_3 \leq 4$, and $x_4 \leq 2$. Now use the I-E principle to get 60.

5. [20 points] Given an 11 by 11 grid



The number of ways to walk from the lower left corner to the upper right corner (in 22 steps) that never dips below the diagonal line equals:

This is the 11th Catalan number $C_{11} = 1/12 \binom{22}{11}$.

(b) The number of these ways which pass through both 4th and 8th diagonal points (counting the lower left point as the first) diagonal points equals:

This equals $C_3 \times C_4 \times C_4$ (multiply not add!)

6. [20 points] **Derive** the recurrence relation $s(p, k) = (p - 1)s(p - 1, k) + s(p - 1, k - 1)$, $1 \leq k \leq p - 1$, for the Stirling numbers of the first kind using its combinatorial interpretation.

See the book.

7. [20 points] **Apply the deferred acceptance algorithm**, with the Big Guys proposing to the little guys, to the preferential ranking matrix below, in order to determine a stable marriage of the Big Guys to the little guys.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>A</i>	5, 1	1, 1	3, 3	2, 2	4, 2
<i>B</i>	4, 4	5, 4	1, 2	2, 4	3, 4
<i>C</i>	4, 2	1, 2	2, 1	5, 5	3, 3
<i>D</i>	1, 5	2, 3	3, 5	4, 3	5, 5
<i>E</i>	1, 3	2, 5	3, 4	4, 1	5, 1

Result is $a : E, b : A, c : C, d : D, e : B$

8. [25 points] How many **nonequivalent** ways are there to color the corners of a regular hexagon with the colors Red and Blue?

The six rotations fix, respectively, $2^6, 2, 2^2, 2^3, 2^2, 2$ colorings. The six reflections fix 2^3 (three times) and 2^4 (three times) colorings. Now use Burnside's theorem – add these numbers up and divide by 12.