MATH 475; EXAM # 2, 100 points, December 1, 2005 (R.A.Brualdi)

TOTAL SCORE (100 points possible):

Name:

1. [15 points] The numbers on the left edge of the difference table for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ are

$$1, 2, 3, 4, 5, 0, 0, 0, 0, \ldots.$$

Determine a formula for $\sum_{k=0}^{n} h_k$.

$$\sum_{k=0}^{n} h_k = 1\binom{n+1}{1} + 2\binom{n+1}{2} + 3\binom{n+1}{3} + 4\binom{n+1}{4} + 5\binom{n+1}{5}$$

2. [5 points] State the Marriage Theorem for the existence of an SDR of a family of sets $A_1, A_2, \ldots, A_n$.

The sets have an SDR if and only if for $k = 1, 2, \ldots, n$, the union of each collection of k of the sets has at least k elements.

3. [15 points] Determine the number of ways to place 6 non-attacking, identical rooks on the following board with ten forbidden positions:

×					
×					
×					
×	×	×	×		
				×	
				×	×

$$6! - 10(5!) + 31(4!) - 34(3!) + 9(2!)$$

4. [15 points] What is the common value of the Stirling number of the second kind $S(n, n-1)$ and the Stirling number of the first kind $s(n, 2)$? What do they count and why are the values the same?

VOID. Should have been $S(n, n-1)$.

5. [15 points] Use either of the recurrence relations we know for the derangement numbers D_n to prove by induction that D_n is even if and only if n is odd, that is, D_n is even if n is odd, and D_n is odd if n is even.

One could use either $D_1 = 0, D_2 = 1, D_n = (n-1)(D_{n-1} + D_{n-2}), (n \geq 3)$ or $D_1 = 0$, $D_n = nD_{n-1} + (-1)^n, n \geq 2$.

Using the second we note that for $n = 1$, it is true (1 is odd and D_1 is 0 an even number). Now assume holds true for $n-1$ and prove true for n.

Case 1: n even. Then $n-1$ is odd and so D_{n-1} is even. By the recursion, we get D_n is even + 1 and so odd.

Case 2: n is odd. Then $n-1$ is even. By the induction hypothesis, D_{n-1} is odd. So by the recursion again we get that D_n is odd $\times$ odd -1 and so even.

6. [20 points] Consider symbols A's, B's, C's, and D's.

(a) What **rational function in simple form** (no infinite series) is the generating function for the sequence h_n, $n = 0, 1, 2, 3, \ldots$, where h_n is the number of n-combinations of these symbols where there are an even number of A's, at most 3 B's, and odd number of C's, and at least 1 D in the n-combinations?

$$\frac{1}{1-x^2}\frac{1-x^4}{1-x}\frac{x}{1-x^2}\frac{x}{1-x}$$

(b) What is a **simple** expression (no infinite series) for the exponential generating function for the sequence h_n, $n = 0, 1, 2, 3, \ldots$, where h_n is the number of n-permutations of these symbols satisfying the above constraints?

$$\frac{e^x + e^{-x}}{2}(1 + x + x^2/2! + x^3/3!)\frac{e^x - e^{-x}}{2}(e^x - 1)$$

7. [15 points] Use the **deferred acceptance algorithm** to determine a stable marriage in case of the preferential ranking matrix below where the BIG GUYS A, B, C, D, E, F propose to the little guys a, b, c, d, e, f:

$$a \to D, b \to A, c \to B,$$

$$d \to E, e \to F, f \to C$$

	a	b	c	d	e	f
A	2, 3	1, 3	5, 1	3, 1	6, 2	4, 6
B	5, 1	1, 4	2, 2	6, 2	4, 4	3, 4
C	1, 5	5, 1	2, 4	6, 3	4, 6	3, 2
D	1, 2	2, 2	3, 3	6, 4	5, 1	4, 5
E	1, 4	3, 6	5, 5	2, 5	4, 3	6, 3
F	2, 6	4, 5	6, 6	1, 6	3, 5	5, 1

What property does a stable marriage have when the deferred acceptance algorithm is applied this way?

There is no **stable** marriage in which any of the BIG GUYS gets a better partner (in terms of his preferences.