Math 240, Fall Semester 2001-02 NAME:
(Prof.) R.A. Brualdi
Exam 1: October 12, 2001, Total Points:
[points in brackets]

1. [15 points] (a) Partition the propositions into groups whereby each pair of propositions in the same group are logically equivalent while two propositions in different groups are not logically equivalent:

$p \to q$,

$\neg q \to \neg p$,

$\neg(p \vee q)$,

$p \vee \neg q$,

$\neg(\neg p \wedge q)$,

$\neg p \vee q$.

$p \to q, \neg q \to \neg p, \neg p \vee q$

$p \vee \neg q, \neg(\neg p \wedge q)$

$\neg(p \vee q)$

(b) Let the variable x in the predicates $P(x)$, $Q(x)$, and $R(x)$ vary over a universe U. Let

$$A = \{x|P(x) \text{ is true}\}, B = \{x|Q(x) \text{ is true}\}, \text{ and } C = \{x|R(x) \text{ is true}\}$$

be the truth sets of $P(x)$, $Q(x)$ and $R(x)$. Using only the set operations union and complement, give the truth sets of:

(i) $(P(x) \wedge Q(x)) \vee \neg R(x)$:

$(A \cap B) \cup (\overline{C})$

(ii) $P(x) \to Q(x)$:

$\overline{A} \cup B$

(ii) $P(x) \leftrightarrow Q(x)$:
$(A \cap B) \cup (\overline{A} \cap \overline{B})$

2. [10 points] (a) How many different Boolean functions of n Boolean variables are there? Justify your answer.

2^{2^n}

(b) Let x, y, z be Boolean variables. Express the Boolean function $f(x, y, z)$ using only the Boolean operations $+$, $\cdot$, and $\overline{}$:

$f(x, y, z) = 1$ if and only if an odd number of x, y, z have value 1.
$x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}z + xyz$

3. [15 points] (a) Which of the following functions are injective (one to one), surjective (onto), bijective (one to one and onto), or none of the above. If bijective, give the inverse. (Here Z is the set of all integers, Z^+ is the set of positive integers, and $\mathcal{P}(S)$ denotes the collection of all subsets of a set S.)

$f : Z^+ \to Z^+,\ f(x) = x + 1.$

injective

$f : Z \to Z,\ f(x) = x + 1.$

bijective; $f^{-1}(x) = x - 1$

$f : \mathcal{P}(S) \to \mathcal{P}(S),\ f(X) = \overline{X}.$

bijective; $f^{-1}(A) = \overline{A}$

$f : \mathcal{P}(S) \to \mathcal{P}(S),\ f(X) = X \cap A$ where A is a fixed subset of S.

neither in general

(b) Consider the function $f : \Re \to Z$ defined by $f(x) = \left\lceil x - \frac{1}{2} \right\rceil$. Determine the inverse image of 3.

$\lceil x - 1/2 \rceil = 3$ iff $2 < x - 1/2 \leq 3$ iff $5/2 < x \leq 7/2$

4. [20 points] (a) Use the Euclidean algorithm to find the multiplicative inverse of 20 modulo 51 (*Trial and error not accepted*).

inverse of 20 mod 51 is 23

(b) Does $20x \equiv 4 \pmod{51}$ have a solution x? If so, find it. Is this solution unique modulo 51? **Explain**.

YES: $x \equiv (20)^{-1} \cdot 4 \; mod \; 51 \equiv 23 \cdot 4 \; mod 51 \equiv 92 \; mod 51 \equiv 41 mod \; 51$.

(c) Note that $11 \cdot 3 \equiv 1 \bmod 32$. Use this fact to find the 11th root of 31 modulo 51, i.e. a number a such that $a^{11} \equiv 31$ modulo 51.

This question was thrown out.

5. [20 points] We have $385 = 5 \cdot 7 \cdot 11$.

(a) Use the Chinese Remainder Theorem to explain how to represent every integer n with $0 \leq n \leq 384$ **uniquely** as a three tuple of integers (a_1, a_2, a_3) where $0 \leq a_1 \leq 2$, $0 \leq a_2 \leq 6$, and $0 \leq a_3 \leq 10$. You must explain the uniqueness.

We did not do this.

(b) What three tuple represents 200?

We did not do this.

(c) Under what conditions does the following hold:

$$a|c \text{ and } b|c \text{ implies } ab|c?$$

(Here a, b, c are positive integers.)

When a and b are relatively prime.

6. [10 points] Consider the functions below and list them from top to bottom (smallest order of magnitude to largest) putting functions f and g in the same line if $f = \Theta(g)$:

$$n^4, \quad 2^n, \quad n \log n, \quad n^2, \quad 5^n, \quad n \log n^4, \quad \sqrt{n}.$$

$\sqrt{n}$, both of $n \log n, n \log n^4, n^2, n^4, 2^n, 5^n$

7. [10 points] Prove by mathematical induction:

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} = \frac{n}{2n+1} \quad (n \geq 1).$$

We haven't done this yet.