Math 475, Spring Semester 2001-02 NAME:SOLUTIONS
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Exam 2: (90 points): Monday, April 8, 2002. Total Points:

1. [48 points, 8 points each] Answer each of the following short questions (show your work):

(a) The sum $\sum_{k=0}^{n}(-1)^k\binom{n}{k}5^k6^{n-k}$ equals:

This is $(6-5)^n = 1$.

(b) One of the recurrence relations satisfied by the derangements numbers D_n is:

$D_n = (n-1)(D_{n-1}+D_{n-2}), (n \geq 3)$ with $D_1 = 0$ and $D_2 = 1$, OR

$D_n = nD_{n-1} + (-1)^n, (n \geq 2)$ with $D_1 = 0$,

and it gives the following value for D_5:

$D_2 = 1, D_3 = 2, D_4 = 9, D_5 = 44$.

(c) The number of permutations of $\{1,2,3,4,5,6,7,8,9\}$ in which exactly 5 integers are in their natural position equals:

$\binom{9}{5}D_4 = 126 \cdot 9 = 1134$.

(d) The **general solution** of the recurrence relation

$$h_n - 7h_{n-1} + 15h_{n-2} - 9h_{n-3} = 0, (n \geq 3)$$

is:

Roots of $x^3 - 7x^2 + 15x - 9 = 0$ are $1, 3, 3$. So the general solution is
$h_n = c1^n + d3^n + en3^n$.

(e) $\frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^7}$ is the (ordinary) generating function of:

The number h_n of nonnegative integer solutions of $2e_1 + e_2 + 5e_3 + 7e_4 = n$.

(f) The nth Catalan number C_n equals: $\frac{1}{n+1}\binom{2n}{n}$ and counts what kind of paths?

Paths of length $2n$ (shortest paths) from $(0,0)$ to (n,n) that never go above the diagonal.

2. [12 points] Determine **one binomial coefficient** that is equal to:

$$\binom{n-3}{k} + 3\binom{n-3}{k-1} + 3\binom{n-3}{k-2} + \binom{n-3}{k-3}.$$

Repeated use of Pascal's formula shows this is $\binom{n}{k}$.

3. [15 points] Determine the **number of ways** to place 7 identical non-attacking rooks on the 7 by 7 board below:

×						
×						
×	×	×				
			×			
			×			
			×	×	×	

$7! - r_1 6! + r_2 5! - r_3 4! + r_4 3! - r_5 2! + r_6 1! - r_7 0!$.

Easy calculations show that $r_1 = 10$, $r_2 = 2 \cdot 2 + 2 \cdot 2 + 5 \cdot 5 = 33$, $r_3 = 4 \cdot 5 + 5 \cdot 4 = 40$, $r_4 = 4 \cdot 4 = 16$, $r_5 = r_6 = r_7 = 0$.

4. [15 points] Find a **closed form expression** for the exponential generating function for the number h_n of n-digit numbers using only the digits 1, 2, 3, 4 in which the number of 1's is odd, the number of 2's is even, the number of 3's is unrestricted, and the number of 4's is at least one.

$\frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2} \cdot e^x \cdot (e^x - 1) = \frac{e^{4x} - e^{3x} + e^{-x} - 1}{4}$.

A simple formula for h_n is:

So $h_0 = 0$ and $h_n = \frac{4^n - 3^n + (-1)^n}{4}$, $n \geq 1$.