MATH 240; EXAM # 1, 100 points, Oct. 7, 2002 (R.A.Brualdi)

TOTAL SCORE (10 problems):

Name: SOLUTIONS **TA (circle): Matthew Petro Dilip Raghavan**

Disc. (circle) TUES. THURS. TIME:

1. [8 points] Calculate the sum $\sum_{i=0}^{n} 10 \cdot 8^i$:

$$10\frac{8^{n+1}-1}{7}$$

2. [8 points] Give the disjunctive normal form (sum of products) of the Boolean function $f(x, y, z, u)$ where $f(x, y, z, u) = 1$ if and only if exactly one of x, y, z, u is 1.

$$f(x, y, z, u) = \bar{x}\bar{y}\bar{z}u + \bar{x}\bar{y}z\bar{u} + \bar{x}y\bar{z}\bar{u} + x\bar{y}\bar{z}\bar{u}$$

3. [8 points] Let $Q(x, y)$ be the predicate: Team x in Conference y has a winning record. Express using quantifiers and the predicate $Q(x, y)$:

Every conference has at least one team with a winning record:

$$\forall y \exists x Q(x, y)$$

There is a conference in which no team has a winning record.

$$\exists y \forall x \neg Q(x, y)$$

4. [8 points] Express the set $\overline{\overline{A} - B}$ in a simple way that does not use the complement:

Using DeMorgan's Law we have,

$$\overline{\overline{A} - B} = \overline{\overline{A} \cap \overline{B}} = A \cup B.$$

5. [12 points] Let $f : A \to B$ be defined as follows:

$A = B = \{x : x \text{ a real number and } 0 \leq x \leq 10\}$ and $f(x) = \lceil x \rceil - \lfloor x \rfloor$.

Answer the following questions:

(i) Is f an injection? Why or why not?
NO, since e.g. $f(0) = f(1) = 0$.
(ii) Is f a surjection? Why or why not?

NO, since e.g. there is no x such that $f(x) = 2$. (or see (iii) below)

(iii) What is the range of f?

Range $= \{0, 1\}$.

6. [8 points] Let $f(x) = \frac{3x^2-4x+1}{x+5}$. Find a very simple function $g(x)$ such that $f(x) = O(g(x))$.

Taking limits we see that $f(x) = O(x)$.

7. [10 points] Recall the identification:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

In the Caeser Cipher defined by $f(p) = p + 5 \bmod 26$, **decrypt** the word **DFP**.

The inverse function of f is $f(p) = p - 5 \bmod 26$, that is, $f(p) = p + 21 \bmod 26$. So DFP is decrypted as YAK

8. [12 points] Use the **Euclidean algorithm and only the Euclidean Algorithm** to find the GCD(330,124):

Using the Euclidean algorithm in a straightforward way we get that the GCD is 2.

9. [12 points] Use the **technique of the Chinese Remainder Theorem** to calculate the smallest positive solution of the system of congruences:

$x \equiv 1 \bmod 4$
$x \equiv 2 \bmod 5$
$x \equiv 3 \bmod 7$.

See pages 142-3 of the text. We have $M = 4 \cdot 5 \cdot 7 = 140$, $M_1 = 5 \cdot 7 = 35$, $M_2 = 4 \cdot 7 = 28$, and $M_3 = 4 \cdot 5 = 20$. We find the inverse of $M_1 = 35 \bmod 4$ to be 3, the inverse of $M_2 = 28 \bmod 5$ to be 2, and the inverse of $M_3 = 20 \bmod 7$ to be 6. The solution is $x \equiv 1(35)(3) + 2(28)(2) + 3(20)6 \bmod 140$, and this gives mod 140 the smallest number 17.

10. [14 points] Let a, b, m be positive integers with $m \geq 2$. Suppose that $a \equiv b \bmod m$. Prove that GCD $(a, m) =$ GCD (b, m).

We have $a = b + qm$ for some integer q. From this we see that any integer that divides both a and m also divides b (so divides both b and m), and any integer that divides both b and m also divides a (so divides both a and m). So $\{a, m\}$ and $\{b, m\}$ have the same divisors and so the same GCD.