MATH 340; FINAL EXAM, 150 points, December 17 , 2007 (R.A.Brualdi)

TOTAL SCORE :

Name: These R. Solutions

I. (24 points; 3 points each) Answer the following short answer questions (no justification wanted). If the information given is not enough to uniquely determine the answer, write *undetermined*.

Let A be an n by n matrix with eigenvalues (including multiplicities) 3, 3, 4, 4, 4.

1. The order n of A is: 5
2. The determinant of A is: $3^2 4^3 = 576$
3. The coefficient of λ^4 in the characteristic polynomial of A is: $-(3+3+4+4+4) = -18$
4. The dimension of the row space of A is: 5
5. The eigenvalues of the matrix A^2 of A are:$3^2, 3^2, 4^2, 4^2, 4^2$.
6. Is A invertible? YES (no eigenvalue is 0)
7. The dimension of the eigenspace of A for the eigenvalue 3 is: Undetermined
8. Is A diagonalizable? Undetermined

II. (12 points; 2 points each) **Circle** whether the following assertions are **T**rue or **F**alse:

1. F: A real, square matrix always has at least one real eigenvalue.
2. T: A finite dimensional vector space with an inner product always has an orthonormal basis.
3. T : Every real, symmetric matrix is diagonalizable..
4. T: If P is an orthogonal matrix, then $|\det P| = 1$.
5. T: If u is orthogonal to vectors v and w, then u is orthogonal to every linear combination of v and w.
6. T (I hope!): I have learned a lot of good stuff in this course.

III (6+12+8 =26 points) Consider the homogeneous system of 2 equation in 4 unknowns with coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

1. Determine a basis for the null space U of A.

   $x_1 = -a - b, x_2 = a, x_3 = -b, x_4 = b$, so null space consists of vectors of the form

$$\begin{bmatrix} -a-b \\ a \\ -b \\ b \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

   and so

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

   form a basis for the null space.

2. Use the Gram-Schmidt process to determine a orthonormal basis of U (**Show your work!**) We have

$$u_1 = v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} -1/2 \\ -1/2 \\ -1 \\ 1 \end{bmatrix}.$$

   The length of u_1 is $\sqrt{2}$ and the length of u_2 is $\sqrt{5/2}$. Dividing by the length gives an orthonormal basis: $w_1 = \frac{1}{\sqrt{2}} u_1$, $w_2 = \frac{\sqrt{2}}{\sqrt{5}} u_2$.

3. Show that the vector $w = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -2 \end{bmatrix}$ is in U and write it as a linear combination of the orthonormal basis found above.

   We easily check that $Aw = 0$ and so w is in the null space. To write w has a linear combination of w_1 and w_2 we need only take dot products: $w \cdot w_1 = \frac{-4}{\sqrt{2}} = -2\sqrt{2}$ and $w \cdot w_2 = \frac{-5\sqrt{2}}{\sqrt{5}} = -\sqrt{10}$.

IV. (10+10+10=30 points) Prove the following three simple assertions:

1. if A is similar to B, and B is similar to C, then A is similar to C.

   Proof:
   We have $B = P^{-1}AP$ and $C = Q^{-1}BQ$ for some invertible P and Q. Then

   $$C = Q^{-1}BQ = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$$

   and so A is similar to B.

2. If 2 is an eigenvalue of A, then 18 is an eigenvalue of $2A^3 - A^2 + 3A$.

   Proof: Let x be a corresponding eigenvector (so nonzero). We then calculate that

   $$(2A^3-A^2+3A)x = 2(A^3)x-(A^2)x+3(A)x = \cdots = 2\cdot 2^3x-2^2x+3\cdot 2x = (16-4+6) = 18x$$

   and so 18 is an eigenvalue of $2A^3 - A^2 + 3A$.

3. If A has n linearly independent eigenvectors $u_1, u_2, \ldots, u_n$, then A is diagonalizable.

   Proof. Let P be the matrix whose columns are the n linearly independent eigenvectors corresponding to eigenvalues $d_1, d_2, \ldots, d_n$. Then

   $$AP = PD \text{ and so } P^{-1}AP = D$$

   where D is the diagonal matrix with $d_1, d_2, \ldots, d_n$.

V. (12 points; 3 points each) Let A be a 5 by 7 matrix of rank 3. Answer the following questions:

1. The dimension of the column space of A is: 3

2. The dimension of the row space of A is:3

3. The dimension of the null space of A is:$7 - 3 = 4$

4. The dimension of the null space of A^T is: $5 - 3 = 2$

VI. (6+8+12=26 points) Consider the linear transformation $L : R^3 \to R^3$ given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 + 2x_3 \\ x_1 + 2x_2 - x_3 \end{bmatrix}.$$

1. What is the matrix of L w.r.t the standard basis $S : e_1, e_2, e_3$ of R^3?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}.$$

2. Consider the basis $T : u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ What is the transition matrix from T to S

   It's

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

3. What is the matrix of L w.r.t the basis T?

   It's $P^{-1}AP$ where $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. Multiplying we get

$$\frac{1}{2} \begin{bmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

VII. (20 points) Determine a least squares solution to

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \\ -1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let A be the coefficient matrix above and b the right column vector. Then we want a solution to

$$A^T Ax = A^T b, \text{ that is, to } \begin{bmatrix} 7 & -2 \\ -2 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Solving this simple system we get $x_1 = \frac{55}{101}$ and $x_2 = \frac{41}{101}$.