MATH 475; EXAM # 1, 100 points, October 12, 2004 (R.A.Brualdi)

TOTAL SCORE (7 problems; 100 points possible):

Name: ANSWERS

1. [10 points] Consider a game of NIM with heaps of size 21, 25, 4, and 6. As the player going first, how many coins should you remove and from what pile?

$$\begin{array}{rcccccc} 21 & : & 1 & 0 & 1 & 0 & 1 \\ 25 & : & 1 & 1 & 0 & 0 & 1 \\ 4 & : & 0 & 0 & 1 & 0 & 0 \\ 6 & : & 0 & 0 & 1 & 1 & 0 \\ \hline & & 0 & 1 & 1 & 1 & 0 \end{array}$$. So remove 2 from pile 25 to balance the game.

2. [10 points] Prove: For any choice of 76 integers, there will always be two whose difference is divisible by 75.

A difference is divisible by 75 iff the numbers have the same remainder when divided by 75. There are 75 different remainders $\{0, 1, 2, \ldots, 74\}$, and so with 76 numbers two will have the same remainder.

3. [15 points] Use the fact that the Ramsey number $r(3, 4) = 9$ to prove that $r(4, 4) \leq 18$.

Conisider K_{18} and one of its points A. From A there are 17 edges, and so at least 9 of one color, say red. Consider the 9 points at the other end of these 9 edges. Since $r(3, 4) = 9$, among them, either there is a red K_3 or a blue K_4. If there is a blue K_4 we are done. If there is a red K_3, then using A we get a red K_4.

4. [15 points] Consider the bit 8-tuple 10011000.

(a) What bit 8-tuple immediately precedes 10011000 in the base 2 algorithm for generating the combinations of an 8-element set?

10010111

(b) What bit 8-tuple immediately follows 10011000 in the reflected Gray code scheme?

10001000

(c) What bit 8-tuple immediately precedes 10011000 in the lexicographic order of bit 8-tuples with exactly 3 1's?

10100001

5. [15 points] 30 toys are to be distributed among 10 children.

(a) How many ways are possible if the toys are all different?

10^{30}

(b) How many ways are possible if the toys are all identical and each child gets at least one toy?

The number of solutions of $x_1 + x_2 + \cdots + x_{10} = 30$ in positive integers, and so the number of solutions of $y_1 + y_2 + \cdots + y_{10} = 20$ in nonnegative integers: $\binom{20+10-1}{20}$.

(c) How many ways are possible if there are 6 different kinds of toys, with 25 of them identical and the other 5 different and special, and no child gets two special toys?

First distribute the 5 special toys — in $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ ways. Then we have to solve an equation of the form $x_1 + x_2 + \cdots + x_{10} = 25$ where 5 of the x's are to be positive and 5 are to be nonnegative. So we get an equation of the form $y_1 + y_2 + \cdots + y_{10} = 20$ where the y's are to be nonnegative — this has $\binom{20+9}{20}$ solutions. Now multiply.

If one doesn't assume that each child gets at least one toy (that would have been OK), then instead of $\binom{20+9}{20}$ one uses $\binom{25+9}{25}$

6. [15 points] Given a 14 by 14 chessboard, partitioned in the obvious way into four 7 by 7 boards (UpperLeft, UpperRight, LowerLeft, LowerRight). How many ways are there to place 9 identical rooks on the board in nonattacking positions so that there are exactly 3 rooks in the UL, exactly 2 rooks in the LR, exactly 2 rooks in the UR, and exactly 2 rooks in the LL?

There are $\binom{7}{3}^2 3! \binom{7}{2}^2 2!$ to place the rooks in the UL and LR. That leaves a 9 by 9 board partitioned as

4 by 4	4 by 5
5 by 4	5 by 5

.

The number of ways to put two rooks in the UR is $\binom{5}{2}\binom{4}{2}2!$ and the same number for the LL. Now multiply everything together.

7 [20 points] Identify each of the following relations as a partial order, equivalence relation, both, or neither.

(a) $=$ on a set X

Both

(b) $\subseteq$ on the subsets of a set S.

Partial Order

(c) $|\ |$ (absolute value) on complex numbers

Equivalence relation

(d) $\equiv$ mod n on the set of integers (n is a positive integer).

Equivalence relation

(e) The intersection of two equivalence relations on a set X. Equivalence relation

If an equivalence relation, how are the equivalence classes of the intersection obtained from the equivalence classes of the individual equivalence relations?)

Nonempty intersections

(f) The intersection of two partial orders on a set X.

Partial order

Is the intersection of two linear orders a linear order?

No