

MATH 240; EXAM # 2, 100 points, November 24, 2003 (R.A.Brualdi)

TOTAL SCORE (7 problems; 100 points possible):

Name:

TA: Josh Davis, (circle) TUES 1:20 TUES 2:25 THURS. 1:20 THUR 2:25

TA: Neil Ramsamooj, (circle) TUES 8:50 TUES 12:05 THUR 8:50 THUR 12:05

1. [10 points] Let the sequence of numbers $a_1, a_2, a_3, \dots, a_n, \dots$ be defined recursively by:

$$a_1 = 6, a_2 = 15, a_n = 7a_{n-1} + 2a_{n-2}, (n \geq 3).$$

Use induction to prove that $a_n \equiv 0 \pmod{3}$ for all $n \geq 1$. [Be sure to label the two steps of induction and to indicate in the inductive step exactly which proposition you are proving and how.]

Basis Step: $a_1 = 6 = 3 \cdot 2$ and $a_2 = 15 = 3 \cdot 5$ and so are divisible by 3.

Inductive Step (For $n \geq 3$, if $P(1), \dots, P(n-1)$ are true, then $P(n)$ is true, where $P(n)$ is “ a_n is divisible by 3.”: DIRECT Proof):

So assume that $P(1), \dots, P(n-1)$ are true. We need to prove that $P(n)$ is true.

Since $P(n-1)$ and $P(n-2)$ are true, a_{n-1} and a_{n-2} are divisible by 3: $a_{n-1} = 3k$ and $a_{n-2} = 3l$. Then by the recursion

$$a_n = 7a_{n-1} + 2a_{n-2} = 7 \cdot 3k + 2 \cdot 3l = 3(7k + 2l),$$

and so a_n is divisible by 3. Hence the result holds by induction.

2. (i) [5 points] The symbol $p\{S\}q$ means (**Choose one**):

- (a) if p is satisfied by the input values to program segment S , and S is executed, then q is satisfied by the output values.
- (b) If p is true, and program segment S doesn't have any bugs, then q is true.
- (c) **YES** if p is satisfied by the input values to program segment S , and S is executed and terminates, then q is satisfied by the output values.
- (d) If program segment S has property p and S is executed and terminates, then S has property q .
- (e) None of the above.

(ii) [5 points] A divide-and-conquer algorithm takes a problem P of size n and divides it into 4 subproblems P_1, P_2, P_3, P_4 each of size $n/10$. If it takes $\log n$ steps to assemble the solutions of the subproblems P_1, P_2, P_3, P_4 into a solution of the problem P , write a recurrence relation for the complexity $f(n)$ of this algorithm to solve problem P .

$$f(n) = 4f(n/10) + \log n.$$

3. (i) [8 points] Determine, with initial conditions, a recurrence relation for the number a_n , ($n \geq 0$) of ways to climb a flight of n stairs if you are allowed to take 1-step, 2-steps, or 4-steps at a time. (Note that we take $a_0 = 1$.)

$$a_n = a_{n-1} + a_{n-2} + a_{n-4}, (n \geq 4); a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3.$$

(ii) [5 points] The general solution of a recurrence relation of order 3 whose characteristic polynomial has roots 5, 5, 8 is:

$$c5^n + dn5^n + e8^n, \text{ where } c, d, e \text{ are arbitrary constants.}$$

4. Answer the following counting questions [**you need not compute factorials or binomial coefficients**].

(i) [5 points] The number of nonempty subsets of a set of 7 elements equals:

$$2^7 - 1.$$

(ii) [8 points] Consider an ordinary deck of 52 cards. A 6-combination of cards is chosen at random. The probability that the 6-combination contains exactly one 8 equals:

$$\frac{4 \cdot \binom{48}{5}}{\binom{52}{6}}.$$

(iii) [10 points] The number of strings of length 12 using the alphabet $\{a, b, c\}$ which contain exactly four a 's or exactly five b 's equals:

Let A be those strings which contain exactly four a 's and B be those that contain exactly 5 b 's. Then we want $|A \cup B|$. But this equals

$$|A \cup B| = |A| + |B| - |A \cap B| = \binom{12}{4} 2^8 + \binom{12}{5} 2^7 - \binom{12}{4} \binom{8}{5}.$$

(iv) [8 points] There are 4 advisors and 100 students at a school. The number of way to have a group of 25 students assigned to each advisor equals:

$$\binom{100}{25} \binom{75}{25} \binom{50}{25} \binom{25}{25} = \frac{100!}{25!^4}.$$

(v) [10 points] An ice-cream store has 5 different varieties of ice cream: Vanilla, Chocolate, Strawberry, Banana, and Pistachio. You have an overwhelming craving for ice cream and you order a dish with 15 scoops of ice cream. How many possible dishes are there, if you want at least one scoop of each of V, C, and S, and at least two of B?

Number of nonnegative integral solutions of

$$x_V + x_C + x_S + x_B + x_P = 15, \text{ where } x_V \geq 1, x_C \geq 1, x_S \geq 1, x_B \geq 2.$$

This is the same as the number of nonnegative integral solution sof:

$$y_V + y_C + y_S + y_B + y_P = 10,$$

and this equals

$$\binom{14}{4} = \binom{14}{10}.$$

5. [5 points] Let R and S be the relations with matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The matrix of the relation $S \circ R$ is:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

6. [8 points] The transitive closure of a relation on a set of 5 elements $\{1, 2, 3, 4, 5\}$ is being determined by **Warshall's algorithm**. The matrix W_2 is shown below. Determine W_3 .

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

7. (i) [8 points] Consider the relation R on the integers $\{-3, -2, -1, 0, 1, 2\}$ defined by $a R b$ if and only if $|a| = |b|$ (absolute value). Say, by circling, whether this relation is

reflexive-YES, symmetric-YES, Anti-symmetric-NO, transitive-YES,

Is this relation an equivalence relation? **YES it is**

If it is an equivalence relation, determine the equivalence classes $[-2]_R$, $[0]_R$, and $[-3]_R$:

$$[-2]_R = \{-2, 2\}$$

$$[0]_R = \{0\}$$

$$[-3]_R = \{-3\}$$

(ii) [5 points] Give an example of a relation on the set $\{a, b, c\}$ which is symmetric, anti-symmetric, and transitive, but **not** reflexive. (You can give the relation by drawing its digraph.)

Here is the matrix of such a relation: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$