TOTAL SCORE (135 points possible):

MATH 340; FINAL EXAM, May 12, 2006 (R.A.Brualdi)

Discussion Section (**circle one**): Mon 8:50 Mon 12:05 Wed 8:50 Wed 12:05

NAME:

1. [27 points] A and B are **real matrices of order** 4 with determinants 3, and 7 respectively. Answer the following questions:

1. $\det(-A) = (-1)^4 3 = 3$

2. $\det A^T = 3$

3. $\det B^{-1} = 7^{-1} = 1/7$

4. $\det(AB) = 3 \cdot 7 = 21$

5. If $1 + i$ and $2 - 3i$ are complex eigenvalues of A, what are its other two eigenvalues?

   $1 - i$ and $2 + 3i$.

6. The row space of A equals: R^4

7. The product of the eigenvalues of A^T equals $\det A^T = 3$:

8. The product of the eigenvalues of B^{-1} equals $(-1)^4 \det B^{-1} = 1/7$:

9. The dot product of the first column vector of A with the second row vector of A^{-1} equals: 0 (since $A^{-1} \cdot A = I_4$)

2. [14 points] Let $A = \begin{bmatrix} 8 & 3 \\ -1 & 4 \end{bmatrix}$. Determine an **invertible matrix** Q that diagonalizes A: $Q^{-1}AQ = D$ and the **diagonal matrix** D.

The characteristic plynomial of A equals

$$\lambda^2 - 12\lambda + 35 = (\lambda - 5)(\lambda - 7).$$

Hence the eigenvalues of A are $5, 7$. We need to find an eigenvector of A for each of these eigenvalues.

$$5I_2 - A = \begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix}.$$

Hence an eigenvector is $[1 \ -1]^T$.

$$7I_2 - A = \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}$$

. Hence an eigenvector is $[3 \ -1]^T$ Let

$$Q = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}.$$

The Q is invertible with

$$Q^{-1} = 1/2 \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}.$$

We must have

$$Q^{-1}AQ = D = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}.$$

3. [12 points] Let A and B be orthogonal matrices of order n. Let x and y be n-tuples written as column vectors.

1. **Prove**:
$$(Ax) \cdot (Ay) = x \cdot y.$$
We have
$$(Ax) \cdot (Ay) = (Ax)^T(Ay) = x^T A^T Ay = x^T I_n y = x^T y = x \cdot y,$$
since A is orthogonal and so $A^{-1} = A^T$.

2. **Prove**: AB is an orthogonal matrix.
We have
$$(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}.$$
Thus AB is an orthogonal matrix.

4. [15 points] Let A and B be 4 by 4 matrices that have the same column space. (Note we are **not** assuming that A and B have the same columns, only that they have the same column space.)

(i) Are A and B sure to have the same number of pivots?

YES: WHY? Because the number of pivots equals the dimension of the column space.

NO: Exhibit an Example

(ii) Are A and B sure to have the same row space?

YES: WHY?

NO: Exhibit an Example.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

have the the same column space but clearly different row spaces. (The first row of A is a basis for the row space of A. and the first row of B is a basis for the row space of B, and clearly these row spaces are different.)

(iii) If A is invertible, are you sure that B is invertible?

YES: WHY? Then the dimension of column space of A, and so of B equals 4. So rank of B is 4 and B is invertible too.

NO: Exhibit an Example.

5. [20 points]

(i) Find an **orthonormal basis for the null space** U of the equation $x + y + z = 0$.

Clearly a basis of the null space U is $x = [1\ -1\ 0]^T$ and $y = [1\ 0\ -1]^T$. Applying the G-S process, we get

$$u_1 = 1/\sqrt{2}[1\ -1\ 0]^T.$$

projecting y onto the space spanned by u and taking the difference vector we get

$$[1\ 0\ -1]^T - 1/\sqrt{2}u_1 = [1/2\ 1/2\ -1]^T.$$

Dividing by its length $\sqrt{6}/2$, we get

$$u_2 = 2/\sqrt{6}[1/2\ 1/2\ -1]^T = 1/\sqrt{6}[1\ 1; -2]^T.$$

(ii) Find the **orthogonal projection** of the vector $b = [1\ 2\ 6]$ onto U.

It's

$$-1/2[1\ -1\ 0]^T - 9/6[1\ 1\ -2]^T = [-2\ -1\ 3]^T.$$

6. [10 points] Let A be a matrix of order n such that 3 is an eigenvalue of A. **Prove** that 0 is an eigenvalue of $A^2 - 4A + 3I_n$.

Let x be an eigenvector of A (so $x \neq 0$) for its eigenvalue 3. Then

$$(A^2 - 4A + 3I_n)x = A^2x - 4Ax + 3x = A(3x) - 4(3x) + 3x = 9x - 12x + 3x = (9 - 12 + 3)x = 0x,$$

Since $x \neq 0$, x is an eigenvector of the matrix $A^2 - 4A + 3I_n$ for the eigenvalue 0.

7. [17 points] Let A be an m by n matrix and b an m by 1 vector such that $Ax = b$ has **exactly one** solution.

(i) **Prove** that the null space of A equals $\{0\}$, that is, consists only of the zero vector.

Since $Ax = b$ has exactly one solution, the columns of A must be linearly independent. Hence the nullspace, being the linear combinations of the columns which give the zero vector, of A is $\{0\}$.

(ii) What is the **rank** of A?

Since the columns of A are linearly independent, the rank must equal n.

(iii) **Prove** that $A^T y = c$ must have a solution for every n by 1 vector c. Since the rank of

A (and hence of the n by m matrix A^T), is n, the dimension of the column space of A^T equals n. Thus every n-tuple c is a linear combination of the columns of A.

8. [20 points] We want to determine the quadratic curve $y = a + bx + cx^2$ that gives the best fit to the experimental data $(1, 2), (2, 5), (3, 6)$ and $(-1, 4)$.

(i) What **system of linear equations** $Ax = b$ do we want the least squares solution of: specify A and b.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 4 \end{bmatrix}.$$

(ii) Are the columns of A **linearly independent**? Why or why not?

Yes, because e.g. the first three rows give an invertible Vandemonde matrix.

(iii) What is the **system of equations** $By = c$, a solution y of which gives a least squares solution of $Ax = b$? Compute B and c but you are **not** expected to solve $By = c$.

$$B = A^T A = \begin{bmatrix} 4 & 5 & 15 \\ 5 & 15 & 35 \\ 15 & 35 & 99 \end{bmatrix} \text{ and } c = \begin{bmatrix} 17 \\ 26 \\ 80 \end{bmatrix}.$$