MATH 240; EXAM # 2, 100 points, November 23, 2004 (R.A.Brualdi)

TOTAL SCORE (9 problems; 100 points possible):

Name: SOLUTIONS

TA: Darren Neubauer (circle time) Mon 12:05 Mon 1:20 Wed 12:05 Wed. 1:20

Do NOT compute binomial coefficients or factorials in any of the problems

1. [10 points] Determine a **recurrence relation, with initial conditions** for the number $a_n (n \geq 1)$ of ways to tile a 2 by n rectangle with 1 by 2 pieces (dominoes) and 2 by 2 pieces ("square 4-ominoes"). Do **NOT** solve the recurrence relation.

The square in the upper left corner can be covered by a vertical domino (a_{n-1} ways to complete, horizonal domino, with one below it (so a_{n-2} ways to complete, or a square domino (a_{n-2} ways to complete. So,

$$a_n = a_{n-1} + 2a_{n-2}, n \geq 3$$

where $a_1 = 1$ and $a_2 = 3$.

2. [10 points] Derive a **closed formula** for the **general solution** of the recurrence relation:

$$a_n = 2a_{n-1} + 3a_{n-2}, n \geq 2.$$

The characteristic equation is $x^2 - 2x - 3 = 0$ which has (char.) roots 3 and -1. So the general solution is $a_n = C3^n + D(-1)^n$ where C, D are arbitrary constants.

3. [5 points] Give a recurrence relation for the number of comparisons $f(n)$ used in the following Divide and Conquer algorithm to compute both the largest and smallest number is a sequence $a_1, a_2, \ldots, a_n$ of n (even) numbers:

> Divide the sequence in half, compute the largest and smallest of each half, and compare the two smallest numbers and the two largest numbers so obtained.

$$f(n) = 2f(n/2) + 2.$$

4. [5 points] Exhibit a full binary tree which shows how the Fibonacci number f_5 is computed by a recursive algoritm.

[I won't draw this here.]

5. [5 points] The symbol $p\{S\}q$ means (**Choose One**):

(a) If p is true, and the program segment S has exponential complexity, then q is true.

(b) If the input values to program segment S satisfy p, and S is executed, then the output values satisfy q.

(c) If p is a proposition and S is a correct program segment, then q is a proposition.

(d) If program segment S has property p and S is executed and terminates, then S has property q.

(e) None of the above.

Answer is (e).

6. [10 points] There are 500 balls in a bag numbered $1, 2, 3, \ldots, 500$. A ball is selected at random from the bag. What is the probability that the ball selected has a number divisible by 6 or 15?

Use $P(E \cup F = P(E) + P(F) - P(E \cap F)$ to get

$$\lfloor 500/6 \rfloor + \lfloor 500/15 \rfloor - \lfloor 500/30 \rfloor,$$

where 30 is the LCM of 6 and 15.

7. [20 points]

(a) The number of bit-strings of length 10 that have exactly 6 0's is:

$\binom{10}{6}$

(b) A team consists of 8 players one of whom is designated as captain and another of whom is designated as assistant captain. There are 15 available players. The number of different teams that can be formed is:

$(15)(14)\binom{13}{6}$ OR $\binom{15}{8}(80(7)$

(c) A team consists of 8 players two of whom are designated as co-captains, There are 15 available players. The number of different teams that can be formed is:

$\binom{15}{2}\binom{13}{6}$ OR $\binom{15}{8}\binom{8}{2}$.

(d) The coefficient of x^6 when $(3 - x)^9$ is multiplied out is:

$\binom{9}{6}3^3$

8. [25 points]

(a) State Pascal's formula for $\binom{n}{k}$ where $1 \leq k \leq n-1$:

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

(b) The number of 9-digit numbers whose digits are $1, 1, 1, 5, 5, 5, 5, 8, 8$ is:

$9!/(3!4!2!)$

(c) The number of ways to distribute 20 **different** objects among 5 people is:

5^{20}

(d) The number of ways to distribute 20 **identical** objects among 5 people is:

The number of solutions in nonnegative integers of the equation $a+b+c+d+e=20$ and so $\binom{24}{4}$.

9. [10 points] Determine the number of 9 digits numbers with distinct digits $1, 2, \ldots, 9$ having the property that 123, 456, 789 never occur as consecutive digits in the number.

Use the inclusion-exclusion principle to get

$$9! - 3 \cdot 7! + 3 \cdot 5! - 3!.$$