MATH 240; EXAM # 2, 101 points, November 22, 2005 (R.A.Brualdi)

TOTAL SCORE (10 problems; 101 points possible):

Name: THESE R. SOLUTIONS

TA: Anders Hendrickson (circle time) Mon 12:05 Mon 1:20 Wed 12:05 Wed. 1:20

Do NOT compute factorials or binomial coefficients.

1. [10 points] Give a **recursive definition with initial conditions** of:

(a) the function $f(n)$ where $f(n) = 8^n$, $n \geq 1$.

$f(n) = 8f(n-1), n \geq 2$ with $f(1) = 8$.

(b) the number h_n of bit-sequences of length n that do not contain 3 consecutive 1's.

$h_n = h_{n-1} + h_{n-2} + h_{n-3}, n \geq 4$ with $h_1 = 2, h_2 = 4, h_3 = 7$.

2.[15 points] Consider a 2 by n rectangle of unit squares and *pieces* which are either dominoes (two unit squares joined along a side) or squares formed from four unit squares. Let h_n ($n \geq 1$) be the number of ways to tile the 2 by n rectangle with these kind of pieces.

(i) Derive a **recurrence relation for h_n with initial conditions**.

$h_n = h_{n-1} + 2h_{n-2}, n \geq 3$ with $h_1 = 1, h_2 = 3$.

(ii) Use **strong induction** to prove that

$$h_n = \frac{2^{n+1} + (-1)^n}{3}.$$

By substitution we check that the formula gives $h_1 = 1$ and $h_2 = 3$, verifying the basis step of the induction.

For the direct proof in the inductive step we assume that

$$h_k = \frac{2^{k+1} + (-1)^k}{3} \text{ for all } k = 1, 2, \ldots, n-1$$

and prove that this implies that

$$h_n = \frac{2^{n+1} + (-1)^n}{3}.$$

But

$$h_n = h_{n-1} + 2h_{n-2} = \frac{2^n + (-1)^{n-1}}{3} + 2\frac{2^{n-1} + (-1)^{n-2}}{3}.$$

Combining we get $h_n = \frac{2^{n+1}+(-1)^n}{3}$.

3. [5 points] A linear homogeneous recurrence relation with constant coefficients for a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ has characteristic roots 3, 5, 8, 9. Write down the **general solution**.

$a_n = c3^n + d5^n + e8^n + f9^n$ for arbitrary constants c, d, e, f.

4. [5 points] The symbol $p\{S\}q$ means (**Choose One**):

(a) If program segment S has property p and S is executed and terminates, then S has property q.

(b) If the input values to program segment S satisfy p, and S is executed, then the number of output values is q.

(c) If p is a proposition and S is a correct program segment, then q is a proposition.

(d) **THIS ONE** If the input values to program segment S satisfy p, and S is executed and terminates, then the output values satisfy q.

(e) If p is true, and the program segment S has exponential complexity, then q is true.

5. [18 points] Answer the following questions:

(a) The **number** of strings of length 12 that have four 0's, five 1's, and three 2's is:

$\binom{12}{4}\binom{8}{5}\binom{3}{3}$

(b) The **number** of poker hands (from an ordinary 52 deck) that contain exactly two pairs (two cards of one rank, two cards of another rank, and one card of yet another rank) equals:

$\binom{13}{2}\binom{4}{2}\binom{4}{2}44$

(c) Passwords are strings of 5 characters using only the 14 characters A, B, C, D, E, F, 1, 2, 3, 4, 5, 6, 7, 8. The **number** of passwords that contain at least 1 letter is:

Find the number that contain no letter and subtract from the total:

$14^5 - 8^5$

6. [18 points] Answer the following questions:

(a) The 25 letters A, B, C, ..., X, Y are to put into 5 labeled boxes (box I, box II, ..., box VI) with no restrictions whatsoever. The **number** of ways this can be done is:

5^{25}

(b) The 25 letters A, B, C, ... X, Y are to put into 5 labeled boxes (box I, box II, ..., box VI) in such a way that each box gets 5 of the letters. The **number** of ways this can be done is:

$\binom{25}{5}\binom{20}{5}\binom{15}{5}\binom{10}{5}\binom{5}{5}$

(c) You want to buy 20 pints of ice-cream from the UW Dairy of possible flavors vanilla, chocolate, berry alvarez, pumpkin, and cookie-dough. If you want to be sure to have at least three pints of chocolate and at least two pints of pumpkin, the **number** of possible solutions is:

Number of solutions in nonnegative integers of $x + y + z + u + v = 15$ and this is $\binom{19}{15}$

7. [5 points] Ten people attend a party, each wearing a hat, which is then checked. At the end of the evening each person is given a hat *at random*. What is the **approximate probability** that no one gets her or his own hat? (**No derivation necessary**.)

About $1/e$ where $e = 2.718281828459045$

8. [10 points] Three cats A, B, C, four dogs P, Q, R, S and five chickens U, V, W, X, Y are at a pet-show. The **number** of ways they can line up so that the cats do not appear consecutively in alphabetical order ABC, the dogs do not appear consecutively in alphabetical order PQRS, and the chickens do not appear consecutively in alphabetical order UVWXY is:

Application of the inclusion-exclusion proncipile:

$$12! - (10! + 9! + 8!) + (7! + 6! + 5!) - 3!$$

9. [5 points] Let $a_1 < a_2 < \cdots < a_n$ be an increasing list of integers. Let x be an integer. Give a **recurrence relation** for the number $f(n)$ of basic steps used in the divide and conquer algorithm known as binary search for determining whether or not x is on the list. You may assume that n is even.

$f(n) = f(n/2) + 2$

10. [10 points] Consider the recurrence relation $a_n = 3a_{n-1} + 1$, $(n \geq 1)$ where $a_0 = 1$. Use a **top-down calculation** to find a simple formula for a_n.

$$a_n = 3a_{n-1} + 1 = 3(3a_{n-2} + 1) + 1 = 3^2 a_{n-2} + 3 + 1 = 3^2(3a_{n-3} + 1) + 3 + 1$$

$$= 3^3 a_{n-3} + 3^2 + 3 + 1 = \cdots = 3^n a_0 + 3^{n-1} + \cdots + 3 + 1$$

$$= 3^n + 3^{n-1} + \cdots + 3 + 1 = \frac{3^{n+1} - 1}{2}.$$