Math 240, Spring Semester 2001-02 NAME:**SOLUTIONS**

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Exam 1: (90 points): Wed. Feb. 20, 2002. Total Points: 90

1. [48 points, 8 points each] Answer each of the following short questions:

(a) **Your next move** in a NIM game where there are 3 piles, consisting of 21, 19, and 13 coins, respectively, and you want to be sure that you will win.

In binary, 21 is 10101, 19 is 10011, and 13 is 01101. The "NIM sum" of these is 01011. So the only winning move is to take 7 coins from the 13-pile leaving 6.

(b) The **permutation** of $\{1, 2, 3, 4, 5, 6\}$ with inversion sequence $1, 1, 2, 1, 1, 0$.

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(c) The **immediate successor** and the **immediate predecessor** of $1, 0, 1, 0, 1, 0, 0, 0$ in the reflected Gray code.

The immediate successor is $1, 0, 1, 1, 1, 0, 0, 0$; the immediate predecessor is $1, 0, 1, 0, 1, 0, 0, 1$.

(d) If you have 20 bagels of four different varieties (Sesame, Onion, Whole Grain, and Plain), which of the following (possibly both, possibly neither) are you **guaranteed** to have;

(i) 4 S or 6 O or 8 WG or 5 P ?

Guaranteed by SPHP (4 boxes!) since $3 + 6 + 7 + 4 + 1 = 21 > 20$.

(ii) 2 S or 10 O or 8 WG or 4 P ?

NOT guaranteed since $1 + 9 + 7 + 3 = 20$.

(e) The following are true: $K_{25} \to K_4, K_5$ and $K_{24} \not\to K_4, K_5$. Explain in words **what this means**.

If the edges joining 25 points are each colored red or blue, then either there are 4 points with all segments between them red, or 5 points with all segments between them blue. If there are only 24 points, then then there is a way to color the segments between them red or blue without the above conclusion holding.

(f) Identify **whether or not** the following are **partially ordered sets**. **If not, say what defining property is not satisfied**.

(i) Set of **all real numbers** where $a \le b$ means that $|a - b| \le 2$.

NO, e.g. transitive law fails: e.g. $1 \le 3, 3 \le 5$ but NOT $1 \le 5$.

(ii) Set of **all integers from** -9 **to** 9 where $a \le b$ means a divides b.

NO, antisymmetric law fails: e.g. $2 \le -2$ and $-2 \le 2$, but $2 \ne -2$.

2. [12 points] Either **determine a perfect cover** by dominoes of the 5 by 5 board with the square in row 3 and column 4 removed, **or prove that no such perfect cover exists**.

Coloring the squares alternatingly black and white and then removing the indicated square, the number of black squares on the "pruned board" does not equal the number of white squares and so a perfect cover is not possible (since each domino covers one square of each color).

3. [15 points] Determine the **number of ways** to place 10 identical non-attacking rooks on a 15 by 15 board if there are **no rooks in the upper left 5 by 5 board** and **exactly two in the lower right 10 by 10 board**.

First choose the two rooks in the lower right - in $\binom{10}{2}\binom{10}{2}2!$ ways.

Now partition the solutions according to the number of rooks in the 5 by 10 upper right board and the 10 by 5 lower left board: 5;3 and 3,5, and 4,4.

(5;3) $\binom{8}{5}5! \times \binom{8}{3}\binom{5}{3}3!$

(3;5) same as above

(4;4) $\binom{5}{4}\binom{8}{4}4! \times \binom{5}{4}\binom{8}{4}4!$

Now add these last three numbers and multiply by the $\binom{10}{2}\binom{10}{2}2!$.

4. [15 points] A box contains 20 donuts, each of a different variety, arranged in a circular fashion (so that **each donut has a donut touching it on each of its two sides**). One of the donuts is a **chocolate donut**. **How many choices for 6 donuts are there if you must choose the chocolate donut but you will not choose two donuts that touch?**

Picking out 6 donuts, including the chocolate one, let $x_1, x_2, \ldots, x_6$ be the number of donuts between the selected donuts as one goes around the circle starting with the chocolate one. Then we want the number of solutions in integers of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 14$$

where each of the x_i's is at least 1. Doing a change of variable ($y_i = x_i - 1$), this becomes solutions in nonnegative integers of

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 8.$$

The number of such solutions is

$$\binom{8+6-1}{8} = \binom{13}{8}.$$