Math 340, FINAL EXAM (125 points) Name:
2:25 pm, May 14, 2000, B239VV
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1. **[45 points; 5 points each]** Show your work or give your reasoning!

(a) What is the determinant of the matrix $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$?

It's sign(2413) $\times 2 \times 1 \times 4 \times 3 = -24$.

(b) Let $A$ be a 4 by 4 matrix with determinant 10. Let $B$ be obtained from $A$ by adding 5 times row 4 to row 1 and subtracting row 2 from row 1. What is the determinant of $B$?

These EROs do not change the determinant, so its 10.

(c) If the (real) eigenvalues of a matrix $A$ are 1, 2, and 3 what are the (real) eigenvalues of the matrix $10I + A$?

$Ax = \lambda x$ implies $(10I + A)x = (10 + \lambda)x$, so it's 11, 12, and 13.

(d) Give an example of a 3 by 3 matrix $A$ which has 6 as an eigenvalue of multiplicity 3 but only 1 linearly independent eigenvector for 6 (the eigenspace associated with the eigenvalue 6 has dimension 1).

$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix}$ has 6 as eigenvalue 3 times and only one eigenvector (look at null space of $6I - A$.

(e) Let $A$ be a 4 by 5 matrix of rank 3. What is the **simplest** matrix that can be obtained from $A$ by elementary row **and** column operations (i.e. the simplest matrix equivalent to $A$)?

The 4 by 5 matrix with 1's in the first three diagonal positions and 0's elsewhere.

(f) $A$ is a 6 by 8 matrix of rank 4. What are the dimensions of the row space, column space, and null space of $A$?

row space: 4, column space: 4, and null space $8 - 4 = 4$.

(g) Complete to a correct statement:

**The columns of $AB$ are linear combinations of the *columns* of the matrix *$A$*.**

(h) Complete to a correct statement:

**The rows of $AB$ are linear combinations of the *rows* of the matrix *$B$*.**

(i) Consider the Markov process whose transition matrix is $A = \begin{bmatrix} 1/3 & 3/4 \\ 2/3 & 1/4 \end{bmatrix}$.

If the initial state vector is $\begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix}$, what is the state vector after one step?

$A \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.

What is the stationary vector (long term behavior) of the Markov process?

Find the eigenvector corresponding to the eigenvalue 1 of $A$ with sum of its entries equal to 1:
$\begin{bmatrix} 9/17 \\ 8/17 \end{bmatrix}$

2. **[30 points, 10 points each]** Answer the following questions:

(a) Let $A$ be a 4 by 4 matrix such that 3 times column 1, minus 2 times column 2, plus 7 times column 3, plus 8 times column 4 equals the zero vector. Is 0 an eigenvalue of $A$? If so, give a corresponding eigenvector.

So $A \begin{bmatrix} 3 \\ -2 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and 0 is an eigenvalue and the left vector is an eigenvector.

(b) Let $A, B, C$ be $n$ by $n$ matrices with $\det(A) = 2$ and $\det(C) = 3$, such that $A^T B^{-1} = C$. What is $\det(B)$?

$\det(A^T B^{-1}) = \det(C)$
$\det A^T \det B^{-1} = \det(C)$
$\det A(\det B)^{-1} = \det(C)$ and $\det B = \det A / \det C = 2/3$.

(c) Let $\lambda$ be an eigenvalue of the square matrix $A$. Give a proof that $\lambda^2$ is an eigenvalue of $A^2$.

There is a nonzero vector $x$ such that $Ax = \lambda x$. Multiply by $A$ on both sides to get:

$$A(Ax) = A(\lambda x), A^2 x = \lambda A x, A^2 x = \lambda \lambda x, A^2 x = \lambda^2 x.$$

Hence $\lambda^2$ is an eigenvalue for $A^2$.

3. **[15 points]** Let $A$ be a 3 by 3 matrix such that

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(a) Verify that the three column vectors above are linearly independent (any correct(!) way).

E.g. show that the determinant of the matrix whose columns are the three vectors is not zero.

(b) Write the above three equations as one equation.

$$A \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Now determine the matrix $A$ explicitly (actual numbers for entries).

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 3/2 & -1/2 \\ 1/2 & -1/2 & 3/2 \end{bmatrix}.$$

4. **[10 points]** Solve the following system of equations **USING CRAMER'S RULE**:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 2 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Answer is $6/38, -2/38, 4/38$.

5. **[10 points]** For each of the following functions $L : R^3 \to R^2$ say whether $L$ is a linear transformation. If $L$ is a linear transformation, determine the matrix of $L$ with respect to the standard bases of $R^3$ and $R^2$. If $L$ is not a linear transformation, justify your answer.

(a) $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 3a_1 - 2a_2 \\ -a_1 + 3a_3 \end{bmatrix}.$

Linear and matrix is $A = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 0 & 3 \end{bmatrix}.$

(b) $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 - a_2 + 3a_3 \\ 4 \end{bmatrix}.$

Not linear; e.g. $L(\mathbf{0}) \neq \mathbf{0}$.

6. **[15 points]** Let $A$ be the matrix

$$\begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix}.$$

Determine an **non-singular** matrix $P$ such that $P^{-1}AP$ is a diagonal matrix. (To help you get started, both $-2$ and $6$ are an eigenvalue of this matrix.)

The characteristic polynomial is computed to be $\lambda^3 - 2\lambda^2 - 20\lambda - 24 = (\lambda + 2)^2(\lambda - 6)$.

There are two linearly independent eigenvectors for $-2$ and (of course) only one for 6. Calculating these and putting as columns of a matrix we get:

$$P = \begin{bmatrix} -1/2 & 3/2 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

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**EXTRA CREDIT PROBLEM [20 points]** Let $L$ be a linear transformation from a 3-dimensional vector space $V$ to itself. Let $S : v_1, v_2, v_3$ be a basis of $V$ with respect to which $L$ has matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Let $T : w_1, w_2.w_3$ be defined by

$$w_1 = v_1, \; w_2 = v_1 + v_2, \; w_3 = v_1 + v_2 + v_3.$$

(a) Prove that $T$ is a basis of $V$.

Suppose that $c_1w_1+c_2w_2+c_3w_3 = 0$. Then substituting we get $(c_1+c_2+c_3)v_1+(c_2+c_3)v_2+c_3v_3 = 0$. Since the $v$'s are linearly independent, this implies all coefficients are zero, and then this easily implies that $c_1 = c_2 = c_3 = 0$. Thus the $w$'s are linearly independent and since there are three of them in a 3-dimensional space they must be a basis.

(b) Write the basis $S$ in terms of the basis $T$ (the $v$'s as linear combinations of the $w$'s).

Solving for the $v$'s in terms of the $w$'s we get $v_1 = w_1, v_2 = -w_1 + w_2, v_3 = -w_2 + w_3$.

(c) Determine the matrix of $L$ with respect to the basis $T$.

This is $P^{-1}AP$ where

$$P^-1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

from the above equations. Multiplying this gives

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$