KUDLA-RAPOPORT CONJECTURE AT RAMIFIED PRIMES

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Abstract. In this paper, we propose a modified Kudla-Rapoport conjecture for the Krämer model of unitary Rapoport-Zink space over a ramified prime, which is a precise identity relating intersection numbers of special cycles to derivatives of Hermitian local density polynomials. We also introduce the notion of special difference cycles, which has surprisingly simple description. Combining this with induction formulas of Hermitian local density polynomial, we prove the modified Kudla-Rapoport conjecture when $n = 3$.

Contents

1. Introduction

In their seminal work [KR11] and [KR14b], Kudla and Rapoport made a conjectural local arithmetic Siegel-Weil formula (the Kudla-Rapoport conjecture) relating the intersection numbers of special divisors on unitary Rapoport-Zink spaces (at a unramfied prime) to the central derivative of certain local density polynomials. A unitary Rapoport-Zink (RZ) space is a local version of a unitary Shimura variety associated to a general unitary group $\mathrm{GU}(1, n-1)$. The Kudla-Rapoport conjecture plays a central role in the arithmetic Siegel-Weil formula for unitary Shimura varieties, which was first proposed by Kudla in [Kud97] for orthogonal Shimura varieties. When $n = 1$ or 2, the Kudla-Rapoport conjecture was proved in [KR11]. The case when $n = 3$ was proved in [Ter10]. The general case was proved recently in [LZ] by an ingenious induction. The Archimedean analogue of the Kudla-Rapoport conjecture was proved in [Liu11] and [GS19].

Originally the Kudla-Rapoport conjecture was proposed only for good primes, namely inert primes over which the Rapoport-Zink space has hyperspecial level structure. For ramified primes, there are two kinds of well-understood arithmetic models of RZ spaces. One is the exotic smooth model which has good reduction, the other is the Krämer model proposed in [Krä03] which only has semi-stable reduction. We mention that

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the underlying Hermitian space of exotic smooth model is always nonsplit when n is even, see [RSZ17, Lemma 3.5]. The analogue of Kudla-Rapoport conjecture for the even dimensional exotic smooth model was studied in [LL21], in which case the conjecture can be proved by the same strategy as [LZ]. For the Krämer model, however, it was expected that serious modification of the original Kudla-Rapoport conjecture is needed. A precise formulation has not previously been known. One of the main goals of this paper is to formulate a precise conjecture (Conjecture 1.1) based on earlier work of [Shi20] and [HSY] for the case when $n = 2$. We then prove Conjecture 1.1 for $n = 3$. We will try to resolve the general case in a sequel.

1.1. **The naive conjecture.** Let p be an odd prime and F be a ramified quadratic field extension of a p-adic number field F_0 with residue field $\mathbb{F}_q$. Fix an algebraic closure k of $\mathbb{F}_q$. Fix a uniformizer π of F such that $\pi_0 = \pi^2$ is a uniformizer of F_0 and let v_π be the valuation on F such that $\mathrm{v}_\pi(\pi) = 1$. Let $\breve{F}_0$ be the completion of a maximal unramified extension of F_0 and $\breve{F} := F \otimes_{F_0} \breve{F}_0$. Let $\mathcal{O}_{\breve{F}}$ and $\mathcal{O}_{\breve{F}_0}$ be the rings of integers of $\breve{F}$ and $\breve{F}_0$ respectively. For a Hermitian lattice or space M of rank n, we define its sign as

$$\chi(M) = \chi((-1)^{\frac{n(n-1)}{2}}\det(M)) = \pm 1 \tag{1.1}$$

where χ is the quadratic character of $F_0^\times$ associated to F/F_0. We call M split or non-split depending in whether $\chi(M) = 1$ or -1. For a Hermitian matrix T, define $\chi(T)$ to be the sign of its associated Hermitian lattice.

Let $\mathbb{Y}$ and $\mathbb{X}$ be pre-fixed framing Hermitian formal $\mathcal{O}_F$-modules of signature $(0, 1)$ and $(1, n-1)$ respectively over $\operatorname{Spec} k$. Recall that Hermitian formal $\mathcal{O}_F$-modules are a particular kind of formal p-divisible groups with $\mathcal{O}_F$-action, see Section 2.1. The space of special quasi-homomorphisms

$$\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.2}$$

is equipped with a Hermitian form $h(,)$, see (2.2). Let $\epsilon = \chi(\mathbb{V})$. The Rapoport-Zink space $\mathcal{N}^{\mathrm{Kra}}_{n,\epsilon}$ parameterizes certain classes of supersingular Hermitian formal $\mathcal{O}_F$-modules of signature $(1, n-1)$ over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$, see Section 2.1. It is a formal scheme over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$ with semi-stable reduction and can be viewed as a minimal regular model of the formal completion of the corresponding global unitary Shimura variety along its basic locus over p. When n is odd, $\mathcal{N}^{\mathrm{Kra}}_{n,1}$ is isomorphic to $\mathcal{N}^{\mathrm{Kra}}_{n,-1}$. We often write $\mathcal{N}^{\mathrm{Kra}}$ instead of $\mathcal{N}^{\mathrm{Kra}}_{n,\epsilon}$ for simplicity.

For each subset $L \subset \mathbb{V}$, define $\mathcal{Z}^{\mathrm{Kra}}(L)$ to be the formal subscheme of $\mathcal{N}^{\mathrm{Kra}}$ where $\mathbf{x}$ deforms to a homomorphism for any $\mathbf{x} \in L$. Let $L \subset \mathbb{V}$ be an $\mathcal{O}_F$-lattice of rank r. We say L is integral if $h(,)|_L$ is non-degenerate and takes values in $\mathcal{O}_F$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ be a basis of L. We define

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_r)}] \in K_0(\mathcal{N}^{\mathrm{Kra}}) \tag{1.3}$$

where $\otimes^{\mathbb{L}}$ is the derived tensor product of complex of coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and $K_0(\mathcal{N}^{\mathrm{Kra}})$ is the Grothendieck groups of finite complexes of coherent locally free sheaves on $\mathcal{N}^{\mathrm{Kra}}$. By [How19, Corollary C], $^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)$ is independent of the choice of basis of L. When L has rank n, we define the intersection number

$$\mathrm{Int}(L) = \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)) \tag{1.4}$$

where χ is the Euler characteristic.

Let ∂_F be the different ideal of F/F_0. Define

$$\mathrm{Herm}_n(\mathcal{O}_F)^\vee := \{T = (t_{ij}) \in \mathrm{Herm}_n(F) \mid \mathrm{val}_\pi(t_{ii}) \geq 0, \text{ and } \mathrm{val}_\pi(t_{ij}\partial_F) \geq 0\}.$$

Notice that $\mathrm{Herm}_n(\mathcal{O}_F)^\vee$ is the dual of $\mathrm{Herm}_n(\mathcal{O}_F)$ under the pairing $\mathrm{Tr}(XY)$. For two Hermitian lattices L and M of rank n and m respectively, set

$$I(M, L, d) = \{\phi \in \mathrm{Hom}_{\mathcal{O}_F}(L/\pi_0^d L, M/\pi_0^d M) \mid (\phi(x), \phi(y)) = (x, y) \in \pi_0^d \cdot \mathrm{Herm}_n(\mathcal{O}_F)^\vee, \ \forall x, y \in L\}.$$

The following limit

$$\alpha(M, L) = \lim_{d \to \infty} q^{-dn(2m-n)} |I(M, L, d)| \tag{1.5}$$

exists and is called a Hermitian local density. We also use $\alpha(S, T)$ to denote $\alpha(M, L)$ if M (resp. L) is represented by a Gram matrix S (resp. T). Let $\mathcal{H}$ be the (Hermitian) hyperbolic plane with Gram matrix $\mathcal{H} = \begin{pmatrix} 0 & \pi^{-1} \\ -\pi^{-1} & 0 \end{pmatrix}$. One can show that there is a (local density) polynomial $\alpha(M, L, X) \in \mathbb{Q}[X]$ such that

$$\alpha(M \oplus \mathcal{H}^k, L) = \alpha(M, L, q^{-2k}).$$

Define its derivative by

$$\alpha'(M,L) := -\frac{\partial}{\partial X}\alpha(M,L,X)|_{X=1}.$$

Let M be the unique unimodular Hermitian $\mathcal{O}_F$-lattice of rank n with $\chi(M) = -\chi(L)$. The naive analogue of the local Kudla-Rapoport conjecture is

$$\mathrm{Int}(L) = 2\frac{\alpha'(M,L)}{\alpha(M,M)}. \tag{1.6}$$

But this conjectural formula is not even true for $n = 2$ according to the main theorem of [HSY]. The analytic side of the conjecture needs to be modified.

1.2. **The precise conjecture.** By [Shi18, Theorem 1.2], $\mathcal{Z}^{\mathrm{Kra}}(L)$ is empty when L is not integral, so we have

$$\mathrm{Int}(L) = 0.$$

On the analytic side, the right hand side of (1.6) is automatically zero only when $\mathrm{v}(L) := \min\{\mathrm{v}_\pi(h(v,v')) \mid v,v' \in L\} \leq -2$, and is non-zero when $\mathrm{v}(L) = -1$. So there should be correction terms involving Hermitian lattices M with $\mathrm{v}(M) = -1$. By [Jac62], there are $n-1$ equivalent classes of such Hermitian lattices:

$$\mathcal{H}_\epsilon^{n,i} := \mathcal{H}^i \oplus I_{n-2i,\epsilon} \quad \text{for } 1 \leq i \leq \frac{n}{2}, \quad \epsilon = \pm 1 \tag{1.7}$$

where $I_{n-2i,\epsilon}$ is the unimodular Hermitian lattice of rank $n-2i$ with $\chi(I_{n-2i,\epsilon}) = \chi(\mathcal{H}_\epsilon^{n,i}) = \epsilon$. When $n = 2r$ is even, we take $I_{0,\epsilon} = 0$ and $\mathcal{H}_1^{n,r} = \mathcal{H}^r$. Then the local arithmetic Siegel-Weil formula, a.k.a. the KR-conjecture at a ramified prime should be of the following form:

$$\mathrm{Int}(L) = 2\frac{\alpha'(I_{n,-\epsilon},L)}{\alpha(I_{n,-\epsilon},I_{n,-\epsilon})} + \sum_i c_\epsilon^{n,i}\frac{\alpha(\mathcal{H}_\epsilon^{n,i},L)}{\alpha(I_{n,-\epsilon},I_{n,-\epsilon})}, \tag{1.8}$$

where $\epsilon = \chi(L)$. Since $\mathrm{Int}(\mathcal{H}_\epsilon^{n,j}) = 0$, we should have

$$2\frac{\alpha'(I_{n,-\epsilon},\mathcal{H}_\epsilon^{n,j})}{\alpha(I_{n,-\epsilon},I_{n,-\epsilon})} + \sum_i c_\epsilon^{n,i}\frac{\alpha(\mathcal{H}_\epsilon^{n,i},\mathcal{H}_\epsilon^{n,j})}{\alpha(I_{n,-\epsilon},I_{n,-\epsilon})} = 0. \tag{1.9}$$

This system of equations turns out to determine the coefficients $c_\epsilon^{n,i}$ uniquely by Theorem 6.1. We propose the following Kudla-Rapoport conjecture at a ramified prime.

Conjecture 1.1. *The identity* (1.8) *always holds with the coefficients* $c_\epsilon^{n,i}$ *uniquely determined by (1.9).*

For convenience, we set ($\epsilon = \chi(L)$)

$$\partial\mathrm{Den}(L) = 2\frac{\alpha'(I_{n,-\epsilon},L)}{\alpha(I_{n,-\epsilon},I_{n,-\epsilon})} + \sum_i c_\epsilon^{n,i}\frac{\alpha(\mathcal{H}_\epsilon^{n,i},L)}{\alpha(I_{n,-\epsilon},I_{n,-\epsilon})}. \tag{1.10}$$

The conjecture holds for $\mathcal{N}_{2,\pm 1}^{\mathrm{Kra}}$ by results in [Shi20] and [HSY]. In this paper, we will prove the conjecture for $n = 3$ and provide some partial results in general case.

Theorem 1.2. *Conjecture 1.1 is true when* $n = 3$.

1.3. **Special difference cycles.** One of the novelty of the paper is the concept of special difference cycles. Let L_1 be an $\mathcal{O}_F$-lattice of $\mathbb{V}$ of rank n_1. Define the special difference cycle $\mathcal{D}(L_1) \in K_0(\mathcal{N}^{\mathrm{Kra}})$ by

$$\mathcal{D}(L_1) = {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L_1) + \sum_{i=1}^{n_1}(-1)^i q^{i(i-1)/2} \sum_{\substack{L_1 \subset L' \subset \frac{1}{\pi}L_1 \\ \dim_{\mathbb{F}_q}(L'/L_1) = i}} {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') \in K_0(\mathcal{N}^{\mathrm{Kra}}). \tag{1.11}$$

$\mathcal{D}(L_1)$ can be seen as a higher codimensional analogue of the difference divisor first introduced in [Ter10, Definition 2.10]. By the definition and a q-adic linear-algebraic inclusion-exclusion principle, we have (see Lemma 2.16)

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L_1) = \sum_{\substack{L' \text{integral} \\ L_1 \subset L' \subset L_{1,F}}} \mathcal{D}(L'). \tag{1.12}$$

Here $L_F = L \otimes_{\mathcal{O}_F} F$ for an $\mathcal{O}_F$-lattice L. The above summation is in fact finite. Assume that we have a decomposition $L = L_1 \oplus L_2$ of $\mathcal{O}_F$-lattices such that L_i has rank n_i and $n_1 + n_2 = n$. Define

$$\mathrm{Int}(L)^{(n_1)} = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L_1) \cdot \mathcal{Z}^{\mathrm{Kra}}(L_2)) \tag{1.13}$$

where $\cdot$ is the product on $K_0(\mathcal{N}^{\mathrm{Kra}})$ induced by tensor product of complexes. Notice that this definition in fact depends on the decomposition of L.

On analytic side, we define

$$\partial\mathrm{Den}(L)^{(n_1)} \coloneqq \partial\mathrm{Den}(L) - \sum_{i=1}^{n_1} (-1)^{i-1} q^{i(i-1)/2} \sum_{\substack{L_1 \subset L_1' \subset L_{1,F} \\ \dim L_1'/L_1 = n_1}} \partial\mathrm{Den}(L_1' \oplus L_2). \tag{1.14}$$

Again this definition depends on the decomposition of L. The analogue of (1.12) holds for $\partial\mathrm{Den}(L)^{(n_1)}$. As a consequence we have the following theorem (see Theorem 5.6 for a refinement).

Theorem 1.3. *Conjecture 1.1 is true if and only if for every lattice $L = L_1 \oplus L_2$ such that L_i has rank n_i, we have*

$$\mathrm{Int}(L)^{(n_1)} = \partial\mathrm{Den}(L)^{(n_1)}. \tag{1.15}$$

We speculate that $\mathcal{D}(L_1)$ is of a simple form when $n_1 = n - 1$. We prove this when $n = 3$ in the process of proving Theorem 1.2, see Theorem 1.4 below. As for evidences of the speculation for general n, the horizontal part of $\mathcal{D}(L_1)$ is almost irreducible (Proposition 4.5), and the intersection of $\mathcal{D}(L_1)$ with an exceptional divisor in $\mathcal{N}^{\mathrm{Kra}}$ is either ± 1 or 0 (Lemma 3.9).

1.4. **The case $n = 3$.** The proof of Theorem 1.2 is divided into three cases, see Section 11. For $\mathrm{v}(L) < 0$, we show directly $\partial\mathrm{Den} = \mathrm{Int}(L) = 0$. The case $\mathrm{v}(L) = 0$ is reduced to the case $n = 2$, which was proved in [Shi20] and [HSY]. For $\mathrm{v}(L) > 0$, we prove that $\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}$ for some decomposition $L = L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$, and then apply Theorem 1.3 (more precisely Theorem 5.6).

In order to prove $\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}$, we need to understand the decomposition of $\mathcal{D}(L^\flat)$. We say a lattice $\Lambda \subset \mathbb{V}$ is a vertex lattice if $\pi\Lambda \subseteq \Lambda^\sharp \subseteq \Lambda$ where $\Lambda^\sharp$ is dual lattice of Λ with respect to $h(,)$ and we call $t = \dim_{\mathbb{F}_q}(\Lambda/\Lambda^\sharp)$ the type of Λ. This has to be an even integer between 0 and n. We denote the set of vertex lattices of type t by $\mathcal{V}^t$. When $n = 3$, a type 2 lattice Λ_2 corresponds to a line $\tilde{\mathcal{N}}_{\Lambda_2} \cong \mathbb{P}^1_k$ in $\mathcal{N}_3^{\mathrm{Kra}}$ and a type 0 lattice Λ_0 corresponds to a divisor $\mathrm{Exc}_{\Lambda_0} \cong \mathbb{P}^2_k$. Let H_{Λ_0} be the hyperplane class of Exc_{Λ_0}. We have the following theorem.

Theorem 1.4. *If $\mathrm{v}(L^\flat) > 0$, we have the following decomposition of cycles in $\mathrm{Gr}^2 K_0(\mathcal{N}_3^{\mathrm{Kra}})$*

$$\mathcal{D}(L^\flat) = \sum_{\substack{\Lambda_2 \in \mathcal{V}^2 \\ L^\flat \subset \Lambda_2^\sharp}} (2[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\substack{\Lambda_0 \in \mathcal{V}^0 \\ \Lambda_0 \subset \Lambda_2}} H_{\Lambda_0})$$

where $\mathrm{Gr}^\bullet K_0(\mathcal{N}_3^{\mathrm{Kra}})$ is the associated graded ring of $K_0(\mathcal{N}_3^{\mathrm{Kra}})$ with respect to the codimension filtration.

Theorem 1.4 is proved by intersecting $\mathcal{D}(L^\flat)$ with special divisors that are isomorphic to $\mathcal{N}_{2,-1}^{\mathrm{Kra}}$ and computing the intersection numbers in two different ways. One way relates the intersection numbers to the main result of [Shi20]. The other way uses the decomposition in Theorem 1.4 and detects the multiplicity of each component that shows up on the right hand side. We expect some analogue of Theorem 1.4 to be true for general n.

1.5. **Notations.** For the rest of this paper, we use the following notations. For $\mathcal{O}_F$-lattices (resp. $\mathcal{O}_{\breve{F}}$-lattices) L and L', we write $L \overset{t}{\subset} L'$ if $L \subset L' \subset \frac{1}{\pi}L$ and $\dim_{\mathbb{F}_q}(L'/L) = t$ (resp. $\dim_k(L'/L) = t$). We say a vector $v \in L$ is primitive if $\frac{1}{\pi}v \notin L$.

For Hermitian lattices L and L', we use $L \obot L'$ to denote orthogonal direct sum, and $L \oplus L'$ as direct sum of lattices. Given a Hermitian lattice L with Hermitian form $(\, , \,)$, we consider two different dual lattices of L. We use $L^\sharp$ (resp. $L^\vee$) to denote the dual lattice of L with respect to $(,)$ (resp. $\mathrm{tr}_{F/F_0}(,)$). Define the fundamental invariant of L with rank n to be the tuple of integers $(a_1, \ldots, a_n)$ such that $0 \le a_1 \le \cdots \le a_n$ and $L^\sharp/L \approx \mathcal{O}_F/(\pi^{a_1}) \oplus \cdots \oplus \mathcal{O}_F/(\pi^{a_n})$ as $\mathcal{O}_F$-modules. The partial order of $\mathbb{Z}^n$ induces a partial order on the set of fundamental invariants. For each Hermitian lattice L, there exists a Jordan decomposition $L = \boxplus_{i\geq s} L_i$ such that $L_i^\sharp = \pi^{-i} L_i$. We call L integral if $s \geq 0$. For an integral lattice L, we define

$$t(L) := \sum_{i\geq 1} \mathrm{rank}_{\mathcal{O}_F}(L_i).$$

For $t \in \mathcal{O}_{F_0}$, let $\mathrm{v}(t) := \mathrm{val}_{\pi_0}(t)$ and write $t = t_0(-\pi_0)^{\mathrm{v}(t)}$. For $x \in \mathbb{V}$, we set $q(x) = (x, x)$ and $\mathrm{v}(x) = \mathrm{v}(q(x))$. We use $\langle t\rangle$ to denote a lattice $\mathcal{O}_F x$ of rank one with $q(x) = t$.

Let $\mathcal{H}_i = \begin{pmatrix} 0 & \pi^i \\ (-\pi)^i & 0 \end{pmatrix}$, and $\mathcal{H} = \mathcal{H}_{-1}$. For a Hermitian matrix T, we define $\mathrm{v}(T) = \mathrm{v}(L)$ where L is a lattice whose Gram matrix is T. We use $\mathrm{Herm}_n(F)$ to denote the set of Hermitian matrices over F of size n. When there is no confusion, we also simply denote it as Herm_n. For $T, T' \in \mathrm{Herm}_n(F)$, we say T is equivalent to T' if there is a $U \in \mathrm{GL}_n(\mathcal{O}_F)$ such that $U^*TU = T'$, where $U^* = {}^t\bar{U}$. In this case, we denote it as $T \approx T'$.

Unless otherwise specified, for a complex $A^\bullet$ of coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$, we often use the same notation to denote the class in the derived category of coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ represented by $A^\bullet$ to avoid cumbersome notation.

1.6. **The structure of the paper.** The paper is divided into three parts. In Part 1, we prove some facts about special cycles for arbitrary n. More specifically, in Section 2 we recall some basic facts about $\mathcal{N}^{\mathrm{Kra}}$ and define special cycles and special difference cycles on it. In Section 3, we compute the intersection number between special cycles and the exceptional divisors. In Section 4, we prove a decomposition theorem for the horizontal component of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ when $L^\flat$ has rank $n-1$.

Part 2 is about Hermitian local densities. In Section 5, we study induction formulas of local density polynomials and relate the local density polynomials with primitive local densities. In Section 6, we show that the coefficients $c_\epsilon^{n,i}$ in (1.9) are uniquely determined and give an algorithm to compute them. In Sections 7 and 8, we compute the local density polynomials when $n \leq 3$.

In Part 3 we prove Theorem 1.2, i.e. Conjecture 1.1 for $n = 3$. In Section 9, we study the reduced locus of the special cycles for $n = 3$. In Section 10, we decompose ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ for $L^\flat$ of rank 2 and $\mathrm{v}(L^\flat) = 0$, and compute the intersection number of $\tilde{\mathcal{N}}_{\Lambda_2}$ with $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$. Finally, we prove Theorem 1.4 and finish the proof of Theorem 1.2 in Section 11.

In Appendix A, we compute the primitive local densities that are used in Part 2 of the paper.

1.7. **Acknowledgement.** To be added later.

Part 1. **The geometric side**

2. Rapoport-Zink space and special cycle

We denote $\bar{a}$ the Galois conjugate of $a \in F$ over F_0. For a p-adic ring R, let $\mathrm{Nilp}\, R$ be the category of R-schemes S such that p is locally nilpotent on S. For such an S, denote its special fiber by $\bar{S}$. Let σ be the Frobenius element of $\breve{F}_0/F_0$.

2.1. **RZ spaces.** Let $S \in \mathrm{Nilp}\, \mathcal{O}_{\breve{F}}$. A p-divisible strict $\mathcal{O}_{F_0}$-module over S is a p-divisible group over S with an $\mathcal{O}_{F_0}$ action whose induced action on its Lie algebra is via the structural morphism $\mathcal{O}_{F_0} \to \mathcal{O}_S$.

Definition 2.1. A formal Hermitian $\mathcal{O}_F$-module of dimension n over S is a triple (X, ι, λ) where X is a supersingular p-divisible strict $\mathcal{O}_{F_0}$-module over S of dimension n and F_0-height $2n$ (supersingular means the Relative Dieudonné module of X at each geometric point of S has slope $\frac{1}{2}$), $\iota : \mathcal{O}_F \to \mathrm{End}(X)$ is an $\mathcal{O}_F$-action and $\lambda : X \to X^\vee$ is a principal polarization in the category of strict $\mathcal{O}_{F_0}$-modules such that the Rosati involution induced by λ is the Galois conjugation of F/F_0 when restricted on $\mathcal{O}_F$. We say (X, ι, λ) satisfies the signature condition $(1, n-1)$ if for all $a \in \mathcal{O}_F$ we have

(i) $\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X) = (T - s(a)) \cdot (T - s(\bar{a}))^{n-1}$ where $s : \mathcal{O}_F \to \mathcal{O}_S$ is the structure morphism;
(ii) $\wedge^n(\iota(a) - s(a) \mid \mathrm{Lie}\, X) = 0$, $\wedge^2(\iota(a) - s(\bar{a}) \mid \mathrm{Lie}\, X) = 0$.

Definition 2.2. Fix a formal Hermitian $\mathcal{O}_F$-module $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of dimension n over k. The moduli space $\mathcal{N}_n^{\mathrm{Pap}}$ is the functor sending each $S \in \mathrm{Nilp}\, \mathcal{O}_{\breve{F}}$ to the groupoid of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ where (X, ι, λ) is a formal Hermitian $\mathcal{O}_F$-module over S of signature $(1, n-1)$ and $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, k} \bar{S}$

is a morphism of formal $\mathcal{O}_F$-modules of height 0. An isomorphism between two such quardruples $(X, \iota, \lambda, \rho)$ and $(X', \iota', \lambda', \rho')$ is given by an $\mathcal{O}_F$-linear isomorphism $\alpha : X \to X'$ such that $\rho' \circ (\alpha \times_S \bar{S}) = \rho$ and $\alpha^*(\lambda')$ is a $\mathcal{O}_{F_0}^\times$ multiple of λ.

By [RTW14], $\mathcal{N}_n^{\mathrm{Pap}}$ is representable by a formal scheme flat and of relative dimension $n-1$ over $\operatorname{Spf} \mathcal{O}_{\breve{F}}$. When $n = 1$, we have $\mathcal{N}_1^{\mathrm{Pap}} \cong \operatorname{Spf} \mathcal{O}_{\breve{F}}$. Fix a formal Hermitian $\mathcal{O}_F$-module $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ of signature $(0, 1)$ over $\operatorname{Spec} k$. It is unique up to $\mathcal{O}_F$-linear isomorphisms. The universal Hermitian $\mathcal{O}_F$-module over $\mathcal{N}_1^{\mathrm{Pap}}$ is the canonical lifting $(\mathcal{G}, \iota_{\mathbb{G}}, \lambda_{\mathcal{G}}, \rho_{\mathcal{G}})$ of $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ to $\operatorname{Spf} \mathcal{O}_{\breve{F}}$ in the sense of [Gro86]. When $n > 1$, $\mathcal{N}_n^{\mathrm{Pap}}$ is regular outside the set of superspecial points over $\operatorname{Spec} k$, which are the points characterized by the condition $(\iota(\pi) \mid \operatorname{Lie} X) = 0$. The set of superspecial points is in fact the set of type 0 lattices (see Section 2.4), hence is isolated and we denote it by Sing.

Definition 2.3. Fix $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be as in Definition 2.2. The moduli space $\mathcal{N}_n^{\mathrm{Kra}}$ is the functor sending each $S \in \operatorname{Nilp} \mathcal{O}_{\breve{F}}$ to the groupoid of isomorphism classes of quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ where $(X, \iota, \lambda, \rho) \in \mathcal{N}_n^{\mathrm{Pap}}(S)$ and $\mathcal{F}$ is a direct summand of $\operatorname{Lie} X$ of rank 1 as an $\mathcal{O}_S$-module such that $\mathcal{O}_F$ acts on $\mathcal{F}$ by the structural morphism and acts on $\operatorname{Lie} X/\mathcal{F}$ by the Galois conjugate of the structural morphism. An isomorphism between two such quintuples $(X, \iota, \lambda, \rho, \mathcal{F})$ and $(X', \iota', \lambda', \rho', \mathcal{F}')$ is an isomorphism $\alpha : (X, \iota, \lambda, \rho) \to (X', \iota', \lambda', \rho')$ in $\mathcal{N}_n^{\mathrm{Pap}}(S)$ such that $\alpha^*(\mathcal{F}') = \mathcal{F}$.

By [Krä03] (see also [Shi20, Proposition 2.7]), the natural forgetful functor $\Phi : \mathcal{N}_n^{\mathrm{Kra}} \to \mathcal{N}_n^{\mathrm{Pap}}$ forgetting $\mathcal{F}$ is the blow up of $\mathcal{N}_n^{\mathrm{Pap}}$ along its singular locus Sing. For each point $\Lambda \in \mathrm{Sing}$, its inverse image $\Phi^{-1}(\Lambda)$ is an exceptional divisor Exc_Λ isomorphic to $\mathbb{P}_k^{n-1}$.

2.2. **Assoicated Hermitian spaces.** Let $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ be the framing object as in Definition 2.2, and N be its rational relative Dieudonne module. Then N is an $2n$-dimensional $\breve{F}_0$-vector space equipped with a σ-linear operator F and a σ^{-1}-linear operator V. The $\mathcal{O}_F$-action $\iota_{\mathbb{X}} : \mathcal{O}_F \to \operatorname{End}(\mathbb{X})$ induces on N an $\mathcal{O}_F$-action commuting with F and V. We still denote this induced action by $\iota_{\mathbb{X}}$ and denote $\iota_{\mathbb{X}}(\pi)$ by π. Let $\tau := \pi V^{-1}$ and $C := N^\tau$. Then C is an n-dimensional F-vector space equipped with a Hermitian form $(,)_{\mathbb{X}}$. When n is odd, there is a unique choice of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ up to quasi-isogenies that preserves the polarization by a factor in $\mathcal{O}_{F_0}^\times$. When n is even, there are two such choices according to the sign $\epsilon = \chi(C)$ (see (1.1)) of C. See [Shi18, Remark 2.16] and [RTW14, Remark 4.2]. We denote the corresponding spaces $\mathcal{N}_n^{\mathrm{Pap}}$ and $\mathcal{N}_n^{\mathrm{Kra}}$ by $\mathcal{N}_{n,\epsilon}^{\mathrm{Pap}}$ and $\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}$ respectively. When n is odd, two different choices of ϵ give us isomorphic moduli spaces. When n is even, two different choices of ϵ give us two sets of non-isomorphic moduli spaces.

Define

$$\mathbb{V} = \operatorname{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \otimes \mathbb{Q}, \tag{2.1}$$

which is equipped with a Hermitian form

$$h(x, y) = \lambda_{\mathbb{Y}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}} \circ x \in \operatorname{End}^0(\mathbb{Y}) \xrightarrow{\sim} F \tag{2.2}$$

where $y^\vee$ is the dual quasi-homomorphism of y. The Hermitian spaces $(\mathbb{V}, h(,))$ and $(C, (,)_{\mathbb{X}})$ are related by the F-linear isomorphism

$$b : \mathbb{V} \to C, \quad \mathbf{x} \mapsto \mathbf{x}(e) \tag{2.3}$$

where e is a generator of the relative Dieudonné module $M(\mathbb{Y})$ of $\mathbb{Y}$ and $(e, e)_{\mathbb{Y}} \in \mathcal{O}_{F_0}^\times$. By [Shi18, Lemma 3.6], we have

$$h(\mathbf{x}, \mathbf{x})(e, e)_{\mathbb{Y}} = (b(\mathbf{x}), b(\mathbf{x}))_{\mathbb{X}}. \tag{2.4}$$

By scaling the hermitian form $(,)_{\mathbb{Y}}$ we can assume that

$$(e, e)_{\mathbb{Y}} = 1,$$

so $\mathbb{V}$ and C are isomorphic as Hermitian spaces. We will sometimes identify $\mathbb{V}$ and C.

2.3. **Special cycles.**

Definition 2.4. For an $\mathcal{O}_F$-lattice L of $\mathbb{V}$, define $\mathcal{Z}^{\mathrm{Pap}}(L)$ to be the subfunctor of $\mathcal{N}_n^{\mathrm{Pap}}$ sending each $S \in \mathrm{Nilp}\,\mathcal{O}_{\breve{F}}$ to the isomorphism classes of tuples $(X, \iota, \lambda, \rho) \in \mathcal{N}_n^{\mathrm{Pap}}(S)$ such that for any $x \in L$ the quasi-homomorphism

$$\rho^{-1} \circ x \circ \rho_{\mathcal{G}} : \mathcal{G} \times_S \bar{S} \to X \times_S \bar{S}$$

entends to a homomorphism $\mathcal{G} \to X$. For $\mathbf{x} \in \mathbb{V}^m$, we let $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) := \mathcal{Z}^{\mathrm{Pap}}(L)$ where $L = \mathrm{Span}\{\mathbf{x}\}$. Let

$$\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \mathcal{Z}^{\mathrm{Kra}}(L) := \mathcal{Z}^{\mathrm{Pap}}(L) \times_{\mathcal{N}_n^{\mathrm{Pap}}} \mathcal{N}_n^{\mathrm{Kra}}.$$

By Grothendieck-Messing theory $\mathcal{Z}^{\mathrm{Pap}}(L)$ (hence $\mathcal{Z}^{\mathrm{Kra}}(L)$) is a closed formal subscheme of $\mathcal{N}_n^{\mathrm{Pap}}$. We sometimes add the subscript ${}_{n,\epsilon}$ to $\mathcal{Z}^{\mathrm{Pap}}(L)$, $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$, $\mathcal{Z}^{\mathrm{Kra}}(L)$ and $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ to indicate their ambient moduli spaces.

Definition 2.5. For an $\mathcal{O}_F$-lattice $L \subset \mathbb{V}$, define $\tilde{\mathcal{Z}}(L)$ to be the strict transform (see the definition after [Har13, Chapter II, Corollary 7.15]) of $\mathcal{Z}^{\mathrm{Pap}}$ under the blow up $\mathcal{N}_n^{\mathrm{Kra}} \to \mathcal{N}_n^{\mathrm{Pap}}$.

Proposition 2.6. *Suppose $\chi(\mathbb{V}) = \epsilon$. Let L be a self-dual lattice of rank m in $\mathbb{V}$ with $\eta = \chi(L)$. We have*

$$\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(L) \cong \mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Pap}}, \text{ and } \tilde{\mathcal{Z}}_{n,\epsilon}(L) \cong \mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Kra}}.$$

Proof. Let us start with the case $L = \mathrm{Span}\{\mathbf{x}_0\}$. Assume that $u = h(\mathbf{x}_0, \mathbf{x}_0)$. Multiplying the Hermitian form $(,)_{\mathbb{X}}$ on C by u^{-1} does not affect the various moduli spaces involved. So we can perform this and assume that $h(\mathbf{x}_0, \mathbf{x}_0) = 1$. Moreover, the sign of its orthogonal complement in $\mathbb{V}$ becomes

$$\epsilon_1 = \epsilon \cdot \chi(u^{-1}) \cdot \chi(u^{-(n-1)}) \cdot \chi(-1)^{n-1} = \epsilon\chi(u)^n \chi(-1)^{n-1}.$$

Then for $(X, \iota, \lambda, \rho) \in \mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(\mathbf{x}_0)(S)$, we define

$$\mathbf{x}_0^* := \lambda_{\mathcal{G}}^{-1} \circ \mathbf{x}_0^\vee \circ \lambda, \ e := \mathbf{x}_0 \circ \mathbf{x}_0^* \in \mathrm{End}(X).$$

By the fact that $h(\mathbf{x}_0, \mathbf{x}_0) = 1$ we know that e is an idempotent. It is routine to check that

$$((1-e)X, (1-e)\iota, (1-e^\vee)\lambda(1-e), \rho(1-e))$$

is an object in $\mathcal{N}_{n-1,\epsilon_1}^{\mathrm{Pap}}(S)$. Conversely given $(Y, \iota_Y, \lambda_Y, \rho_Y) \in \mathcal{N}_{n-1,\epsilon_1}^{\mathrm{Pap}}(S)$, the object

$$(Y \times \mathcal{G}, \iota_Y \times \iota_{\mathcal{G}}, \lambda_Y \times \lambda_{\mathcal{G}}, \rho_Y \times \rho_{\mathcal{G}})$$

is in $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(\mathbf{x}_0)(S)$ with x given by $x : \mathcal{G} \hookrightarrow Y \times \mathcal{G}$. The above two constrcutions are inverse to each other. This shows that $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(\mathbf{x}_0) \cong \mathcal{N}_{n-1,\epsilon_1}^{\mathrm{Pap}}$. For general L of rank m and determinant u, find a basis with Gram matrix $\{1, \ldots, 1, u\}$ and apply the above result repeatedly. So we have $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(L) \cong \mathcal{N}_{n-m,\epsilon_m}^{\mathrm{Pap}}$ where

$$\epsilon_m = \epsilon\chi(u)^{n-m+1}\chi(-1)^{(n-m)m+\frac{m(m-1)}{2}}.$$

Notice that by scaling the Hermitian form by $(-1)^m u$ again we have $\mathcal{N}_{n-m,\epsilon_m}^{\mathrm{Pap}} = \mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Pap}}$.

It then follows from [Har13, Chapter II, Corollary 7.15] that $\tilde{\mathcal{Z}}_{n,\epsilon}(L)$ is the blow up of $\mathcal{Z}_{n,\epsilon}^{\mathrm{Pap}}(L)$ along its superspecial points, which is $\mathcal{N}_{n-m,\epsilon\eta}^{\mathrm{Kra}}$. □

Corollary 2.7. *Let L be as in Proposition 2.6 and $\mathbf{y} \in \mathbb{V}$ such that $\mathbf{y} \perp L$. Then*

$$\mathcal{Z}_{n,\epsilon}^{\mathrm{Kra}}(\mathbf{y}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(L) \cong \mathcal{Z}_{n-1,\epsilon\eta}^{\mathrm{Kra}}(\mathbf{y}).$$

Remark 2.8. *It follows directly from the definition that $\tilde{\mathcal{Z}}(L)$ is a closed sub formal scheme of $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \cdots \cap \tilde{\mathcal{Z}}(\mathbf{x}_r)$ if $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is a basis of L. However in general these two can not be identified. For example, when $n = 3$ and $r = 2$, the intersection of $\tilde{\mathcal{Z}}(L)$ with Exc_Λ has dimension 0 for any $\Lambda \in \mathcal{V}^0$ while the intersection of $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \tilde{\mathcal{Z}}(\mathbf{x}_2)$ with Exc_Λ might contain components of dimension bigger than 1.*

2.4. **Bruhat-Tits stratification.** Recall the following results.

Proposition 2.9. *(*[RTW14, Proposition 2.2 and 2.4]*) Let $\mathcal{N}(k)$ be the set of $\mathcal{O}_{\breve{F}}$-lattices*

$$\mathcal{N}(k) = \{M \subset C \otimes_F \breve{F} \mid M^\sharp = M, \pi\tau(M) \subset M \subset \pi^{-1}\tau(M),\ \dim_k(M + \tau(M))/M \leq 1\}.$$

Then the map

$$\mathcal{N}^{\mathrm{Pap}}(k) \to \mathcal{N}(k), \quad x = (X, \iota, \lambda, \rho) \mapsto M(x) := \rho(M(X)) \subset N$$

is a bijection.

We say a lattice $\Lambda \subset C$ is a vertex lattice if $\pi\Lambda \subseteq \Lambda^\sharp \subseteq \Lambda$ where $\Lambda^\sharp$ is dual lattice of Λ with respect to $(,)_{\mathbb{X}}$, and we call $t = \dim_{\mathbb{F}_q}(\Lambda/\Lambda^\sharp)$ the type of Λ. We denote the set of vertex lattices (resp. of type t) by $\mathcal{V}$ (resp. $\mathcal{V}^t$). We say two vertex lattice Λ_1 and Λ_2 are neighbours if $\Lambda_1 \subset \Lambda_2$ or $\Lambda_2 \subset \Lambda_1$. Then we can define a simplicial complex $\mathcal{L}$ as follows. When n is odd or when n is even and C is non-split, then an r-simplex is formed by $\Lambda_0, \ldots, \Lambda_r$ if any two members of this set are neighbours. When n is even and C is split, we refer to discussion before [RTW14, 3.4] for the definition of $\mathcal{L}$. We also use $\mathcal{L}_{n,\epsilon}$ to denote $\mathcal{L}$ if C has dimension n and $\chi(C) = \epsilon$. Again when n is odd, $\mathcal{L}_{n,1} = \mathcal{L}_{n,-1}$, hence we use $\mathcal{L}_n$ to denote it.

By results in Sections 4 and 6 of loc. cit., to each $\Lambda \in \mathcal{V}^t$ we can associate a Deligne-Lusztig varieties $\mathcal{N}_\Lambda$ and $\mathcal{N}_\Lambda^\circ$ of dimension $t/2$, such that

$$\mathcal{N}_\Lambda(k) = \{M \in \mathcal{N}(k) \mid M \subset \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}\},$$

and

$$\mathcal{N}_\Lambda^\circ(k) = \{M \in \mathcal{N}(k) \mid \Lambda(M) = \Lambda\}.$$

Here $\Lambda(M)$ is the minimal vertex lattice such that $\Lambda(M) \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$ contains M which always exists by [RTW14, Proposition 4.1]. By Theorem 1.1 of loc.cit., we know that

$$\mathcal{N}_\Lambda := \bigsqcup_{\Lambda' \in \mathcal{L}, \Lambda' \subseteq \Lambda} \mathcal{N}_{\Lambda'}^\circ,$$

and

$$\mathcal{N}_{red}^{\mathrm{Pap}} = \bigsqcup_{\Lambda \in \mathcal{L}} \mathcal{N}_\Lambda^\circ$$

where each $\mathcal{N}_\Lambda$ is a closed subvariety of $\mathcal{N}_{red}^{\mathrm{Pap}}$. By loc. cit., we also know that

$$\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'} = \begin{cases} \mathcal{N}_{\Lambda \cap \Lambda'} & \text{if } \Lambda \cap \Lambda' \in \mathcal{L}, \\ \emptyset & \text{otherwise.} \end{cases}$$

For a lattice $L \subset \mathbb{V}$, define

$$\mathcal{V}(L) := \{\Lambda \in \mathcal{V} \mid L \subseteq \Lambda^\sharp\}, \text{ and } \mathcal{V}^t(L) := \{\Lambda \in \mathcal{V}^t \mid L \subseteq \Lambda^\sharp\}. \tag{2.5}$$

When $L = \mathrm{Span}\{\mathbf{x}\}$ we also denote $\mathcal{V}(L)$ (resp. $\mathcal{V}^t(L)$) by $\mathcal{V}(\mathbf{x})$ (resp. $\mathcal{V}^t(\mathbf{x})$). For any subset S of $\mathcal{V}$, we define $\mathcal{L}(S)$ to be the subcomplex of $\mathcal{L}$ such that a simplex is in $\mathcal{L}(S)$ if and only if every vertex in it is in S. For a lattice L of $\mathcal{V}$ and $\mathbf{x} \in C$, define

$$\mathcal{L}(L) = \mathcal{L}(\mathcal{V}(L)). \tag{2.6}$$

When $L = \mathrm{Span}\{\mathbf{x}\}$ we also denote $\mathcal{L}(L)$ by $\mathcal{L}(\mathbf{x})$.

2.5. **Horizontal and vertical part.** A formal scheme X over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ is called horizontal (resp. vertical) if it is flat over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ (resp. π is locally nilpotent on $\mathcal{O}_X$). For a formal scheme X over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$, its horizontal part is canonically defined by torsion sections on $\mathcal{O}_X$. If X is noetherian, then we have the following decomposition by primary decomposition

$$X = X_h \cup X_v \tag{2.7}$$

as a union of horizontal and vertical formal subschemes.

Lemma 2.10. *For a lattice $L^\flat \subset \mathbb{V}$ of rank greater or equal to $n-1$ with nondegenerate Hermitian form, $\mathcal{Z}^{\mathrm{Kra}}(L)$ is noetherian.*

Proof. The lemma can be proved as in [LZ, Lemma 2.9.2]. □

Lemma 2.11. *For a rank $n-1$ lattice $L^\flat \subset \mathbb{V}$ with nondegenerate Hermitian form, $\mathcal{Z}^{\mathrm{Kra}}(L)_v$ is supported on the reduced locus $\mathcal{N}^{\mathrm{Kra}}_{red}$ of $\mathcal{N}^{\mathrm{Kra}}$, i.e., $\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L)_v}$ is annihilated by a power of the ideal sheaf of $\mathcal{N}^{\mathrm{Kra}}_{red}$.*

Proof. The proof is the same as that of [LZ, Lemma 5.1.1]. □

2.6. **Derived special cycles.** For a locally noetherian formal scheme X together with a formal subscheme Y, denote by $K_0^Y(X)$ the Grothendieck group of finite complexes of coherent locally free $\mathcal{O}_X$-modules acyclic outside Y. For such a complex $A^\bullet$, denote by $[A^\bullet]$ the element in $K_0^Y(X)$ represented by it. We use $K_0(X)$ to denote $K_0^X(X)$. Denote by $\mathrm{F}^iK_0^Y(X)$ the codimension i filtration on $K_0^Y(X)$ and $\mathrm{Gr}^iK_0^Y(X)$ its i-th graded piece. We have a cup product $\cdot$ on $K_0^Y(X)_\mathbb{Q}$ defined by tensor product of complexes:

$$[A_1^\bullet]\cdot[A_2^\bullet]=[A_1^\bullet\otimes A_2^\bullet].$$

When X is a scheme, the cup product satisfies ([SABK94, Section I.3, Theorem 1.3])

$$\mathrm{F}^iK_0^Y(X)_\mathbb{Q}\cdot \mathrm{F}^jK_0^Y(X)_\mathbb{Q}\subset \mathrm{F}^{i+j}K_0^Y(X)_\mathbb{Q}. \tag{2.8}$$

It is expected that (2.8) is also true when X is a formal scheme. We will only need special cases of this fact which can be checked directly, see for example Lemma 3.4 and 10.1.

Let $K_0'(Y)$ be the Grothendieck group of coherent sheaves of $\mathcal{O}_Y$-modules on Y. When X is regular we have the following isomorphism

$$K_0^Y(X)\cong K_0'(Y). \tag{2.9}$$

In particular, $K_0(X)\cong K_0'(X)$. There is a map

$$\alpha:\mathrm{CH}^i(X)\to \mathrm{Gr}^iK_0'(X),\ \mathcal{Z}\mapsto[\mathcal{O}_\mathcal{Z}]$$

where $\mathcal{Z}$ is a formal subscheme of pure codimension i and $[\mathcal{O}_\mathcal{Z}]$ is the class in $\mathrm{Gr}^iK_0(X)$ represented by $\mathcal{O}_\mathcal{Z}$. When X is a regular scheme, we know that α becomes an isomorphism after tensoring with $\mathbb{Q}$ ([SABK94, Section I.3, Theorem 1.3]):

$$\mathrm{CH}^i(X)_\mathbb{Q}\xrightarrow{\alpha_\mathbb{Q}}\mathrm{Gr}^iK_0'(X)_\mathbb{Q}.$$

Recall that for $\mathbf{x}\in\mathbb{V}$, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a divisor ([How19, Proposition 4.3]).

Definition 2.12. For $\mathbf{x}=(\mathbf{x}_1,\ldots,\mathbf{x}_r)\in\mathbb{V}^r$, define ${}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ to be

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)}\otimes^\mathbb{L}\cdots\otimes^\mathbb{L}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_r)}]\in K_0^{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}(\mathcal{N}^{\mathrm{Kra}}) \tag{2.10}$$

where $\otimes^\mathbb{L}$ is the derived tensor product of complexes of coherent locally free sheaves on $\mathcal{N}^{\mathrm{Kra}}$. By [How19, Theorem B], ${}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ only depends on $L:=\mathrm{Span}\{\mathbf{x}\}$, hence can be denoted as ${}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(L)$.

Definition 2.13. When L has rank n, we define the intersection number

$$\mathrm{Int}(L)=\chi(\mathcal{N}^{\mathrm{Kra}},{}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(L)) \tag{2.11}$$

where χ is the Euler characteristic.

Lemma 2.14. *$\mathcal{Z}^{\mathrm{Kra}}(L)$ is properly supported on $\mathcal{N}^{\mathrm{Kra}}_{red}$. In particular,* $\mathrm{Int}(L)$ *is finite.*

Proof. This can be proved exactly the same way as [LZ, Lemma 2.10.1]. □

2.7. **Special Difference cycles.** Conjecture 1.1 and Theorem 5.2 motivate us to make the following definition.

Definition 2.15. For $L\subset\mathbb{V}$ a rank ℓ lattice, define the special difference cycle $\mathcal{D}(L)\in K_0^{\mathcal{Z}^{\mathrm{Kra}}(L)}(\mathcal{N}^{\mathrm{Kra}})$ by

$$\mathcal{D}(L)={}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(L)+\sum_{i=1}^{\ell}(-1)^iq^{i(i-1)/2}\sum_{\substack{L\subset L'\subset\frac{1}{\pi}L\\ \dim_{\mathbb{F}_q}(L'/L)=i}}{}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(L'). \tag{2.12}$$

One interesting observation is the following decomposition of ${}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(L)$.

Lemma 2.16. *For $L\subset\mathbb{V}$ a lattice of rank ℓ, we have the following identity in $K_0^{\mathcal{Z}^{\mathrm{Kra}}(L)}(\mathcal{N}^{\mathrm{Kra}})$ where the summation is finite.*

$${}^\mathbb{L}\mathcal{Z}^{\mathrm{Kra}}(L)=\sum_{\substack{L'\,\mathrm{integral}\\ L\subset L'\subset L_F}}\mathcal{D}(L').$$

Proof. First of all, if L is not integral, neither is L' if $L \subset L'$. In this case ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = 0$ and the summation index on the right hand side of the identity in the lemma is empty. This proves the lemma when $\mathrm{v}(L) < 0$. We can now prove the identity by induction on the fundamental invariant of L. Assume that the lemma is proved for all $L' \subset L_F$ with $L \subsetneq L'$.

For L' with $L \subsetneq L' \subset \frac{1}{\pi}L$, we have

$$ {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') = \sum_{L' \subset L'' \subset L'_F} \mathcal{D}(L'') $$

by the induction hypothesis. Combining this with (2.12), we can write

$$ {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = \sum_{L \subset L' \subset L_F} m(L')\mathcal{D}(L') $$

where $m(L') \in \mathbb{Z}$. Notice that $m(L) = 1$. For any L'' such that $L \subset L'' \subsetneq L_F$, let $M' = \frac{1}{\pi}L \cap L''$ and $m = \dim_{\mathbb{F}_q}(M'/L)$. We have

$$ m(L'') = -\sum_{i=1}^{m} (-1)^i q^{i(i-1)/2} \sum_{\substack{L \subset L' \subset M' \\ \dim_{\mathbb{F}_q}(L'/L) = i}} 1 = 1 \tag{2.13} $$

by evaluating the identity in Corollary to Lemma 12 of [Tam63] at $t = 1$. □

Remark 2.17. *When $\ell = 1$ and $L = \mathrm{Span}\{\mathbf{x}\}$, the Cartier divisor*

$$ \mathcal{D}(L) = \mathcal{Z}(\mathbf{x}) - \mathcal{Z}(\frac{1}{\pi}\mathbf{x}) $$

is the difference divisor $\mathcal{D}(\mathbf{x})$ defined in [Ter10, Definition 2.10].

Definition 2.18. Assume $L = L_1 \oplus L_2$, where L_i is of rank n_i and $n_1 + n_2 = n$. We define

$$ \mathrm{Int}(L)^{(n_1)} = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L_2)). \tag{2.14} $$

Notice that $\mathrm{Int}(L)^{(n_1)}$ depends on the decomposition $L = L_1 \oplus L_2$.

3. Special cycles and exceptional divisors

For a formal subscheme $\mathcal{Z}$ of $\mathcal{N}^{\mathrm{Kra}}$, we use the notation $\otimes_{\mathcal{Z}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{Z}}$) instead of $\otimes_{\mathcal{O}_{\mathcal{Z}}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{Z}}}$). We also simply write $\otimes$ (resp. $\otimes^{\mathbb{L}}$) instead of $\otimes_{\mathcal{N}^{\mathrm{Kra}}}$ (resp. $\otimes^{\mathbb{L}}_{\mathcal{N}^{\mathrm{Kra}}}$). Let us first recall the following distribution law of derived tensor product. In this section, we identify $\mathbb{V}$ with C by the isomorphism b defined in (2.3).

Lemma 3.1. *Assume that $\mathcal{A}_i$ $(1 \le i \le k)$ is in the derived category of bounded coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and $i : \mathcal{Z} \to \mathcal{N}^{\mathrm{Kra}}$ is a closed embedding of formal subscheme. Then the following identity holds in the derived category of bounded coherent sheaves on $\mathcal{Z}$.*

$$ i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) = i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) \otimes^{\mathbb{L}}_{\mathcal{Z}} \ldots \otimes^{\mathbb{L}}_{\mathcal{Z}} i^*(\mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}). $$

Proof. We can take locally free representatives of $A_i^\bullet$ of $\mathcal{A}_i$. Then $A_1^\bullet \otimes \ldots \otimes A_k^\bullet$ is again a complex of locally free sheaves on $\mathcal{N}^{\mathrm{Kra}}$, hence a locally free representatives of $\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k$. Hence $i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \ldots \otimes^{\mathbb{L}} \mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}})$ can be represented by $A_1^\bullet \otimes \ldots \otimes A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}}$. Meanwhile $A_i^\bullet \otimes \mathcal{O}_{\mathcal{Z}}$ is a representative of $\mathcal{A}_i \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}$ in the derived category of bounded coherent sheaves on $\mathcal{N}^{\mathrm{Kra}}$ and is also a complex of locally free sheaves on $\mathcal{Z}$. Hence $i^*(\mathcal{A}_1 \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}}) \otimes^{\mathbb{L}}_{\mathcal{Z}} \ldots \otimes^{\mathbb{L}}_{\mathcal{Z}} i^*(\mathcal{A}_k \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}})$ can be represented by $(A_1^\bullet \otimes \mathcal{O}_{\mathcal{Z}}) \otimes_{\mathcal{Z}} \ldots \otimes_{\mathcal{Z}} (A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}})$. Now by the distribution law of tensor products we have

$$ A_1^\bullet \otimes \ldots \otimes A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}} = (A_1^\bullet \otimes \mathcal{O}_{\mathcal{Z}}) \otimes_{\mathcal{Z}} \ldots \otimes_{\mathcal{Z}} (A_k^\bullet \otimes \mathcal{O}_{\mathcal{Z}}). $$

This finishes the proof of the lemma. □

Proposition 3.2. *Assume that the dimension of $\mathbb{V}$ is $n \ge 2$. Then for each $\mathbf{x} \in \mathbb{V}$, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a divisor. Moreover, we have the following decomposition of Cartier divisors*

$$ \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) = \tilde{\mathcal{Z}}(\mathbf{x}) + \sum_{\Lambda \in \mathcal{V}^0, \mathbf{x} \in \Lambda} (m_\Lambda(\mathbf{x}) + 1)\mathrm{Exc}_\Lambda \tag{3.1} $$

where $m_\Lambda(\mathbf{x})$ is the largest integer m such that $\pi^{-m} \cdot \mathbf{x} \in \Lambda$.

Proof. The fact that $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ is a divisor is due to [How19, Proposition 4.3]. By [Shi18, Proposition 3.7], the superspecial point corresponding to a type zero lattice Λ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$ if and only if $\mathbf{x} \in \Lambda$. Hence $\mathrm{Exc}_\Lambda \subset \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ if and only if $\mathbf{x} \in \Lambda$. Since $\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}$ is regular, we must have a decomposition as in (3.1) and the only job left is to determine the multiplicity of each Exc_Λ.

Fix a type zero lattice Λ and let $m := m_\Lambda(\mathbf{x})$. Then $\pi^{-m} \cdot \mathbf{x}$ is a primitive vector in Λ. By Lemma A.3, there exists a decomposition

$$\Lambda = \Lambda_2 \obot \Lambda'$$

where Λ_2 and Λ' are unimodular lattices of rank 2 and $n-2$ respectively and $\pi^{-m} \cdot \mathbf{x} \in \Lambda_2$. Let $\eta = \chi(\Lambda')$. By applying Proposition 2.6, we see that $\tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') \cong \mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. Moreover we have the following proper intersections

$$\mathcal{Z}_{n,\epsilon}^{\mathrm{Kra}}(\mathbf{x}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \mathcal{Z}_{2,\epsilon\eta}^{\mathrm{Kra}}(\mathbf{x}),\ \tilde{\mathcal{Z}}_{n,\epsilon}(\mathbf{x}) \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x}),\ \mathrm{Exc}_\Lambda \cap \tilde{\mathcal{Z}}_{n,\epsilon}(\Lambda') = \mathrm{Exc}_{\Lambda_2}$$

where Exc_{Λ_2} is the exceptional divisor in $\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$ corresponding to the vertex lattice Λ_2. Hence the multiplicity of Exc_Λ in $\mathcal{Z}_{n,\epsilon}^{\mathrm{Kra}}(\mathbf{x})$ is the same as the multiplicity of Exc_{Λ_2} in $\mathcal{Z}_{2,\epsilon\eta}^{\mathrm{Kra}}(\mathbf{x})$. Now the proposition follows from [Shi20, Theorem 4.5] and [HSY, Theorem 4.1]. □

The Chow ring $\mathrm{CH}^\bullet(\mathrm{Exc}_\Lambda) \cong \mathrm{Gr}^\bullet K_0(\mathrm{Exc}_\Lambda)$ is isomorphic to $\mathbb{Z}[H_\Lambda]/(H_\Lambda^{n-1}-1)$ where H_Λ is the hyperplane class of Exc_Λ represented by any $\mathbb{P}^{n-2}$ in Exc_Λ.

Proposition 3.3. *Assume* $\dim \mathbb{V} = n \geq 2$. *Assume* $\mathbf{x} \in \mathbb{V}$ *such that* $h(\mathbf{x},\mathbf{x}) \neq 0$ *and* Λ *is a type* 0 *vertex lattice containig* $\mathbf{x}$. *Let* $m := m_\Lambda(\mathbf{x})$ *as in Proposition 3.2. Then* $\tilde{\mathcal{Z}}(\mathbf{x})$ *and* Exc_Λ *intersect properly and*

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})\cap \mathrm{Exc}_\Lambda}] = (2m+1)H_\Lambda \in \mathrm{CH}^1(\mathrm{Exc}_\Lambda).$$

Proof. First $\tilde{\mathcal{Z}}(\mathbf{x})$ and Exc_Λ are Cartier divisors with no common component, so they intersect properly. Let $m = m_\Lambda(\mathbf{x})$ and $\mathbf{x}' := \pi^{-m} \cdot \mathbf{x}$. By assumption $m \geq 0$. By Proposition 5.9, we have

$$\{v \in \Lambda \mid h(\mathbf{x}', v) = 0\} = \mathrm{Span}\{\mathbf{y}\} \obot \Lambda'$$

where $\mathrm{v}(\mathbf{y}) = \mathrm{v}(\mathbf{x}')$ and Λ' is unimodular. Let $\eta = \chi(\Lambda')$ and

$$\Lambda_2 := \{v \in \Lambda \mid v \perp \Lambda'\}.$$

Λ_2 is rank 2 unimodular and contains $\mathbf{x}'$.

By Proposition 2.6, we have $\tilde{\mathcal{Z}}(\Lambda') \cong \mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. In particular, $\tilde{\mathcal{Z}}(\Lambda')$ is regular. By Corollary 2.7, we know that $\tilde{\mathcal{Z}}(\Lambda') \cap \tilde{\mathcal{Z}}(\mathbf{x}) = \tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x})$. In particular $\tilde{\mathcal{Z}}(\Lambda')$ and $\tilde{\mathcal{Z}}(\mathbf{x})$ intersect properly as $\tilde{\mathcal{Z}}_{2,\epsilon\eta}(\mathbf{x})$ is a divisor in $\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. On the other hand $\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda$ is the exceptional divisor Exc_{Λ_2} in $\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. Since $\mathrm{Exc}_\Lambda \cong \mathbb{P}^{n-1}$, it is also regular. Our strategy is to compute the intersection number

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')})$$

in two different ways. By Lemma 3.1, one way is

$$\chi(\tilde{\mathcal{Z}}(\Lambda'), \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')\cap\tilde{\mathcal{Z}}(\mathbf{x})} \otimes^{\mathbb{L}}_{\tilde{\mathcal{Z}}(\Lambda')} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')\cap \mathrm{Exc}_\Lambda}) \tag{3.2}$$

where we use the fact that the intersections $\tilde{\mathcal{Z}}(\Lambda') \cap \tilde{\mathcal{Z}}(\mathbf{x})$ and $\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda$ are proper (see for example [Zha21, Lemma B.2]). The other way is, by Lemma 3.1,

$$\chi(\mathrm{Exc}_\Lambda, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x})\cap \mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \mathcal{O}_{\tilde{\mathcal{Z}}(\Lambda')\cap \mathrm{Exc}_\Lambda}). \tag{3.3}$$

When $\epsilon\eta = -1$, by Proposition 3.11 and Theorem 4.5 of [Shi20], we know that (3.2) is equal to $2m+1$. When $\epsilon\eta = 1$, by Lemma 3.10, Theorem 4.1 and Lemma 5.2 of [HSY], we know that (3.2) is equal to $2m+1$ as well. Since the intersection number of H_Λ with $\mathrm{Exc}_{\Lambda_2} \cong \mathbb{P}^1$ in Exc_Λ is 1, the proposition follows. □

3.1. Intersection numbers involving the exceptional divisors.

Lemma 3.4. *The class of* $\underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{m}$ *in* $\mathrm{CH}^{m-1}(\mathrm{Exc}_\Lambda)$ *is* $(-2H_\Lambda)^{m-1}$.

Proof. To study this intersection, it suffices to consider the local model N^{Kra} constructed in [Krä03]. Let N_s^{Kra} be its special fiber. Recall by equation (4.11) loc. cit., we have

$$N_s^{\mathrm{Kra}} = \mathrm{Exc} + Z_2$$

as Cartier divisors where Exc is the exceptional divisor of N^{Kra} and Z_2 is a divisor in N^{Kra} which intersect properly with Exc. Their intersection is $2H$ where H is the hyperplane class of Exc. Since Exc is properly supported on N^{Kra}, we have

$$[\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}} \mathcal{O}_{N_s^{\mathrm{Kra}}}] = 0$$

Hence

$$\begin{aligned} 0 &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{N_s^{\mathrm{Kra}}}] \\ &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{\mathrm{Exc}}] + [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{Z_2}] \\ &= [\mathcal{O}_{\mathrm{Exc}} \otimes^{\mathbb{L}}_{N^{\mathrm{Kra}}} \mathcal{O}_{\mathrm{Exc}}] + 2H \end{aligned}$$

This proves the lemma when $m = 2$. The general case now follows from Lemma 3.1. □

Corollary 3.5. *Let $\Lambda \in \mathcal{V}^0$ and $\mathbf{x} \in \Lambda$. Then we have the following identity in $\mathrm{CH}^1(\mathrm{Exc}_\Lambda)$:*

$$[\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}] = -H_\Lambda.$$

Proof. By Propositions 3.2, 3.3 and Lemma 3.4, we have the following identity in $\mathrm{CH}^1(\mathrm{Exc}_\Lambda)$:

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}] = [(2m_\Lambda(\mathbf{x}) + 1) - 2(m_\Lambda(\mathbf{x}) + 1)]H_\Lambda = -H_\Lambda.$$

This finishes the proof of the corollary. □

Corollary 3.6. *Assume that $n - m \geq 1$ and $\mathrm{Exc}_\Lambda \subset \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cap \ldots \cap \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)$, then*

$$\chi(\mathcal{N}_n^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \ldots \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)} \otimes^{\mathbb{L}} \underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{n-m}) = (-1)^{n-1} \cdot 2^{n-m-1}.$$

Proof. By Corollary 3.5, Lemmas 3.1 and 3.4, we have

$$\begin{aligned} &\chi(\mathcal{N}_n^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \ldots \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_m)} \otimes^{\mathbb{L}} \underbrace{\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}}_{n-m}) \\ &= \chi(\mathrm{Exc}_\Lambda, \underbrace{(-H_\Lambda) \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \cdots \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} (-H_\Lambda)}_{m} \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \underbrace{(-2H_\Lambda) \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} \cdots \otimes^{\mathbb{L}}_{\mathrm{Exc}_\Lambda} (-2H_\Lambda)}_{n-m-1}) \\ &= (-1)^m \cdot (-2)^{n-m-1}. \end{aligned}$$

□

For $\Lambda \in \mathcal{V}^0$, let $\mathbb{P}^1_\Lambda$ be any $\mathbb{P}^1$ in Exc_Λ, and

$$\mathrm{Int}_\Lambda(\mathbf{x}) = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}). \tag{3.4}$$

Corollary 3.7. *For $\Lambda \in \mathcal{V}^0$, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) = -2. \tag{3.5}$$

Proof. By Lemma 3.4, we have

$$\begin{aligned} &\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) \\ &= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} (\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda})) \\ &= \chi(\mathrm{Exc}_\Lambda, (\mathcal{O}_{\mathrm{Exc}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda}) \\ &= -2\chi(\mathrm{Exc}_\Lambda, H_\Lambda \cdot [\mathcal{O}_{\mathbb{P}^1_\Lambda}]) \\ &= -2. \end{aligned}$$

□

Corollary 3.8. *For $\Lambda \in \mathcal{V}^0$, we have*

$$\mathrm{Int}_\Lambda(\mathbf{x}) = -1_\Lambda(\mathbf{x}).$$

Proof. If $\mathbf{x} \notin \Lambda$, then the intersection number is apparently 0. Otherwise we have by Corollary 3.5

$$\begin{aligned}&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1_\Lambda})\\ &= \chi(\mathrm{Exc}_\Lambda, (\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) \otimes_{\mathcal{O}_{\mathrm{Exc}_\Lambda}} \mathcal{O}_{\mathbb{P}^1_\Lambda})\\ &= -\chi(\mathrm{Exc}_\Lambda, H_\Lambda \cdot [\mathcal{O}_{\mathbb{P}^1_\Lambda}])\\ &= -1.\end{aligned}$$

□

The above results suggest that the difficulty to compute $\mathrm{Int}(L)$ mainly lies in computing

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(x_n)}).$$

We end this section by studying the intersection number of difference cycle with exceptional divisors.

Lemma 3.9. *If $L^\flat$ has rank $n-1$, then for any $\Lambda \in \mathcal{V}^0(L^\flat)$, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = \begin{cases} (-1)^{n-1} & \text{if } L^\flat = \Lambda \cap L^\flat_F, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.10. $L^\flat = \Lambda \cap L^\flat_F$ *if and only if* $L^\flat$ *is of type (see* (4.2) *and Lemma 4.2 below)* 1 *or* 0 *and* Λ *is at the boundary of the* $\mathcal{L}(L^\flat)$.

Proof. Define

$$M' := \frac{1}{\pi} L^\flat \cap \Lambda \text{ and } m := \dim_{\mathbb{F}_q}(M'/L^\flat).$$

Then for L' such that $L^\flat \subset L' \subset \frac{1}{\pi} L^\flat$, we know that $\mathcal{Z}^{\mathrm{Kra}}(L')$ intersects Exc_Λ if and only if $L' \subset M'$. For such L', by Corollary 3.6, we have

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = (-1)^{n-1}. \tag{3.6}$$

Hence

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\mathrm{Exc}_\Lambda}]) = (-1)^{n-1}[1 + \sum_{i=1}^{m} (-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat \subset L' \subset M' \\ \dim_{\mathbb{F}_q}(L'/L^\flat) = i}} 1].$$

Notice that $m = 0$ if and only if $M' = L^\flat$ which is equivalent to the condition $L^\flat = \Lambda \cap L^\flat_F$. In this case the summation in (4.12) is over an empty set hence (4.12) is equal to 1. If $m > 0$ we know (4.12) is equal to 0 by (2.13). □

4. Horizontal components of special cycles

Given an integral Hermitian lattice L we can have its Jordan decomposition:

$$L = \oplus_{t \geq 0} L_t \tag{4.1}$$

where L_t is π^t-modular, see [Jac62]. Define the type of L to be

$$t(L) = \sum_{t \geq 1} \mathrm{rank}_{\mathcal{O}_F}(L_t). \tag{4.2}$$

4.1. **Quasi-canonical lifting cycles.** Assume that $\dim(\mathbb{V}) = 2$. When $\chi(\mathbb{V}) = -1$, for $\mathbf{y} \in \mathbb{V}$, by [Shi20, Theorem 4.5], we have

$$\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y}) = \mathcal{Z}_0 + \sum_{s=1}^{\mathrm{v}(\mathbf{y})} (\mathcal{Z}_s^+ + \mathcal{Z}_s^-)$$

where $\mathcal{Z}_0 \cong \mathrm{Spf}\, \mathcal{O}_{\breve{F}}$. Define

$$\tilde{\mathcal{Z}}_{2,-1}(\mathbf{y})^o := \begin{cases} \mathcal{Z}^+_{\mathrm{v}(\mathbf{y})} + \mathcal{Z}^-_{\mathrm{v}(\mathbf{y})} & \text{if } \mathrm{v}(\mathbf{y}) > 0, \\ \mathcal{Z}_0 & \text{if } \mathrm{v}(\mathbf{y}) = 0. \end{cases}$$

When $\chi(\mathbb{V}) = 1$, for $\mathbf{y} \in \mathbb{V}$ such that $\mathrm{v}(\mathbf{y}) = 0$, by [HSY, Theorem 4.1], we have

$$\tilde{\mathcal{Z}}_{2,1}(\mathbf{y}) = \tilde{\mathcal{Z}}_0.$$

Define $\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})^o$ to be $\tilde{\mathcal{Z}}_{2,1}(\mathbf{y})$ itself.

4.2. **Horizontal cycles.** Let $L^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$. Define

$$\mathrm{Mod}_i := \{M^\flat \text{ is a rank } n-1 \text{ lattice in } \mathbb{V} \mid M^\flat \subseteq (M^\flat)^\sharp,\ t(M^\flat) = i\}, \tag{4.3}$$

and

$$\mathrm{Mod}_i(L^\flat) := \{M^\flat \in \mathrm{Mod}_i \mid L^\flat \subseteq M^\flat\}. \tag{4.4}$$

For $M^\flat \in \mathrm{Mod}_1$, fix a Jordan decompostion of $M^\flat = \oplus_{t\geq 0} M_t^\flat$ as in (4.1), define

$$\chi(M^\flat, \mathbb{V}) := \begin{cases} \chi(\mathbb{V}) & \text{if } n = 2, \\ \chi((M^\flat_{0,F})^\perp) & \text{if } n \geq 3. \end{cases} \tag{4.5}$$

The definition of $\chi(M^\flat, \mathbb{V})$ is independent of the choice of Jordan decomposition. Define

$$\mathrm{Mod}_h := \mathrm{Mod}_0 \cup \{M^\flat \in \mathrm{Mod}_1 \mid \chi(M^\flat, \mathbb{V}) = -1\}. \tag{4.6}$$

Define

$$\mathrm{Mod}_h(L^\flat) := \{M^\flat \in \mathrm{Mod}_h \mid L^\flat \subseteq M^\flat\}. \tag{4.7}$$

For $M^\flat \in \mathrm{Mod}_h(L^\flat)$, we can decompose $M^\flat$ as $M_0^\flat \oplus \mathrm{Span}\{\mathbf{y}\}$. Then Proposition 2.6 and its corollary imply that

$$\tilde{\mathcal{Z}}(M^\flat) \cong \tilde{\mathcal{Z}}_{2,\chi(M^\flat,\mathbb{V})}(\mathbf{y}).$$

Define

$$\tilde{\mathcal{Z}}(M^\flat)^o \cong \tilde{\mathcal{Z}}_{2,\chi(M^\flat,\mathbb{V})}(\mathbf{y})^o. \tag{4.8}$$

Theorem 4.1. *Let $L^\flat$ be a rank $n-1$ integral lattice in $\mathbb{V}$, then*

$$\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h = \bigcup_{M^\flat \in \mathrm{Mod}_h(L^\flat)} \tilde{\mathcal{Z}}(M^\flat)^o. \tag{4.9}$$

In particular, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ is of pure dimension 1. Moreover we have the following identity in $\mathrm{Gr}^{n-1}K_0(\mathcal{N}^{\mathrm{Kra}})$:

$$[\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h}] = \sum_{M^\flat \in \mathrm{Mod}_h(L^\flat)} [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^o}].$$

Proof. The proof largely follows [LZ, Section 4.4]. Let K be a finite extension of $\breve{F}$. Assume that z is an irreducible component of $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)(\mathcal{O}_K) = \mathcal{Z}^{\mathrm{Pap}}(L^\flat)(\mathcal{O}_K)$, and let G be the corresponding formal $\mathcal{O}_F$-module over $\mathcal{O}_K$. Define

$$L := \mathrm{Hom}_{\mathcal{O}_F}(T_p\mathcal{G}, T_pG)$$

where $\mathcal{G}$ is the canonical lifting and T_p is the integral p-adic Tate module. L is an $\mathcal{O}_F$-module of rank n equipped with the Hermitian form

$$\{x, y\} = \lambda_{\mathcal{G}}^\vee \circ y^\vee \circ \lambda_G \circ x,$$

under which it is self-dual. We have two inclusions (preserving Hermitian forms)

$$i_K : \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G)_F \to L_F,$$

and

$$i_k : \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G)_F \to \mathbb{V},$$

By Lemma 4.4.1 of loc.cit., we have

$$\mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G) = i_K^{-1}(L). \tag{4.10}$$

Let

$$M^\flat := (L_F^\flat) \cap i_k(i_K^{-1}(L)) \cong \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}, G).$$

Then $z \subset \mathcal{Z}(M^\flat)(\mathcal{O}_K)$. Combining Lemma 4.2 with Proposition 2.6, we know that z is one of the irreducible component of $\tilde{\mathcal{Z}}(M^\flat)^o$. It remains to prove that z has multiplicity 1 in $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$. Consider R-points of both sides of (4.9), where $R := \mathcal{O}_K[x]/(x^2)$. As in [Krä03] (see [RZ96, Appendix of Chapter 3]) we know

$$\mathbb{D}(\mathcal{G})(R) \cong \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R, \text{ and } \mathbb{D}(G)(R) \cong (\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R)^n$$

where $\mathbb{D}$ is the $\mathcal{O}_{F_0}$-relative Dieudonné crystal. Define

$$\tilde{e}_0 = 1 \otimes 1 \in \mathbb{D}(\mathcal{G})(R), \quad \tilde{f}_0 = \pi \otimes 1 \in \mathbb{D}(\mathcal{G})(R).$$

Then the Hodge submodule $\mathcal{F}_0$ of $\mathbb{D}(\mathcal{G})(R)$ is spanned by

$$(1 \otimes \pi)\tilde{e}_0 + \tilde{f}_0.$$

$\mathbb{D}(G)(R)$ is equipped with an $\mathcal{O}_F$-invariant symplectic form $\langle , \rangle$ and we can assume that $\mathbb{D}(G)(R)$ has a basis $\{\tilde{e}_1, \ldots, \tilde{e}_n, \tilde{f}_1, \ldots, \tilde{f}_n\}$ such that

$$(\pi \otimes 1)\tilde{e}_i = \tilde{f}_i, \quad \langle \tilde{e}_i, \tilde{f}_j \rangle = \delta_{ij}.$$

Since any element in $L^\flat$ is $\mathcal{O}_F$-linear, we can arrange a change of basis if necessary and assume that

$$L^\flat((1 \otimes \pi)\tilde{e}_0 + \tilde{f}_0) = \mathrm{Span}_R\{(1 \otimes \pi^{a_1})((1 \otimes \pi)\tilde{e}_1 + \tilde{f}_1), \ldots, (1 \otimes \pi^{a_{n-1}})((1 \otimes \pi)\tilde{e}_{n-1} + \tilde{f}_{n-1})\}.$$

Now $\mathbb{D}(G)(\mathcal{O}_K) = \mathbb{D}(G)(R) \otimes_R \mathcal{O}_K$. Let $e_i = \tilde{e}_i \otimes 1$ and $f_i = \tilde{f}_i \otimes 1$ respectively. There is an exact sequence of free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_K$-modules (the Hodge filtration)

$$0 \to \mathcal{F} \to \mathbb{D}(G)(\mathcal{O}_K) \to \mathrm{Lie}\, G \to 0$$

where $\mathcal{F}$ is isotropic with respect to $\langle , \rangle$. We must have $L^\flat((1 \otimes \pi)e_0 + f_0) \subset \mathcal{F}$. Hence we have

$$(1 \otimes \pi)e_1 + f_1, \ldots, (1 \otimes \pi)e_{n-1} + f_{n-1} \subset \mathcal{F}.$$

Since $\mathcal{F}$ is isotropic and by the signature condition, we have

$$\mathcal{F} = \mathrm{Span}_{\mathcal{O}_K}\{(1 \otimes \pi)e_1 + f_1, \ldots, (1 \otimes \pi)e_{n-1} + f_{n-1}, (1 \otimes \pi)e_n - f_n\}.$$

Since $(x) \subset R$ has a nilpotent p.d. structure, by Grothendieck-Messing theory, a lift $\tilde{z}$ of z to $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)(R)$ corresponds to a lift of $\mathcal{F}$ to an isotropic $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} R$-module $\tilde{\mathcal{F}}$ in $\mathbb{D}(G)(R)$ containing the image of $L^\flat$. By the same reasonning as above, we must have

$$\tilde{\mathcal{F}} = \mathrm{Span}_R\{(1 \otimes \pi)\tilde{e}_1 + \tilde{f}_1, \ldots, (1 \otimes \pi)\tilde{e}_{n-1} + \tilde{f}_{n-1}, (1 \otimes \pi)\tilde{e}_n - \tilde{f}_n\}.$$

Hence such lift is unique. This implies that the multiplicity of z in $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ is one. □ □

Lemma 4.2. *Let L be a self-dual Hermitian lattice of rank n and W be a $n-1$ dimensional subspace of L_F. Then $t(M^\flat) \leq 1$ for $M^\flat = L \cap W$.*

Proof. This is exactly the same as the proof of [LZ, Lemma 4.5.1]. Notice that in our case we may need some blocks $\begin{pmatrix} 0 & \pi^a \\ (-\pi)^a & 0 \end{pmatrix}$ in the upper left $(n-1) \times (n-1)$ block of T as in loc.cit. Alternatively, see [LL21, Lemma 2.24(2)]. □

We end this subsection with the following lemma.

Lemma 4.3. *Assume $M^\flat \in \mathrm{Mod}_h$. Then $\tilde{\mathcal{Z}}(M^\flat)^o$ intersects the special fiber of $\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}$ at a unique Exc_Λ for some $\Lambda \in \mathcal{V}^0$. Moreover*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^o} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) = 1.$$

Proof. By the definition of Mod_h, we can find a decomposition of $M^\flat$

$$M^\flat = \Lambda' \obot \{\mathbf{x}\}$$

such that Λ' is self dual and $\chi(\Lambda') = \eta$. By Proposition 2.6, $\tilde{\mathcal{Z}}(\Lambda') \cong \mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. Moreover $\tilde{\mathcal{Z}}(\Lambda') \cap \mathrm{Exc}_\Lambda = \mathbb{P}^1$ is an exceptional divisor in $\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}$. Hence by Lemma 3.1, we have

$$\chi(\mathcal{N}_{n,\epsilon}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^o} \otimes^{\mathbb{L}} \mathcal{O}_{\mathrm{Exc}_\Lambda}) = \chi(\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)^o} \otimes^{\mathbb{L}}_{\mathcal{N}_{2,\epsilon\eta}^{\mathrm{Kra}}} \mathcal{O}_{\mathbb{P}^1}).$$

Now the lemma follows from [HSY, Lemma 5.2] when $\epsilon\eta = 1$, and from [Shi20, Proposition 3.11] when $\epsilon\eta = -1$. □

4.3. **The horizontal part of special difference cycles.** Definition 2.15 motivates us to make the following definition.

Definition 4.4. When $L^\flat$ is a rank $n-1$ integral lattice, define $\mathcal{D}(L)_h \in \mathrm{Gr}^{n-1}(\mathcal{N}^{\mathrm{Kra}})$ by

$$\mathcal{D}(L)_h = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h}] + \sum_{i=1}^{n-1} (-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat \subset L' \subset \frac{1}{\pi} L^\flat \\ \dim_{\mathbb{F}_q}(L'/L^\flat) = i}} [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(L')_h}]. \tag{4.11}$$

Proposition 4.5. *Assume $L^\flat$ is a rank $n-1$ integral lattice, then*

$$\mathcal{D}(L^\flat)_h = \begin{cases} \tilde{\mathcal{Z}}(L^\flat)^o & \text{if } L^\flat \in \mathrm{Mod}_h, \\ 0 & \text{if } L^\flat \notin \mathrm{Mod}_h. \end{cases}$$

Proof. By Theorem 4.1, it suffices to compute the multiplicity of an irreducible components in $\tilde{\mathcal{Z}}(M^\flat)^o$ in $\mathcal{D}(L)_h$ for all $M^\flat \in \mathrm{Mod}_h(L^\flat)$ (see (4.7)). For such a $M^\flat$, define

$$M' := \frac{1}{\pi} L^\flat \cap M^\flat \text{ and } m := \dim_{\mathbb{F}_q}(M'/L^\flat).$$

Then for a lattice L' with $L^\flat \subset L' \subset \frac{1}{\pi} L^\flat$, we know that $\tilde{\mathcal{Z}}(M^\flat)^o$ is in $\mathcal{Z}^{\mathrm{Kra}}(L')_h$ if and only if $L' \subset M'$. Hence the multiplicity of an irreducible components in $\tilde{\mathcal{Z}}(M^\flat)^o$ in $\mathcal{D}(L)_h$ is

$$1 + \sum_{i=1}^{m} (-1)^i q^{i(i-1)/2} \sum_{\substack{L^\flat \subset L' \subset M' \\ \dim_{\mathbb{F}_q}(L'/L^\flat) = i}} 1. \tag{4.12}$$

Notice that $m = 0$ if and only if $M' = M^\flat = L^\flat$, in this case the summation in (4.12) is over an empty set, hence (4.12) is equal to 1. If $m > 0$, (4.12) is equal to 0 by (2.13). □

Part 2. The analytic side

5. Induction formula and primitive local density

In this section, we study various induction formulas of local density polynomials. Let M be a Hermitian $\mathcal{O}_F$-lattice of rank m with a Gram matrix S, and let $M_k = \mathcal{H}^k \oplus M$ for an integer $k \geq 0$. Let L be a Hermitian $\mathcal{O}_F$-lattice of rank n with a Gram matrix T.

There is a polynomial $\alpha(M, L, X)$ of X—the local density polynomial—such that

$$\alpha(M, L, q^{-2k}) := \int_{\mathrm{Herm}_n(F)} \int_{M_k^n} \psi(\langle Y, T(\mathbf{x}) - T \rangle) d\mathbf{x}\, dY, \tag{5.1}$$

where $T(\mathbf{x})$ is the moment matrix of $\mathbf{x}$, $d\mathbf{x}$ is the Haar measure on M_k^n with total volume 1, dY is the Haar measures on $\mathrm{Herm}_n(F)$ such that $\mathrm{Herm}_n(\mathcal{O}_F)$ has total volume 1, and ψ is an additive character of F_0 with conductor $\mathcal{O}_{F_0}$. Finally, we define $\langle X, Y \rangle = \mathrm{Tr}(XY)$ on Herm_n. We will also denote $\alpha(M, L) = \alpha(M, L, 1)$ and

$$\alpha'(M, L) = -\frac{\partial}{\partial X} \alpha(M, L, X)|_{X=1}. \tag{5.2}$$

The dual of $\mathrm{Herm}_n(\mathcal{O}_F)$ with respect to $\langle X, Y \rangle$ is given by

$$\mathrm{Herm}_n(\mathcal{O}_F)^\vee := \{T = (t_{ij}) \in \mathrm{Herm}_n(F) \mid \mathrm{v}_\pi(t_{ii}) \geq 0, \text{ and } \mathrm{v}_\pi(t_{ij}\partial_F) \geq 0\}.$$

Set

$$I(M, L, d) = \{\phi \in \mathrm{Hom}_{\mathcal{O}_F}(M/\pi_0^d M, L/\pi_0^d L) \mid (\phi(x), \phi(y)) = (x, y) \in \pi_0^d \cdot \mathrm{Herm}_n(\mathcal{O}_F)^\vee, \ \forall x, y \in M\}. \tag{5.3}$$

Then a simple calculation ([Shi20, Lemma 6.1]) shows that

$$\alpha(M, L) = q^{-dn(2m-n)} |I(M, L, d)| \tag{5.4}$$

for sufficiently large integers $d > 0$. Since $\alpha(M, L, X)$ only depends on S and T, we also denote it by $\alpha(S, T, X)$.

Now we define primitive local density polynomials. For $1 \le \ell \le n$, let

$$M_k^{n,(\ell)} = \{\mathbf{x} = (x_1, \cdots, x_n) \in M_k^n \mid \dim \operatorname{Span}\{x_1, \cdots, x_\ell\} = \ell \text{ in } M_k/\pi M_k\}, \tag{5.5}$$

and $L = L_1 \oplus L_2$, where $L_1 = \operatorname{Span}\{l_1, \cdots, l_\ell\}$ and $L_2 = \operatorname{Span}\{l_{\ell+1}, \cdots, l_n\}$. We define the local ℓ-primitive density to be

$$\beta(M_k, L_1 \oplus L_2)^{(\ell)} = \int_{\mathrm{Herm}_n(F)} \int_{M_k^{n,(\ell)}} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x}\, dY. \tag{5.6}$$

When $\ell \neq n$, the above definition depends on a choice of $L = L_1 \oplus L_2$. Hence we always fix such a decomposition $L = L_1 \oplus L_2$ in this case. When $L = L_1 \obot L_2$, and L_i is represented by T_i, we also denote $\beta(M, L_1 \obot L_2)^{(\ell)}$ as $\beta(S, \mathrm{Diag}(T_1, T_2))^{(\ell)}$. When $\ell = n$, we simply denote $\beta(M, L_1 \oplus L_2)^{(\ell)}$ as $\beta(M, L)$.

Lemma 5.1. *Assume $L = L_1 \oplus L_2$ where $\operatorname{rank}(L_1) = n_1$. Then*

$$\alpha(M, L, X) = \sum_{L_1 \subset L_1' \subset L_{1,F}} (q^{n-m} X)^{\ell(L_1'/L_1)} \beta(M, L_1' \oplus L_2, X)^{(n_1)},$$

where $\ell(L_1'/L_1) = \operatorname{length}_{\mathcal{O}_F} L_1'/L_1$.

Proof. This is the analogue of [Kit83, Lemma 3]. Let $G = \mathrm{GL}_{n_1}(F) \cap \mathrm{M}_{n_1,n_1}(\mathcal{O}_F)$ and $U = \mathrm{GL}_{n_1}(\mathcal{O}_F)$. By choosing a basis $\{l_1, \cdots, l_{n_1}\}$ of L_1, we may identify $U \backslash G$ with $\{L_1' \mid L_1 \subset L_1' \subset L_{1,F}\}$ by sending g to $L_1 \cdot g^{-1}$. Then the identity we want to prove is equivalent to

$$\alpha(M, L, X) = \sum_{g \in U \backslash G} |\det g|^{2k+m-n} \beta(M, L_1 \cdot g^{-1} \obot L_2, X)^{(n_1)},$$

where $|\pi| = q^{-1}$. By a partition of M_k^n, we have

$$\begin{aligned} \alpha(M, L, X) &= \int_{\mathrm{Herm}_n(F)} dY \int_{M_k^n} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x} \\ &= \sum_{g \in U \backslash G} \int_{\mathrm{Herm}_n(F)} dY \int_{M_k^{n,(n_1)} \cdot g_1} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x}, \end{aligned}$$

where $g_1 = \mathrm{Diag}(g, I_{n-n_1})$, and the action of g_1 is simply matrix multiplication on the n components of $M^{n,(n_1)}$. Now

$$\begin{aligned} &\int_{\mathrm{Herm}_n(F)} dY \int_{M_k^{n,(n_1)} \cdot g_1} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{M_k^{n,(n_1)}} \psi(\langle Y, T(\mathbf{x} g_1) - T\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{M_k^{n,(n_1)}} \psi(\langle Y, (T(\mathbf{x}) - T[g_1^{-1}])[g_1]\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m} \int_{\mathrm{Herm}_n(F)} dY \int_{M_k^{n,(n_1)}} \psi(\langle Y[g_1^*], T(\mathbf{x}) - T[g_1^{-1}]\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m-n} \int_{\mathrm{Herm}_n(F)} dY \int_{M_k^{n,(n_1)}} \psi(\langle Y, T(\mathbf{x}) - T[g_1^{-1}]\rangle) d\mathbf{x} \\ &= |\det g_1|^{2k+m-n} \beta(M_k, L \cdot g_1^{-1})^{(n_1)}. \end{aligned}$$

Here $T[g] \coloneqq g^* T g$. Now the lemma is clear. □

Theorem 5.2. *Let L be as in Lemma 5.1. Then*

$$\alpha(M, L, X) = \sum_{i=1}^{n_1} (-1)^{i-1} q^{i(i-1)/2 + i(n-m)} X^i \sum_{\substack{L_1 \subset L_1' \subset \pi^{-1} L_1 \\ \dim L_1'/L_1 = i}} \alpha(M, L_1' \oplus L_2, X) + \beta(M, L, X)^{(n_1)}.$$

Proof. This is an analogue of [Kat99, Proposition 2.1]. The proof follows from a combination of the argument (in a reverse order) in 2.16 and Lemma 5.1. □

Motivated by Theorem 5.2, we give the following definition.

Definition 5.3. Let $L = L_1 \oplus L_2$ be as in Lemma 5.1. We define

$$\partial\mathrm{Den}(L)^{(n_1)} := \partial\mathrm{Den}(L) - \sum_{i=1}^{n_1}(-1)^{i-1}q^{i(i-1)/2} \sum_{\substack{L_1\subset L_1'\subset L_{1,F}\\ \dim L_1'/L_1=n_1}} \partial\mathrm{Den}(L_1'\oplus L_2). \tag{5.7}$$

Corollary 5.4. *Let $L = L_1 \oplus L_2$ be as in Lemma 5.1, and $\epsilon = \chi(L)$. Then*

$$\partial\mathrm{Den}(L)^{(n_1)} = \frac{1}{\alpha(M,M)}\left(2\beta'(M,L)^{(n_1)} + \sum_j c_\epsilon^{n,i}\beta(\mathcal{H}_\epsilon^{n,i}, L)^{(n_1)}\right).$$

As a corollary of Lemma 5.1, we have

Corollary 5.5. *Let $L = L_1 \oplus L_2$ be as in Lemma 5.1. Then we have the following identity where the summation is finite:*

$$\partial\mathrm{Den}(L) = \sum_{L_1\subset L_1'\subset L_{1,F}} \partial\mathrm{Den}(L_1'\oplus L_2)^{(n_1)}.$$

We may reduce the identity $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$ to a primitive version as the following theorem shows.

Theorem 5.6. *Let $L = L_1 \oplus L_2 \subset \mathbb{V}$ be as in Lemma 5.1.*

(1) Conjecture 1.1 is true for L if for every $L_1 \subset L_1' \subset L_{1,F}$, we have

$$\mathrm{Int}(L_1'\oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1'\oplus L_2)^{(n_1)},$$

(2) If Conjecture 1.1 holds for all lattices $L' = L_1' \oplus L_2$ of $\mathbb{V}$ of rank n with $L_1 \subset L_1' \subset L_{1,F}$, then

$$\mathrm{Int}(L_1\oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1\oplus L_2)^{(n_1)}.$$

(3) For $1 \le n_1 \le n$, Conjecture 1.1 is true if and only if for every lattice $L = L_1 \oplus L_2 \subset \mathbb{V}$ with $\mathrm{rank}(L_1) = n_1$, one has

$$\mathrm{Int}(L_1\oplus L_2)^{(n_1)} = \partial\mathrm{Den}(L_1\oplus L_2)^{(n_1)}.$$

Proof. (1) follows from Lemma 2.16 and Corollary 5.5. (2) follows from Definitions 2.15 and 5.3. (3) follows from (1) and (2). □

For the rest of this section, we assume that $M = I_{m,\epsilon}$ is unimodular, and let $S = \mathrm{Diag}(I_{m-1}, \nu)$ be a Gram matrix so $\epsilon = \chi((-1)^{\frac{m(m-1)}{2}}\nu)$. To go further with the calculation of $\alpha(M, L, X)$, we need an induction formula for $\beta(M, L, X)^{(\ell)}$ as follows. The proof is essentially the same as that of Corollary 9.11 of [KR11], and is left to the reader.

Proposition 5.7. *Let $L = L_1 \obot L_2$, where L_j is of rank n_j. Let $C(M_k, L_1)$ be the $\mathrm{U}(M_k)$-orbits of sublattices $M(i) \subset M_k$ such that $M(i)$ is isometric to L_1, and write $C(M_k, L_1) = \sqcup_{i\in J}\{M(i)\}$.*

$$\beta(M,L,X)^{(n_1)} = \sum_{i\in J} |M : M(i) \perp M(i)^\perp|^{-n_2} |M(i)^\vee : M(i)|^{n_2} \beta_i(M, L_1, X)\alpha(M(i)^\perp, L_2, X),$$

where

$$\beta_i(M, L_1, X) = \lim_{d\to\infty} q^{-dn_1(2m+4k-n_1)} |\{\phi \in I(M_k, L_1, d)^{(n_1)} \mid \exists\ \Phi \in \mathrm{U}(M) \text{ with } \phi(L_1) = \Phi(M(i))\}|,$$

and

$$I(L, M_1, d)^{(n_1)} := \{\phi \in I(L, M_1, d) \mid \mathrm{rank}_{\mathbb{F}_q}\phi(M_1)\otimes_{\mathcal{O}_F}\mathbb{F}_q = n_1\}.$$

Recall that $I(M_k, L_1, d)$ is defined in (5.3).

One special case is that $L = \mathcal{H}^i \obot L_2$. Since any sublattice of $M_k = M \obot \mathcal{H}^k$ represented by $\mathcal{H}^i$ is always a direct summand of M_k and $\alpha(M, L, X) = \beta(M, L, X)^{(2i)}$, the above proposition specializes to

Corollary 5.8. *Assume $L = \mathcal{H}^i \obot L_2$, then*

$$\alpha(M, L, X) = \alpha(M, \mathcal{H}^i, X)\alpha(M, L_2, q^{2i}X). \tag{5.8}$$

We end this section with two more special cases of Proposition 5.7. Proofs are given in Appendix A.

Proposition 5.9. *Let the notation be as in Proposition 5.7. Assume $n_1 = 1$ and $L_1 = \langle t \rangle$ where $t \in \mathcal{O}_{F_0}$.*

(1) There always exists a primitive vector $M(1) \in \mathcal{H}^k$ with $q(M(1)) = t$, and

$$M(1)^\perp \cong \mathcal{H}^{k-1} \oplus I_{m,\epsilon} \oplus \langle -t \rangle$$

Here $\langle t \rangle$ stands for a lattice $\mathcal{O}_F v$ of rank one with $(v, v) = t$.

(2) If $\mathrm{v}(t) = 0$, then there exist a primitive vector $M(0) \in M$ with $q(M(0)) = t$, and

$$M(0)^\perp \cong \mathcal{H}^k \oplus I_{m-2,\epsilon_{m-2}} \oplus \langle \nu t \rangle.$$

Here $\epsilon_{m-2} = \chi((-1)^{\frac{(m-2)(m-3)}{2}})$.

(3) If $\mathrm{v}(t) > 0$, then there exist a primitive vector $M(0) \in M$ with $q(M(0)) = t$ only when M is isotropic (i.e. $\exists\, v \in M$ with $q(v) = 0$). In this case,

$$M(0)^\perp \cong \mathcal{H}^k \oplus I_{m-2,\epsilon} \oplus \langle -t \rangle.$$

Assuming the existence of $M(1)$ and $M(0)$, we have

$$|M_k : M(i) \perp M(i)^\perp|^{-1} |M(i)^\vee : M(i)| = \begin{cases} 1 & \text{if } i = 1, \\ q & \text{if } i = 0. \end{cases}$$

(4) Under the action of $\mathrm{U}(M_k)$, v is either in the same orbit of a fixed vector $M(1) \in \mathcal{H}^k$ or a fixed vector $M(0) \in M$.

(5) We have the following induction formula:

$$\begin{aligned} &\beta(M, L, X)^{(1)} \\ &= \beta_1(M, L_1, X)\alpha(M(1)^\perp, L_2, X) + q^{n-1}\beta_0(M, L_1, X)\alpha(M(0)^\perp, L_2, X). \end{aligned}$$

Moreover,

(a) For any L_1,

$$\beta_1(M, L_1, X) = 1 - X.$$

(b) Assume $\mathrm{v}(t) = 0$, then

$$\beta_0(M, L_1, X) = \begin{cases} (1 + \chi(M)\chi(L) q^{-\frac{m-1}{2}})X & \text{if } m \text{ is odd,} \\ (1 - \chi(M) q^{-\frac{m}{2}})X & \text{if } m \text{ is even.} \end{cases}$$

(c) Assume $\mathrm{v}(t) > 0$, then

$$\beta_0(M, L_1, X) = \begin{cases} (1 - q^{1-m})X & \text{if } m \text{ is odd,} \\ \left(1 - q^{1-m} + \chi(M)(q-1)q^{-\frac{m}{2}}\right) X & \text{if } m \text{ is even.} \end{cases}$$

Proof. Parts (1)—(4) are proved in subsection A.1. The induction formula for $\beta(M, L, X)^{(1)}$ follows from Proposition 5.7. For the formula of $\beta_i(M, L_1, X)$, see Corollaries A.10 and A.12. □

Proposition 5.10. *Let the notation be as in Proposition 5.7. Assume $\mathrm{v}(L_1) > 0$ and $n_1 = 2$. Then we have a partition of $C(M_k, L_1) = \bigsqcup_{i=0}^{2} C_i(M_k, L_1)$ such that for any $M(i) \in C_i(M_k, L_1)$, $M(i)^\perp$ is isometric to*

$$(-L_1) \oplus \mathcal{H}^{k-i} \oplus M^{(i)}.$$

Here $M^{(i)}$ is a unimodular $\mathcal{O}_F$-lattice of rank $m - 2(2 - i)$ and has determinant $(-1)^i \det L$.

Moreover, we have

$$\beta(M, L, X)^{(2)} = \sum_{i=0}^{2} q^{(2-i)(n-2)} \beta_i(M, L_1, X)\alpha(M(i)^\perp, L_2, X), \tag{5.9}$$

where

$$\begin{aligned} &\beta_2(M, L_1, X) = (1 - X)(1 - q^2 X), \\ &\beta_1(M, L_1, X) = q(q + 1)\left((1 - q^{1-m}) + \delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}}\right) X(1 - X), \end{aligned}$$

and

$$\beta_0(M, L_1, X) = \begin{cases} q(1-q^{1-m})(1-q^{3-m})X^2 & \text{if } m \text{ is odd,} \\ q\left((1-q^{2-m}) + \chi(M)(q^2-1)q^{-\frac{m}{2}}\right)(1-q^{2-m})X^2 & \text{if } m \text{ is even.} \end{cases}$$

Here $\delta_e(m) = 1$ or 0 depending on whether m is even or odd.

Proof. Equation (5.9) follows from Proposition 5.7 and Proposition A.5. For the formula of $\beta_i(S, L_1, X)$, see Corollaries A.10, A.13 and Proposition A.14. □

6. The modified Kudla-Rapoport conjecture

Theorem 6.1. *Let $r_\epsilon = \frac{n-1}{2}$ when n is odd, and $r_\epsilon = \lfloor \frac{n+\epsilon}{2} \rfloor$ when n is even. In the following we just write r_ϵ as r.*

$$A_\epsilon = (A_\epsilon^{(i,j)}) = \begin{pmatrix} \alpha(\mathcal{H}_\epsilon^{n,1}, \mathcal{H}_\epsilon^{n,1}) & \alpha(\mathcal{H}_\epsilon^{n,2}, \mathcal{H}_\epsilon^{n,1}) & \cdots & \alpha(\mathcal{H}_\epsilon^{n,r}, \mathcal{H}_\epsilon^{n,1}) \\ 0 & \alpha(\mathcal{H}_\epsilon^{n,2}, \mathcal{H}_\epsilon^{n,2}) & \cdots & \alpha(\mathcal{H}_\epsilon^{n,r}, \mathcal{H}_\epsilon^{n,2}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \alpha(\mathcal{H}_\epsilon^{n,r}, \mathcal{H}_\epsilon^{n,r}) \end{pmatrix},$$

$$B_\epsilon = {}^t(\alpha'(I_{n,-\epsilon}, \mathcal{H}_\epsilon^{n,1}), \cdots, \alpha'(I_{n,-\epsilon}, \mathcal{H}_\epsilon^{n,r})),$$

and

$$C_\epsilon = {}^t(c_\epsilon^{n,1}, \cdots, c_\epsilon^{n,r}),$$

where $c_\epsilon^{n,i}$ is as in Conjecture 1.1.

Then C_ϵ is the solution of the equation

$$A_\epsilon C_\epsilon = -2B_\epsilon. \tag{6.1}$$

Moreover,

$$A_\epsilon^{(j,j)} = 2q^{\frac{(n-2j)(n-2j-1)}{2}} \prod_{0<s\le j}(1-q^{-2s}) \prod_{1\le s\le [\frac{n-2j-1}{2}]}(1-q^{-2s}) \cdot \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1-\epsilon q^{-\frac{n-2j}{2}} & \text{if } n \text{ is even.} \end{cases} \tag{6.2}$$

For $i < j$,

$$A_\epsilon^{(i,j)} = A_\epsilon^{(j,j)} \cdot \begin{cases} I(n-2i, \frac{n-2i-1}{2}, j-i) & \text{if } n \text{ is odd,} \\ I(n-2i, \frac{n-2i-1+\epsilon}{2}, j-i) & \text{if } n \text{ is even,} \end{cases} \tag{6.3}$$

where

$$I(n,d,k) := \prod_{s=1}^{k} \frac{(q^{d-s+1}-1)(q^{n-d-s}+1)}{q^s-1}.$$

Proof. First notice that $\alpha(\mathcal{H}_\epsilon^{n,i}, \mathcal{H}_\epsilon^{n,j}) = 0$ if $i < j$. So (1.9) is indeed equivalent to (6.1), and there exists a unique solution C_ϵ.

Now we compute $A_\epsilon^{(j,j)}$ explicitly. Corollary 5.8 and Lemma A.15 imply that

$$\alpha(\mathcal{H}_\epsilon^{n,j}, \mathcal{H}_\epsilon^{n,j}) = \alpha(\mathcal{H}^j, \mathcal{H}^j)\alpha(I_{n-2j,\epsilon}, I_{n-2j,\epsilon}).$$

According to Lemma A.9,

$$\alpha(\mathcal{H}^j, \mathcal{H}^j) = \prod_{0<s\le j}(1-q^{-2s}).$$

By Lemma A.11,

$$\alpha(I_{n-2j,\epsilon}, I_{n-2j,\epsilon}) = |\mathrm{O}(\overline{M}_{I_{n-2j,\epsilon}})(\mathbb{F}_q)|.$$

Now (6.2) follows from the well-known formula:

$$|\mathrm{O}(\overline{M}_{I_{n-2j,\epsilon}})(\mathbb{F}_q)| = \begin{cases} 2q^{\frac{(n-2j)(n-2j-1)}{2}} \prod_{s=1}^{\frac{n-2j-1}{2}}(1-q^{-2s}) & \text{if } n \text{ is odd,} \\ 2q^{\frac{(n-2j)(n-2j-1)}{2}}(1-\epsilon q^{-\frac{n-2j}{2}}) \prod_{s=1}^{\frac{n-2j}{2}-1}(1-q^{-2s}) & \text{if } n \text{ is even.} \end{cases}$$

To obtain (6.3), notice that (Corollary 5.8)

$$\alpha(\mathcal{H}_\epsilon^{n,j}, \mathcal{H}_\epsilon^{n,i}) = \alpha(\mathcal{H}_\epsilon^{n,j}, \mathcal{H}^i)\alpha(\mathcal{H}_\epsilon^{n-2i,j-i}, I_{n-2i,\epsilon}),$$

and

$$\alpha(\mathcal{H}_\epsilon^{n,j}, \mathcal{H}_\epsilon^{n,j}) = \alpha(\mathcal{H}_\epsilon^{n,j}, \mathcal{H}^i)\alpha(\mathcal{H}_\epsilon^{n-2i,j-i}, \mathcal{H}_\epsilon^{n-2i,j-i}).$$

Hence

$$\frac{A_\epsilon^{(i,j)}}{A_\epsilon^{(j,j)}} = \frac{\alpha(\mathcal{H}_\epsilon^{n,j}, \mathcal{H}_\epsilon^{n,i})}{\alpha(\mathcal{H}_\epsilon^{n,j}, \mathcal{H}_\epsilon^{n,j})} = \frac{\alpha(\mathcal{H}_\epsilon^{n-2i,j-i}, I_{n-2i,\epsilon})}{\alpha(\mathcal{H}_\epsilon^{n-2i,j-i}, \mathcal{H}_\epsilon^{n-2i,j-i})}.$$

Fix an $\mathcal{O}_F$-lattice L that is represented by $I_{n-2i,\epsilon}$. According to Lemma 6.2, to compute $\frac{A_\epsilon^{(i,j)}}{A_\epsilon^{(j,j)}}$, we need to count the number of lattices L' in L_F such that contain $L \subset L'$ and $L' \cong \mathcal{H}_\epsilon^{n-2i,j-i}$, which is equivalent to the following condition:

$$\pi L \overset{j-i}{\subset} \pi L' \overset{n-2j}{\subset} L'^{,\sharp} \overset{j-i}{\subset} L \overset{j-i}{\subset} L'.$$

Since L' and $\pi L'$ determine each other, we just need to count $\pi L'$ satisfying the above condition. We regard $\pi L'/\pi L$ as a $(j-i)$-dimensional subspace of $L/\pi L$, where $L/\pi L$ is equipped with quadratic form $(x, y)/\pi$.

Claim: The condition

$$\pi L' \subset L'^{,\sharp}$$

is equivalent to the condition that $\pi L'/\pi L$ is an isotropic subspace of $L/\pi L$.

Indeed, assume $\pi L'/\pi L$ is an isotropic subspace of $L/\pi L$. Then $(\pi x, \pi y) \in \pi\mathcal{O}_F$ for any $x, y \in L'$, which is equivalent to $(x, \pi y) \in \mathcal{O}_F$ for any $x, y \in L'$. The latter condition is the same as $L' \subset L'^{,\sharp}$. The other direction is clear.

Therefore $\frac{A_\epsilon^{(i,j)}}{A_\epsilon^{(j,j)}}$ is the number of $(j-i)$-dimensional isotropic subspaces of $L/\pi L$. According to [LZ21, Lemma 3.2.2], it equals to

$$\begin{cases} I(n-2i, \frac{n-2i-1}{2}, j-i) & \text{if } n \text{ is odd}, \\ I(n-2i, \frac{n-2i-1+\epsilon}{2}, j-i) & \text{if } n \text{ is even}. \end{cases}$$

□

Lemma 6.2. *Let F/F_0 be a quadratic p-adic field extension, and let L and M be two $\mathcal{O}_F$-Hermitian lattices of rank n. then $\frac{\alpha(M,L)}{\alpha(M,M)}$ is equal to the number of lattices L' in L_F containing L and isometric to M.*

Proof. The proof is a generalization of that of Proposition 10.2 of [KR14b] and works for both inert and ramified primes.

Let us assume that there is an isometric embedding from L into M, otherwise both sides of the identity in the lemma are zero. In this case, we have a fixed $L_F \cong M_F$. Let α (resp. β) be a top degree translation invariant form on L_F (resp. $\mathrm{Herm}_n(F)$). Let $\nu_p = \alpha/h^*(\beta)$ where

$$h: \quad L_F^n \to \mathrm{Herm}_n(F) \quad x \mapsto (x, x).$$

Define X to be the set of linear isometric embeddings from L into M. By the argument in Section 3 of [GY00] (in particular Lemma 3.4), we know that

$$\alpha(M, L) = \mathrm{vol}(X, d\nu_p)\frac{\mathrm{vol}(\mathrm{Herm}_n(\mathcal{O}_F), d\beta)}{\mathrm{vol}((M)^n, d\alpha)}. \tag{6.4}$$

For any $\phi \in X$ regarded as a linear isometry from L_F to itself, the lattice $L_\phi := \phi^{-1}(M)$ is a lattice containing L. Conversely, for any L' containing L and isometric to M, there is a $\phi \in X$ such that $L_\phi = L'$. Hence we have a partition

$$X = \bigsqcup_{L \subset L'} X_{L'}, \quad X_{L'} := \{\phi \in X \mid L_\phi = L'\}.$$

Since each L' is isomorphic to M, all the $X_{L'}$ have the same volume as that of X_M. Specializing (6.4) to $L = M$, we see

$$\alpha(M, M) = \mathrm{vol}(X_M, d\nu_p)\frac{\mathrm{vol}(\mathrm{Herm}_n(\mathcal{O}_F), d\beta)}{\mathrm{vol}((M)^n, d\alpha)}. \tag{6.5}$$

Dividing equation (6.4) by (6.5), we prove the lemma. □

Remark 6.3. *When F/F_0 is unramified and M is unimodular, the lemma was proved by equation (3.6.1.1) of* [LZ].

In order to solve C_ϵ, we also need to know B_ϵ, which can be calculated by applying Corollary 5.8 and Proposition 5.9 inductively.

Lemma 6.4. *Assume $n = 3$ and $\epsilon = \chi(L)$. Then*

$$\partial Den(L) = 2\frac{\alpha'(I_{3,-\epsilon}, L)}{\alpha(I_{3,-\epsilon}, I_{3,-\epsilon})} + \frac{q^2}{1+q}\frac{\alpha(\mathcal{H}_\epsilon^{3,1}, L)}{\alpha(I_{3,-\epsilon}, I_{3,-\epsilon})}.$$

Proof. First of all, according to Theorem 6.1,

$$\alpha(\mathcal{H}_\epsilon^{3,1}, \mathcal{H}_\epsilon^{3,1}) = 2(1-q^{-2}). \tag{6.6}$$

By Corollary 5.8, we have

$$\alpha(I_{3,-\epsilon}, \mathcal{H}_\epsilon^{3,1}, X) = \alpha(I_{3,-\epsilon}, \mathcal{H}, X)\alpha(I_{3,-\epsilon}, I_{1,\epsilon}, q^2X).$$

According to Lemmas A.15 and A.9,

$$\alpha(I_{3,-\epsilon}, \mathcal{H}, X) = \alpha(\mathcal{H}^k, \mathcal{H}) = \beta(\mathcal{H}^k, \mathcal{H}) = 1 - X.$$

Lemma 7.1 gives that

$$\alpha(I_{3,-\epsilon}, I_{1,\epsilon}, q^2X) = 1 - qX.$$

Hence

$$\alpha(I_{3,-\epsilon}, \mathcal{H}_\epsilon^{3,1}, X) = (1-X)(1-qX),$$

and

$$\alpha'(I_{3,-\epsilon}, \mathcal{H}_\epsilon^{3,1}) = 1 - q.$$

Combining this with (6.6), we solve (6.1) and obtain

$$c_\epsilon^{3,1} = \frac{q^2}{1+q}.$$

Now the lemma follows from (1.10). □

7. Local density formula when rank$(T) \le 2$

The main purpose of this section is to give an explicit formula for $\alpha(M, L, X)$ when $M = I_{m,\epsilon}$ is unimodular of rank m, and L has rank 2. Assume that M has a Gram matrix $S = \mathrm{Diag}(I_{m-1}, \nu)$ with $\nu \in \mathcal{O}_{F_0}^\times$, and that L has a Gram matrix T. As the explicit formula depends on the form of S and T, we will write the local density function as $\alpha(S, T, X) = \alpha(M, L, X)$.

7.1. The case $T = (t)$.

In order to apply induction formulas to calculate $\alpha(S, T, X)$ for T with rank$(T) = 2$, we need to consider the case $T = (t)$ and M a little more general than unimodular. Write $t = t_0(-\pi_0)^{\mathrm{v}(t)}$ for $t_0 \in \mathcal{O}_{F_0}^\times$, and

$$S_{a,b} = \mathrm{Diag}(S, \nu_1(-\pi_0)^a, \nu_2(-\pi_0)^b) = \mathrm{Diag}(s_1, \ldots, s_{m+2}) \tag{7.1}$$

for integers $0 \le a \le b$.

Lemma 7.1. *Assume $0 \le a \le b \le \mathrm{v}(t)$.*

(1) If m is odd, then

$$\alpha(S_{a,b}, (t), X) = 1 + \chi(S)\chi(-\nu_1)(q-1)\sum_{s=a+1}^{b} q^{-ms+a+\frac{m-1}{2}}X^s + \chi(S_{a,b})\chi(t_0)q^{-(m+1)\mathrm{v}(t)+a+b-\frac{m+1}{2}}X^{\mathrm{v}(t)+1}.$$

(2) If m is even, then

$$\alpha(S_{a,b},(t),X) = 1 + \chi(S)(q-1)\sum_{s=1}^{a} q^{-(m-1)s+\frac{m}{2}-1}X^s + \chi(S_{a,b})\left((q-1)\sum_{s=b+1}^{\mathrm{v}(t)} q^{-(m+1)s+a+b+\frac{m}{2}}X^s - q^{-(m+1)\mathrm{v}(t)+a+b-1-\frac{m}{2}}X^{\mathrm{v}(t)+1}\right).$$

Proof. Direct calculation gives

$$\begin{aligned}\alpha(S_{a,b},(t),X) &= \int_{F_0} dY \int_{\mathcal{O}_F^{2k+m+2}} \psi(\langle Y, \mathrm{Diag}(\mathcal{H}^k, S_{a,b})[\mathbf{x}] - t\rangle)\, d\mathbf{x} \\ &= \int_{F_0}\int_{\mathcal{O}_F^{2k}\times\mathcal{O}_F^{m+2}} \psi(Y\sum_{i=1}^{k}\mathrm{tr}(\frac{1}{\pi}x_i\bar{y}_i) + Y\sum_{l=1}^{m+2} s_l z_l \bar{z}_l)\psi(-tY)\prod_i dx_i dy_i \prod_l dz_l\, dY \\ &= 1 + \sum_{s=1}^{\infty}\int_{v(Y)=-s} I_k(Y) I_{S_{a,b}}(Y)\psi(-tY)\, dY.\end{aligned}$$

Here, according to [Shi20, Lemma 7.6],

$$I_k(Y) = \int_{\mathcal{O}_F^{2k}} \psi(Y\sum_{i=1}^{k}\mathrm{tr}(\frac{1}{\pi}x_i\bar{y}_i))\prod dx_i dy_i = q^{-2ks},$$

and

$$I_{S_{a,b}}(Y) = \int_{\mathcal{O}_F^{m+2}} \psi(Y\sum_{l=1}^{m+2} s_l z_l\bar{z}_l)\prod dz_l = \prod_{l=1}^{m+2} J(s_l Y),$$

where

$$J(t) = \int_{\mathcal{O}_F} \psi(tz\bar{z})dz = \begin{cases} 1 & \text{if } \mathrm{v}(t)\geq 0, \\ q^{\mathrm{v}(t)}\chi(-t_0)g(\chi,\psi_{\frac{1}{\pi_0}}) & \text{if } \mathrm{v}(t) < 0, \end{cases} \tag{7.2}$$

and

$$g(\chi,\psi_{\frac{1}{\pi_0}}) = \sum_{x\in\mathcal{O}_{F_0}/\pi_0}\chi(x)\psi(\frac{x}{\pi_0})$$

is the Gauss sum. Write $\psi' = \psi_{\frac{1}{\pi_0}}$. Then

$$\begin{aligned}\alpha(S_{a,b},(t),X) = 1 &+ \sum_{s=1}^{a} q^s \int_{\mathcal{O}_{F_0}^\times} q^{-2ks}\cdot q^{-ms}\chi(\nu(-Y)^m)g(\chi,\psi')^m\psi(-(-\pi_0)^s Yt)dY \\ &+ \sum_{s=a+1}^{b}\int_{\mathcal{O}_{F_0}^\times} q^{-2ks}\cdot q^{-ms+a}\chi(\nu_1\nu(-Y)^{m+1})g(\chi,\psi')^{m+1}\psi(-(-\pi_0)^{-s}Yt)\, dY \\ &+ \sum_{s=b+1}^{\infty}\int_{\mathcal{O}_{F_0}^\times} q^{-2ks}\cdot q^{-(m+1)s+a+b}\chi(\nu_1\nu_2\nu(-Y)^{m+2})g(\chi,\psi')^{m+2}\psi(-(-\pi_0)^{-s}Yt)\, dY.\end{aligned}$$

Recall the well-known facts that

$$\begin{aligned} g(\chi,\psi')^2 &= \chi(-1)\cdot q, \\ \int_{\mathcal{O}_{F_0}^\times}\psi((-\pi_0)^{-s}Yt)dY &= \mathrm{Char}(\pi_0^s\mathcal{O}_{F_0})(t) - q^{-1}\,\mathrm{Char}(\pi_0^{s-1}\mathcal{O}_{F_0})(t), \\ \int_{\mathcal{O}_{F_0}^\times}\chi(Y)\psi((-\pi_0)^{-s}Yt)dY &= \chi(-t_0)q^{-1}g(\chi,\psi')\,\mathrm{Char}(\pi_0^{s-1}\mathcal{O}_{F_0}^\times)(t).\end{aligned} \tag{7.3}$$

When m is odd, we have

$$\alpha(S_{a,b},(t),X)=1+\chi((-1)^{\frac{m+1}{2}}\nu_1\nu)(q-1)\sum_{s=a+1}^{b}q^{-ms+a+\frac{m-1}{2}}X^s + \chi((-1)^{\frac{m+1}{2}}\nu_1\nu_2\nu t_0)q^{-(m+1)(\mathrm{v}(t)+1)+a+b+\frac{m+1}{2}}X^{\mathrm{v}(t)+1}.$$

When m is even, we have

$$\alpha(S_{a,b},(t),X)=1+\chi((-1)^{\frac{m}{2}}\nu)(q-1)\sum_{s=1}^{a}q^{-(m-1)s+\frac{m}{2}-1}X^s+\chi((-1)^{\frac{m+2}{2}}\nu_1\nu_2\nu) \cdot\left((q-1)\sum_{s=b+1}^{\mathrm{v}(t)}q^{-(m+1)s+a+b+\frac{m}{2}}X^s-q^{-(m+1)(\mathrm{v}(t)+1)+a+b+\frac{m}{2}}X^{\mathrm{v}(t)+1}\right).$$

Finally, notice that for S of rank m we have

$$\chi(S)=\begin{cases}\chi((-1)^{\frac{m-1}{2}}\nu) & \text{if } m \text{ is odd},\\ \chi((-1)^{\frac{m}{2}}\nu) & \text{if } m \text{ is even}.\end{cases}$$

Now the lemma is clear. □

Similarly, we have the following lemma.

Lemma 7.2. *Let S be unimodular with odd rank m. Then*

$$\alpha(S\oplus\mathcal{H}_i,(t),X)=\begin{cases}1+\chi(S)\chi(t_0)q^{-(\mathrm{v}(t)+1)(m+1)+\frac{m+1}{2}+i}X^{\mathrm{v}(t)+1} & \text{if } i\le 2\mathrm{v}(t),\\ 1+\chi(S)\chi(t_0)q^{-(\mathrm{v}(t)+1)(m-1)+\frac{m-1}{2}}X^{\mathrm{v}(t)+1} & \text{if } i> 2\mathrm{v}(t).\end{cases}$$

7.2. **The case $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b)$.** In this subsection, we compute $\alpha(S,T,X)$ for S unimodular of rank $m\ge 2$ and $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b)$ with $0\le a\le b$. Notice that $\alpha(S,T,X)=0$ when $a<0$.

Proposition 7.3. *Assume $T=\mathrm{Diag}(u_1(-\pi_0)^a,u_2(-\pi_0)^b)$ and that S is isotropic of even rank $m\ge 2$, then*

$$\begin{aligned}&\alpha(S,T,X)\\ &=(1-X)\left(\sum_{i=0}^{a}(q^{2-m}X)^i+\gamma_e(S,T,X)\right)+qX(q^{2-m}X)^a(1-\chi(S)q^{-\frac{m}{2}})(1+\chi(S)\chi(T)q^{\frac{m-2}{2}}(q^{2-m}X)^{b+1})\\ &\quad+\left(1-q^{-(m-1)}+(q-1)\chi(S)q^{-\frac{m}{2}}\right)X\cdot\left(q\sum_{i=0}^{a-1}(q^{2-m}X)^i+\gamma_e(S,T,X)-\chi(S)\chi(T)q^{\frac{m}{2}}(q^{2-m}X)^{a+b+1}\right),\end{aligned}$$

where

$$\gamma_e(S,T,X)=\chi(S)q^{\frac{m}{2}}\left(\sum_{d=1}^{a}(q^d-1)(q^{2-m}X)^d+\chi(T)q^a(q^{2-m}X)^{b+1}\sum_{i=0}^{a}(q^{1-m}X)^i\right).$$

Proof. Since S is of even rank, $u_1\cdot S_k\approx S_k$. Hence for $T'=\mathrm{Diag}((-\pi_0)^a,u_1u_2(-\pi_0)^b)$, we have $\alpha(S,T)=\alpha(S,T')$. As a result, we may assume T is of the form $\mathrm{Diag}((-\pi_0)^a,u(-\pi_0)^b)$ without loss of generality.

According to Theorem 5.2 and Proposition 5.9, we have

$$\begin{aligned}\alpha(S,T,X)&=\beta_1(S,(-\pi_0)^a,X)\alpha(M(1)^\perp,u(-\pi_0)^b)+q\beta_0(S,(-\pi_0)^a,X)\alpha(M(0)^\perp,u(-\pi_0)^b)\\ &\quad+q^{2-m}X\alpha(S,\mathrm{Diag}((-\pi_0)^{a-1},u(-\pi_0)^b),X)\end{aligned}$$

where $M(1)^\perp=\mathrm{Diag}(\mathcal{H}^{k-1},-(-\pi_0)^i,S)$ and $M(0)^\perp=\mathrm{Diag}(\mathcal{H}^k,\underbrace{-(-\pi_0)^i,1,\dots,1,-\nu}_{m-1})$. Continuing this process, we obtain

$$\alpha(S,T,X)=\sum_{i=0}^{a}(q^{2-m}X)^{a-i}\left(\beta_1(S,(-\pi_0)^i,X)\alpha(M(1)^\perp,u(-\pi_0)^b)+q\beta_0(S,(-\pi_0)^i,X)\alpha(M(0)^\perp,u(-\pi_0)^b)\right).$$

By the formulas in Proposition 5.9 and Lemma 7.1, the above equals to

$$\sum_{i=0}^{a}(q^{2-m}X)^{a-i}(1-X)\left(1+(q-1)\chi(S)q^{\frac{m-2}{2}}\sum_{s=-i}^{-1}(q^{(m-1)}(q^2X)^{-1})^s+\chi(S)\chi(T)q^{i+\frac{m}{2}}(q^{2-m}X)^{b+1}\right)$$
$$+q(q^{2-m}X)^aX(1-q^{-\frac{m}{2}}\chi(S))(1+\chi(S)\chi(T)q^{-(b+1)(m-2)+\frac{m-2}{2}}X^{b+1})$$
$$+q\sum_{i=1}^{a}(q^{2-m}X)^{a-i}X\left((1-q^{-(m-1)})+(q-1)\chi(S)q^{-(m-1)+\frac{m-2}{2}}\right)$$
$$\cdot\left(1+(q-1)\chi(S)q^{\frac{m-4}{2}}\sum_{s=-i}^{-1}(q^{(m-3)}X^{-1})^s+\chi(S)\chi(T)q^{i+\frac{m-2}{2}}(q^{2-m}X)^{b+1}\right).$$

Now the transformation

$$\sum_{i=0}^{a}\sum_{s=1}^{i}q^s(q^{2-m}X)^{a-i+s}=\sum_{d=1}^{a}\sum_{s=1}^{d}q^s(q^{2-m}X)^d$$

and some calculation give us the result we want. □

The case that S is anisotropic (i.e. when $m=2$ and $\chi(S)=-1$) can be computed similarly and is simpler. We omit the detail here. In particular, we may recover the following formula.

Proposition 7.4. *(*[Shi20, Theorem 6.2(1)]*) Assume $S=\mathrm{Diag}(1,\nu)$, then*

$$\begin{aligned}&\alpha(S,T,X)\\&=(1-X)(1+\chi(S)+q\chi(S))\sum_{e=0}^{\alpha}(qX)^e-\chi(T)q^{\alpha+1}X^{\beta+1}(1-X)\sum_{e=0}^{\alpha}(q^{-1}X)^e\\&\quad-\chi(T)(1+q)(X^{\alpha+\beta+2}+\chi(S)\chi(T))+(1+\chi(S))q^{\alpha+1}X^{\alpha+1}(1+\chi(T)X^{\beta-\alpha}).\end{aligned}$$

Moreover, a similar computation yields the following, and we leave the detail to reader.

Proposition 7.5. *Assume that S is unimodular of odd rank $m\geq 3$. Then*

$$\begin{aligned}&\alpha(S,T,X)\\&=(1-X)\left(\sum_{i=0}^{a}(q^{2-m}X)^i+\gamma_{o,1}(S,T,X)\right)+(1-q^{-(m-1)})X\left(q\sum_{i=0}^{a-1}(q^{2-m}X)^i+\gamma_{o,0}(S,T,X)\right)\\&\quad+qX(q^{2-m}X)^a\left(1+\chi(S)\chi(u_1)q^{-\frac{m-1}{2}}\right)\left(1-\chi(S)\chi(u_1)q^{(2-m)b-\frac{m-1}{2}}X^{b+1}\right),\end{aligned}$$

where

$$\gamma_{o,1}(S,T,X)=\chi(S)\chi(u_1)q^{\frac{m-1}{2}}\left(\sum_{d=a+1}^{a+b}(q^{a+b+1-d}-1)(q^{2-m}X)^d-\sum_{i=b+1}^{a+b+1}(q^{2-m}X)^i\right),$$

and

$$\gamma_{o,0}(S,T,X)=\chi(S)\chi(u_1)q^{\frac{m-1}{2}}\left(\sum_{d=a+1}^{a+b}(q^{a+b+1-d}-q)(q^{2-m}X)^d-\sum_{i=b+1}^{a+b}(q^{2-m}X)^i\right).$$

8. Local density formula when $\mathrm{rank}(T)=3$

In this section, we compute the analytic side explicitly for the case $n=3$. Similarly to the previous section, we will write local density function in terms of S and T instead of M and L. We assume again that M is unimodular. We treat the case $\mathrm{v}(T)\leq -1$ in the first subsection. In the second subsection, we deal with the case when $T=\mathrm{Diag}(1,T_2)$ for T_2 diagonal. In the last subsection, instead of $\partial\mathrm{Den}(T)$, we compute $\partial\mathrm{Den}(T)^{(2)}$ for T of the form not covered by previous sections.

8.1. $\partial\mathrm{Den}(T)$ **for** T **with** $\mathrm{v}(T) \leq -1$**.**

Proposition 8.1. *If* $\mathrm{v}(T) \leq -1$*, then* $\mathrm{Int}(T) = \partial\mathrm{Den}(T) = 0$.

Proof. If $\mathrm{v}(T) < -1$, then $\partial\mathrm{Den}(T) = 0$ since $\mathrm{v}(S_k) \geq -1$. If $\mathrm{v}(T) = -1$, then T is of the form $\mathrm{Diag}(\mathcal{H}, (u_3(-\pi_0)^c))$. In this case, according to Corollary 5.8, Lemmas A.8 and A.9, we have

$$\alpha(S, T, X) = (1 - X)\alpha(S, (u_3(-\pi_0)^c), q^2X).$$

Similarly, we have

$$\begin{aligned}\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T) &= \beta(\mathcal{H}^{3,1}_{\chi(T)}, \mathcal{H})\alpha(u_3, (u_3(-\pi_0)^c)) \\ &= (1 - q^{-2})\alpha(u_3, (u_3(-\pi_0)^c)).\end{aligned}$$

Hence,

$$\begin{aligned}\partial\mathrm{Den}(T) &= 2\alpha(S, (u_3(-\pi_0)^c), q^2) + \frac{q^2}{1+q}(1 - q^{-2})\alpha((u_3), (u_3(-\pi_0)^c)) \\ &= 2(1 - q) + 2(q - 1) \\ &= 0,\end{aligned}$$

according to Lemma 7.1. □

8.2. $\partial\mathrm{Den}(T)$ **for** $T = \mathrm{Diag}(1, T_2)$ **with** T_2 **diagonal.** In this subsection, we assume $T = \mathrm{Diag}(1, T_2)$, where $T_2 = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$ with $0 \leq a \leq b$. Let $u = u_1u_2$. Also, let $S = \mathrm{Diag}(1, 1, \nu)$ and $S_2 = \mathrm{Diag}(1, \nu)$. We compare $\partial\mathrm{Den}(T)$ and $\partial\mathrm{Den}(T_2)$ in this subsection.

Recall that

$$\partial\mathrm{Den}(T) = 2\frac{\alpha'(S, T)}{\alpha(S, S)} + \frac{q^2}{1+q}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T)}{\alpha(S, S)}.$$

Moreover, according to [Shi20, Theorem 1.3] and [HSY, Theorem 1.1], the analytic side in the case $n = 2$ is

$$\partial\mathrm{Den}(T_2) = 2\frac{\alpha'(S_2, T_2)}{\alpha(S_2, S_2)} - \frac{2q^2}{q^2 - 1}\frac{\alpha(\mathcal{H}, T_2)}{\alpha(S_2, S_2)}.$$

Proposition 8.2.

$$\partial\mathrm{Den}(T) - \partial\mathrm{Den}(T_2) = \begin{cases} 1 + 2\sum_{i=1}^a q^i & \text{if } \chi(T) = 1, \\ 1 & \text{if } \chi(T) = -1. \end{cases}$$

Proof. Proposition 5.9 implies that

$$\alpha(S, T, X) = (1 - X)\alpha(\mathrm{Diag}(-1, S), T_2, q^2X) + q^2(1 + q^{-1}\chi(S))X\alpha(S_2, T_2, X).$$

Hence

$$\alpha'(S, T) = \alpha(\mathrm{Diag}(-1, S), T_2, q^2) + q^2(1 + q^{-1}\chi(S))\alpha'(S_2, T_2). \tag{8.1}$$

According to Lemma A.11, one can check that $\alpha(S, S) = \beta(S, S) = 2q(q^2 - 1)$, and $\alpha(S_2, S_2) = 2(q - \chi(S_2))$. Then

$$\frac{\alpha'(S, T)}{\alpha(S, S)} - \frac{\alpha'(S_2, T_2)}{\alpha(S_2, S_2)} = \frac{\alpha(\mathrm{Diag}(-1, S), T_2, q^2)}{\alpha(S, S)}. \tag{8.2}$$

Hence we just need to check that

$$2\frac{\alpha(\mathrm{Diag}(-1, S), T_2, q^2)}{\alpha(S, S)} + \frac{q^2}{1+q^2}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T)}{\alpha(S, S)} + \frac{2q^2}{q^2 - 1}\frac{\alpha(\mathcal{H}, T_2)}{\alpha(S_2, S_2)} = \begin{cases} 1 + 2\sum_{i=1}^a q^i & \text{if } \chi(T) = 1, \\ 1 & \text{if } \chi(T) = -1. \end{cases}$$

By Proposition 7.3, we may check that

$$2\alpha(\mathrm{Diag}(-1, S), T_2, q^2) = \begin{cases} 2(2q^{a+2} - (q+1)^2)(q - 1) & \text{if } \chi(T) = 1, \\ 2(q - 1)(q^2 - 1) & \text{if } \chi(T) = -1. \end{cases} \tag{8.3}$$

To compute $\frac{q^2}{1+q}\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T)$, we may choose $\mathcal{H}^{3,1}_{\chi(T)} = \mathrm{Diag}(\mathcal{H}, 1)$ when $\chi(T) = 1$. By Corollary 5.8, Proposition 7.4, and a direct calculation, we have

$$\frac{q^2}{1+q}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T)}{\alpha(S,S)} = \frac{1}{2q(q^2-1)} \cdot \begin{cases} (q-1)\alpha(\mathrm{Diag}(-1,1), T_2) + \frac{2q^2}{q-1}\alpha(\mathcal{H}, T_2) & \text{if } \chi(T) = 1 \\ (q-1)\alpha(\mathrm{Diag}(-1,-\delta), T_2) & \text{if } \chi(T) = -1. \end{cases}$$

Combining this with the formulas in [HSY, Theorem 6.1], we have

$$\frac{q^2}{1+q^2}\frac{\alpha(\mathcal{H}^{3,1}_{\chi(T)}, T)}{\alpha(S,S)} + \frac{2q^2}{q^2-1}\frac{\alpha(\mathcal{H}, T_2)}{\alpha(S_2,S_2)} = \frac{1}{q(q^2-1)} \cdot \begin{cases} 4q^{a+2} - q^2 - 2q - 1 & \text{if } \chi(T) = 1 \\ (q^2-1) & \text{if } \chi(T) = -1. \end{cases} \tag{8.4}$$

Now a direct computation combined with (8.3) and (8.4) proves the proposition. □

Corollary 8.3. *Assume L is a Hermitian lattice with Gram matrix T, then*

$$\partial\mathrm{Den}(T) - \partial\mathrm{Den}(T_2) = |\{\mathcal{V}^0(L)\}|. \tag{8.5}$$

Proof. We can write $L = L^\flat \obot \mathcal{O}_F\mathbf{x}$ where $q(\mathbf{x}) = 1$. If $L^\flat$ is non-split, then $|\{\mathcal{V}^0(L)\}| = 1$.

If $L^\flat$ is split, then $|\{\mathcal{V}^0(L)\}| = 1 + 2\sum_{i=1}^a q^i$ since $\mathcal{L}_3(L)$ can be identified with $\mathcal{L}_{2,1}(L^\flat)$, which is a ball in $\mathcal{L}_{2,1}$ centered at a vertex lattice of type 0 with radius a (see [HSY] for more detail). Here $\mathcal{L}_{2,1}$ is the Bruhat-Tits tree associated with $\mathcal{N}^{\mathrm{Kra}}_{2,1}$ and $\mathcal{L}_{2,1}(L^\flat)$ is the subtree of $\mathcal{L}_{2,1}$ associated with $L^\flat$. □

8.3. $\partial\mathrm{Den}(T)^{(2)}$**.** In this subsection, we compute $\partial\mathrm{Den}(T)^{(2)}$ for $T = \mathrm{Diag}(T_2, u_3(-\pi_0)^c)$ where $\mathrm{v}(T_2) > 0$. Recall that by definition, $\partial\mathrm{Den}(T)^{(2)} = \partial\mathrm{Den}(L^\flat \obot \mathcal{O}_F\mathbf{x})^{(2)}$ where the Gram matrix of $L = L^\flat \obot \mathcal{O}_F\mathbf{x}$ is T. We consider two cases separately in Propositions 8.4 and 8.5.

Proposition 8.4. *Let $T = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ where $0 < a \le b \le c$. Then*

$$\partial\mathrm{Den}(T)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}.$$

Proof. Recall that

$$\partial\mathrm{Den}(T)^{(2)} = \frac{1}{2q(q^2-1)}\left(2\beta'(S,T)^{(2)} + \frac{q^2}{1+q}\beta(\mathcal{H}^{3,1}_{\chi(T)}, T)^{(2)}\right).$$

We compute $\beta'(S,T)^{(2)}$ first. According to Proposition 5.10, $\beta_0(S, T_2, X) = 0$ and

$$\begin{aligned}\beta(S,T,X)^{(2)} &= \beta_2(S,T_2,X)\alpha(\mathrm{Diag}(S,-T_2), u_3(-\pi_0)^c, q^4X) + q\beta_1(S,T_2,X)\alpha(\mathrm{Diag}(-\nu,-T_2), u_3(-\pi_0)^c, q^2X) \\ &= (1-X)(1-q^2X)\alpha(\mathrm{Diag}(S,-T_2), u_3(-\pi_0)^c, q^4X) \\ &\quad + (q+1)(q^2-1)X(1-X)\alpha(\mathrm{Diag}(-\nu,-T_2), u_3(-\pi_0)^c, q^2X).\end{aligned}$$

According to Lemma 7.1,

$$\begin{aligned}&\alpha(\mathrm{Diag}(S,-T_2), u_3(-\pi_0)^c, q^4X) \\ &= 1 + \chi(S)\chi(u_1)(q-1)\sum_{s=a+1}^{b} q^{-3s+a+1}(q^4X)^s + \chi(u_1u_2u_3\nu)q^{-4(c+1)+a+b+2}(q^4X)^{c+1},\end{aligned}$$

and

$$\begin{aligned}&\alpha(\mathrm{Diag}(-\nu,-T_2), u_3(-\pi_0)^c, q^2X) \\ &= 1 + \chi(S)\chi(u_1)(q-1)\sum_{s=a+1}^{b} q^{-s+a}(q^2X)^s + \chi(u_1u_2u_3\nu)q^{-2(c+1)+a+b+1}(q^2X)^{c+1}.\end{aligned}$$

The relation $\chi(u_1u_2u_3\nu) = \chi(S)\chi(T) = -1$ and a direct calculation show that

$$\beta'(S,T_2)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}.$$

Finally, $\beta(\mathcal{H}^{3,1}_{\chi(T)}, T)^{(2)} = 0$ by Proposition 5.10. The proposition is proved. □

Proposition 8.5. *Let $T = \mathrm{Diag}(\mathcal{H}_a, u_3(-\pi_0)^c)$ where a is a positive odd integer and $c \ge 0$. Then*

$$\partial\mathrm{Den}(T)^{(2)} = \begin{cases} (1 - q^a) & \text{if } a \le 2c, \\ (1 - q^{2c+1}) & \text{if } a > 2c. \end{cases}$$

Proof. Recall that

$$\partial\mathrm{Den}(T)^{(2)} = \frac{1}{2q(q^2-1)}\left(2\beta'(S,T)^{(2)} + \frac{q^2}{1+q}\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}\right).$$

We need to compute $\beta'(S,T)^{(2)}$ and $\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}$.

According to Proposition 5.10, $\beta_0(S,T_2,X)=0$ and

$$\begin{aligned}&\beta(S,T,X)^{(2)}\\&=\beta_2(S,\mathcal{H}_a,X)\alpha(\mathrm{Diag}(S,\mathcal{H}_a),u_3(-\pi_0)^c,q^4X)+q\beta_1(S,\mathcal{H}_a,X)\alpha(\mathrm{Diag}(-\nu,\mathcal{H}_a),u_3(-\pi_0)^c,q^2X)\\&=(1-X)\left((1-q^2X)\alpha(\mathrm{Diag}(S,\mathcal{H}_a),u_3(-\pi_0)^c,q^4X)+(q+1)(q^2-1)X\alpha(\mathrm{Diag}(-\nu,\mathcal{H}_a),u_3(-\pi_0)^c,q^2X)\right).\end{aligned}$$

According to Lemma 7.2,

$$\alpha(\mathrm{Diag}(S,\mathcal{H}_a),u_3(-\pi_0)^c,q^4X)=\begin{cases}1+\chi(S)\chi(u_3)q^{2+a}X^{c+1} & \text{if } a\le 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+3}X^{c+1} & \text{if } a>2c,\end{cases}$$

and

$$\alpha(\mathrm{Diag}(-\nu,\mathcal{H}_a),u_3(-\pi_0)^c,q^2X)=\begin{cases}1+\chi(S)\chi(u_3)q^{1+a}X^{c+1} & \text{if } a\le 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+2}X^{c+1} & \text{if } a>2c.\end{cases}$$

A short computation shows that

$$\beta'(S,T)^{(2)}=q(q^2-1)\cdot\begin{cases}1+\chi(S)\chi(u_3)q^{a} & \text{if } a\le 2c,\\ 1+\chi(S)\chi(u_3)q^{2c+1} & \text{if } a>2c.\end{cases}$$

Notice that $\chi(S)\chi(u_3)=\chi(S)\chi(T)=-1$. Finally, $\beta(\mathcal{H}^{3,1}_{\chi(T)},T)^{(2)}=0$ by Proposition 5.10. The proposition is proved. ∎

Part 3. The dimension three case

9. Reduced locus of special cycle

9.1. **The Bruhat-Tits building for $n=3$.** From now on we assume $n=3$ and $\mathcal{L}=\mathcal{L}_3$ as in Section 2.4.

Lemma 9.1. *(1) For every $\Lambda_2\in\mathcal{V}^2$, $\mathcal{N}_{\Lambda_2}$ is isomorphic to the projective line $\mathbb{P}^1$ over k. Its $q+1$ rational points correspond to all $\Lambda_0\in\mathcal{V}^0$ contained in Λ_2.*

(2) Every $\Lambda_0\in\mathcal{V}^0$ is contained in $q+1$ type 2 lattices. In other words, there are $q+1$ projective lines in $(\mathcal{N}_3^{\mathrm{Pap}})_{red}$ passing through the superspecial point $\mathcal{N}_{\Lambda_0}(k)$. Moreover

$$\bigcap_{\Lambda_2\in\mathcal{V}_2,\,\Lambda_0\subset\Lambda_2}\Lambda_2^\sharp=\pi\Lambda_0. \tag{9.1}$$

Proof. Suppose $z\in\mathcal{N}(k)$ and $M:=M(z)\subset N$ is defined as in Proposition 2.9. Since $n=3$, by [RTW14, Proposition 4.1] we have $\Lambda(M)\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}=M+\tau(M)$.

Proof of (1): Suppose $z\in\mathcal{N}_{\Lambda_2}(k)$, i.e. $M\subset\Lambda_2$.

If $M=\tau(M)$, then $M=\Lambda_0\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}$ for some $\Lambda_0\in\mathcal{V}^0$ contained in Λ_2.

If $M\neq\tau(M)$, then by taking the dual of $M\subset\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}$ we have the following sequence of inclusions

$$(\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F})^\sharp\overset{1}{\subset}M\overset{1}{\subset}M+\tau(M)=\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}. \tag{9.2}$$

In both cases the class of M in $\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F}/(\Lambda_2\otimes_{\mathcal{O}_F}\mathcal{O}_{\breve F})^\sharp\cong k^2$ is a line. This finishes the proof of (1).

Proof of (2): For each $\Lambda_0\in\mathcal{V}^0$ we just need to count the number of lattices $\Lambda_2\in\mathcal{V}^2$ that contains Λ_0. We have the following sequence of inclusions

$$\pi\Lambda_0\overset{2}{\subset}\Lambda_2^\sharp\overset{1}{\subset}\Lambda_0\overset{1}{\subset}\Lambda_2.$$

With respect to the quadratic form $(,)\pmod{\pi}$ on $\Lambda_0/\pi\Lambda_0$, the dual lattice $\Lambda_2^\sharp$ corresponds to the 2-dimensional subspaces $U:=\Lambda_2^\sharp/\pi\Lambda_0$ in $\Lambda_0/\pi\Lambda_0$ such that $U^\perp\overset{1}{\subset}U$. So we just need to count the number of isotropic lines $U^\perp$. Assume that $\{e_1, e_2, e_3\}$ is a basis of $\Lambda_0/\pi\Lambda_0$ whose Gram matrix with respect to the quadratic form $(,)_{\mathbb{X}}$ (mod π) is

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & \epsilon \end{pmatrix}.$$

It is easy to see that the isotropic lines are $\mathrm{Span}\{e_1\}$, $\mathrm{Span}\{e_2\}$ and $\mathrm{Span}\{e_1 - \frac{\epsilon a^2}{2} e_2 + a e_3\}$ ($a \in \mathbb{F}_q^\times$). Finally, equation (9.1) can be checked directly using this basis. □

It is well-known that $\mathcal{L}$ is a tree, see for example [Bro89, Section 3 of Chapter VI]. More specifically, the vertices of $\mathcal{L}$ correspond to vertex lattices of type 2 or 0. There is an edge between $\Lambda \in \mathcal{V}^2$ and $\Lambda_0 \in \mathcal{V}^0$ if $\Lambda_0 \subset \Lambda$. We give each edge length $\frac{1}{2}$. This defines a metric $d(,)$ on $\mathcal{L}$. Recall that we have defined $\mathcal{L}(L)$ in (2.6). Then the boundary of $\mathcal{L}(L)$ is the set

$$\mathcal{B}(L) = \{\Lambda \in \mathcal{V}^0(L) \mid \exists \Lambda_2 \in \mathcal{V}^2 \text{ such that } \Lambda \subset \Lambda_2, \Lambda_2 \notin \mathcal{L}(L)\}. \tag{9.3}$$

Recall we have the isomorphism $b : \mathbb{V} \to C$ defined in (2.3). Recall from [KR14a] or [HSY] that the vertices of $\mathcal{L}_{2,1}$ correspond to vertex lattices of type 2, and an edge corresponds to a vertex lattice of type 0. Each vertex of $\mathcal{L}_{2,1}$ is contained in $q+1$ edges and each edge connects exactly two vertices. For $\mathbf{x} \in \mathbb{V}$ with $\mathrm{v}(\mathbf{x}) = 0$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ split, let $\mathcal{L}_{2,1}(\mathbf{x})$ be the Bruhat-Tits tree of $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}_{2,1}^{\mathrm{Pap}}$. We have a natural embedding $\mathcal{L}_{2,1}(\mathbf{x}) \hookrightarrow \mathcal{L}$ defined as follows. First we send each vertex of $\mathcal{L}_{2,1}(\mathbf{x})$ corresponding to a vertex lattice $\Lambda \subset \mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ of type 2 to the vertex of $\mathcal{L}$ corresponding to the type 2 lattice $\Lambda \oplus \mathrm{Span}\{b(\mathbf{x})\}$. An edge corresponding to a type zero lattice $\Lambda_0 \subset \mathrm{Span}_F\{b(\mathbf{x})\}^\perp$ is broken into two pieces evenly and sent to the union of the two edges in $\mathcal{L}$ joining the two vertices corresponding to $\Lambda \oplus \mathrm{Span}\{b(\mathbf{x})\}$ and $\Lambda' \oplus \mathrm{Span}\{b(\mathbf{x})\}$ where Λ and Λ' are the two type 2 lattices containing Λ_0.

9.2. **Rank 1 case.**

Lemma 9.2. *A point $z \in \mathcal{N}_3^{\mathrm{Pap}}(k)$ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $b_{\mathbf{x}} \in M(z)$.*

(1) Assume $\Lambda_0 \in \mathcal{V}^0$, then the superspecial point $\mathcal{N}_{\Lambda_0}(k)$ is in $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $b_{\mathbf{x}} \in \Lambda_0$.

(2) Assume $\Lambda_2 \in \mathcal{V}^2$, then

$$\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k) \cap \mathcal{N}_{\Lambda_2}(k) = \begin{cases} \mathcal{N}_{\Lambda_2}(k) & \text{if } b_{\mathbf{x}} \in \Lambda_2^\sharp, \\ \text{a superspecial point in } \mathcal{N}_{\Lambda_2}(k) & \text{if } b_{\mathbf{x}} \in \Lambda_2 \backslash \Lambda_2^\sharp, \\ \emptyset & \text{if } b_{\mathbf{x}} \notin \Lambda_2. \end{cases}$$

Proof. By Dieudonné theory, $z \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ if and only if $\mathbf{x}(M(\mathbb{Y})) \subset M(z)$ if and only if $b_{\mathbf{x}} \in M(z)$ since e is a generator of $M(\mathbb{Y})$. For $z = \mathcal{N}_{\Lambda_0}(k)$ where $\Lambda_0 \in \mathcal{V}^0$, we have $M(z) = \Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$. Hence (1) immediately follows.

Now we proceed to prove (2). If $b_{\mathbf{x}} \in \Lambda^\sharp$, then (9.2) tells us that $z \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$ for any $z \in \mathcal{N}_{\Lambda_2}^\circ(k)$. The fact that $\Lambda_2^\sharp \subset \Lambda_0$ for any $\Lambda_0 \in \mathcal{L}^0$ contained in Λ_2 implies that $\mathcal{N}_{\Lambda_0}(k) \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k)$. So $\mathcal{N}_{\Lambda_2}(k) \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$.

If $b_{\mathbf{x}} \in \Lambda_2 \backslash \Lambda_2^\sharp$, then $\Lambda_0 := \Lambda^\sharp + \mathrm{Span}\{b_{\mathbf{x}}\}$ is a type 0 lattice contained in Λ_2 and $\mathcal{N}_{\Lambda_0}(k) \in \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$. On the other hand, since $\tau(\Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}) = \Lambda_0 \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$, equation (9.2) tells us that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})$ does not contain any point in $\mathcal{N}_{\Lambda_2}^\circ(k)$.

If $b_{\mathbf{x}} \notin \Lambda_2$, then $b_{\mathbf{x}} \notin M(z)$ for any $z \in \mathcal{N}_{\Lambda_2}(k)$, hence $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})(k) \cap \mathcal{N}_{\Lambda_2}(k) = \emptyset$. □

Corollary 9.3. *Let $L \subset \mathbb{V}$. Assume $z \in \mathcal{Z}^{\mathrm{Pap}}(L)(k)$ and $z \in \mathcal{N}_\Lambda(k)$ where $\Lambda \in \mathcal{V}^2$. Then $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\pi L)$.*

Corollary 9.4. *Assume $\mathbf{x} \in \mathbb{V}$ and $\mathrm{v}(\mathbf{x}) > 0$. Assume $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{red}$ where $\Lambda \in \mathcal{V}^2$, then either $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\frac{1}{\pi}\mathbf{x})_{red}$ or $\mathcal{N}_\Lambda \cap \mathcal{Z}^{\mathrm{Pap}}(\frac{1}{\pi}\mathbf{x})_{red}$ is a unique superspecial point.*

Lemma 9.5. *For $L \subset \mathbb{V}$ a lattice of arbitrary rank, $\mathcal{Z}^{\mathrm{Pap}}(L)_{red}$ is connected.*

Proof. Suppose U_1 and U_2 are two different connected components of $\mathcal{Z}^{\mathrm{Pap}}(L)_{red}$. Since $\mathrm{SU}(\mathbb{V})$ acts transitively on $\mathcal{L}$, we can find a $\mathbf{x} \in \mathbb{V}$ such that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}_{2,1}^{\mathrm{Pap}}$ (i.e. $\{\mathbf{x}\}^\perp$ is split) and $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cap U_i \neq \emptyset$ for $i = 1, 2$. Hence the reduced locus of

$$\mathcal{Z}^{\mathrm{Pap}}(L \oplus \mathrm{Span}\{\mathbf{x}\}) \cong \mathcal{Z}_{2,1}^{\mathrm{Pap}}(L')$$

is not connected where L' is the orthogonal projection of L onto $\{\mathbf{x}\}^\perp$. This contradicts Corollaries 3.13, 3.15 and Lemma 3.16 of [HSY]. □

Recall that for a lattice $L \subset \mathbb{V}$ (resp. $\mathbf{x} \in \mathbb{V}$), we have defined $\mathcal{V}(L)$ and $\mathcal{L}(L)$ (resp. $\mathcal{V}(\mathbf{x})$ and $\mathcal{L}(\mathbf{x})$) in Section 2.4.

Proposition 9.6. *Assume that $\mathbf{x} \in \mathbb{V}$ such that $h(\mathbf{x}, \mathbf{x}) \neq 0$. Then we have*

$$\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x})_{red} = \bigcup_{\Lambda \in \mathcal{V}(\mathbf{x})} \mathcal{N}_\Lambda,$$

where $\mathcal{V}(\mathbf{x})$ is given as follows.

1. *When $\mathrm{v}(\mathbf{x}) = 0$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is non-split, there is a unique vertex lattice $\Lambda_{\mathbf{x}} \in \mathcal{V}^0$ containing $b_{\mathbf{x}}$. In this case $\mathcal{V}(\mathbf{x}) = \{\Lambda_{\mathbf{x}}\}$.*
2. *When $\mathrm{v}(\mathbf{x}) = d$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is non-split, we have*
$$\mathcal{V}(\mathbf{x}) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d\}$$
where $\Lambda_{\mathbf{x}/\pi^d}$ is as in (1).
3. *When $\mathrm{v}(\mathbf{x}) = 0$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is split, $\mathcal{L}(\mathbf{x})$ is the tree $\mathcal{L}_{2,1}(\mathbf{x})$ described after (9.3).*
4. *When $\mathrm{v}(\mathbf{x}) = d$ and $\mathrm{Span}_F\{\mathbf{x}\}^\perp$ is split, we have*
$$\mathcal{V}(\mathbf{x}) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\mathbf{x}/\pi^d)) \leq d\}$$
where $\mathcal{L}(\mathbf{x}/\pi^d)$ is as in (3).
5. *When $h(\mathbf{x}, \mathbf{x}) \notin \mathcal{O}_{F_0}$, $\mathcal{V}(\mathbf{x})$ is empty.*

Proof. Proof of (1): This is a direct consequence of Proposition 2.6 and the fact that $\mathcal{N}_{2,-1}^{\mathrm{Pap}}$ has only one reduced point, see [Shi20, Section 2] or [RSZ18, Section 8]. Alternatively since $\mathrm{Span}_F\{b_{\mathbf{x}}\}^\perp$ is non-split of dimension 2, it contains a unique self dual lattice Λ', then $\Lambda_{\mathbf{x}} := \mathrm{Span}\{b_{\mathbf{x}}\} \oplus \Lambda'$ is the unique type 0 lattice containing $b_{\mathbf{x}}$.

Proof of (3): Applying Proposition 2.6,we see that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}) \cong \mathcal{N}_{2,1}^{\mathrm{Pap}}$ is the Drinfeld p-adic half space, see [KR14a] and [HSY]. The required properties of $\mathcal{L}(\mathbf{x})$ and $\mathcal{V}(\mathbf{x})$ follow.

Proof of (2): We prove this by induction. The case $d = 0$ is just (1). Now we assume $d > 0$ and that the statement holds for $d - 1$, i.e.

$$\mathcal{V}(\mathbf{x}/\pi) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d - 1\}.$$

Then applying Corollary 9.3 to the lattice $L = \mathrm{Span}\{\mathbf{x}/\pi\}$ we have

$$\bigcup_{\Lambda \in \mathcal{V}^2,\ d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d} \mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}).$$

Corollary 9.4 and the induction hypothesis imply that every $\Lambda_2 \in \mathcal{V}^2(\mathbf{x})$ satisfies $d(\Lambda, \Lambda_{\mathbf{x}/\pi^d}) \leq d$. By Lemma 9.5 there is no isolated $\Lambda_0 \in \mathcal{V}^0(\mathbf{x})$, i.e. every $\Lambda_0 \in \mathcal{V}^0(\mathbf{x})$ is contained in some $\Lambda_2 \in \mathcal{V}^2(\mathbf{x})$ if $\mathrm{v}(\mathbf{x}) > 0$. This finishes the proof of (2).

Similarly we can prove (4) by an induction on d, the case $d = 0$ is just (3).

(5) follows directly from Lemma 9.2. □

9.3. **Rank 2 case.**

Proposition 9.7. *Assume that $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\} \subset \mathbb{V}$ is integral of rank 2. Then*

$$\mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{red} = \bigcup_{\Lambda \in \mathcal{V}(L^\flat)} \mathcal{N}_\Lambda$$

is a finite union, where $\mathcal{V}(L^\flat)$ is the set of vertices of the tree $\mathcal{L}(L^\flat)$ described as follows.

1. *Assume $L^\flat \approx \mathcal{H}_{2a+1}$ for some $a \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{L}(L^\flat)$ is a ball centered at a vertex lattice of type 2 with radius $\frac{2a+1}{2}$.*
2. *Assume $L^\flat = \mathrm{Span}\{\pi^a \mathbf{x}_1, \pi^a \mathbf{x}_2\}$ where $\mathrm{v}(\mathbf{x}_1) = 0$, $\mathrm{v}(\mathbf{x}_2) \geq 0$ and $\mathrm{Span}_F\{\mathbf{x}_1\}^\perp$ is nonsplit. Then $\mathcal{L}(L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius a.*

(3) *Assume* $L^\flat = \mathrm{Span}\{\pi^a \mathbf{x}_1, \pi^{a+r}\mathbf{x}_2\}$ *where* $\mathbf{x}_1 \perp \mathbf{x}_2$, $\mathrm{v}(\mathbf{x}_1) = \mathrm{v}(\mathbf{x}_2) = 0$, $r \geq 0$ *and* $\mathrm{Span}_F\{\mathbf{x}_1\}^\perp$ *is split. Then*

$$\mathcal{L}(L^\flat) = \{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a\},$$

where

$$\mathcal{L}(\pi^{-a}L^\flat) = \{\Lambda \in \mathcal{L}(\mathbf{x}_1) \mid d(\Lambda, \Lambda_0) \leq r\},$$

$\mathcal{L}(\mathbf{x}_1)$ *is described in (3) of Proposition 9.6 and* Λ_0 *is the unique type* 0 *vertex lattice containing* $\{x_1, x_2\}$.

Proof. As in the proof of Proposition 9.2, for a $\Lambda \in \mathcal{V}$, $\mathcal{N}_\Lambda \subset \mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{red}$ if and only if $\Lambda^\sharp$ contains $b_{\mathbf{x}_1}, b_{\mathbf{x}_2}$.

We first prove (1) when $a = 0$. Suppose $\Lambda \in \mathcal{V}^2(L^\flat)$. Extend $\{b_{\mathbf{x}_1}, b_{\mathbf{x}_2}\}$ to a basis $\{b_{\mathbf{x}_1}, b_{\mathbf{x}_2}, b_3\}$ of $\mathbb{V}$ with Gram matrix $\mathcal{H}_1 \oplus \{-\epsilon\}$. Choose a basis $\{v_1, v_2, v_3\}$ of $\Lambda^\sharp$ with the same Gram matrix $\mathcal{H}_1 \oplus \{-\epsilon\}$. Then $b_{\mathbf{x}_i} \in \Lambda^\sharp$ $(i = 1, 2)$ by Lemma 9.2 and

$$b_{\mathbf{x}_i} = a_{i1}v_1 + a_{i2}v_2 + a_{i3}v_3$$

where $a_{ij} \in \mathcal{O}_F$ $(j = 1, 2, 3)$. The fact that $(b_{\mathbf{x}_i}, b_{\mathbf{x}_j})_{1\leq i,j\leq 2} = T$ implies $a_{i3} \in \pi\mathcal{O}_F$ for $i = 1, 2$ and $(a_{ij})_{1\leq i,j\leq 2}$ is in $\mathrm{GL}_2(\mathcal{O}_F)$. This guarantees that $L^\flat$ is a direct summand of $\Lambda^\sharp$ by Gram-Schmit process. Hence $\Lambda^\sharp$ is in fact the lattice $\mathrm{Span}_{\mathcal{O}_F}\{b_{\mathbf{x}_1}, b_{\mathbf{x}_2}, b_3\}$. The fact that all $\Lambda_0 \in \mathcal{V}^0(L^\flat)$ are in Λ follows from Lemma 9.5.

When $a = 0$, (2) follows from the fact that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1) = \mathcal{N}^{\mathrm{Pap}}_{2,-1}$ (by Proposition 2.6) and $\mathcal{Z}^{\mathrm{Pap}}(L^\flat)_{red} = \mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1)_{red}$ is a unique superspecial point. Similarly, (3) follows from the fact that $\mathcal{Z}^{\mathrm{Pap}}(\mathbf{x}_1) = \mathcal{N}^{\mathrm{Pap}}_{2,1}$ and [HSY, Corollary 3.13].

Now we prove (1), (2) and (3) for general a. First of all, $\mathcal{L}(L^\flat) = \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2)$ by definition. By Corollary 9.3 we have

$$\{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a\} \subset \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2).$$

Notice that for a sub-tree $\mathcal{L}'$ of a tree $\mathcal{L}$ and a vertex $x \in \mathcal{L} \setminus \mathcal{L}'$, there is a unique geodesic segment joining x with $\mathcal{L}'$. Given $\Lambda \in \mathcal{L}(L^\flat) = \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2)$, let γ be the unique geodesic segment joining Λ with $\mathcal{L}(\pi^{-a}L^\flat)$. Assume that γ intersects $\mathcal{L}(\pi^{-a}L^\flat)$ at $\Lambda(L^\flat)$. Since $\mathcal{L}(\pi^{-a}L^\flat) = \mathcal{L}(\mathbf{x}_1) \cap \mathcal{L}(\mathbf{x}_2)$, γ necessarily intersects both $\mathcal{L}(\mathbf{x}_1)$ and $\mathcal{L}(\mathbf{x}_2)$. Without loss of generality we assume that γ intersects $\mathcal{L}(\mathbf{x}_1)$ at $\Lambda(\mathbf{x}_1)$ first. Hence the intersection of γ with $\mathcal{L}(\mathbf{x}_2)$ is $\Lambda(L^\flat)$ and

$$d(\Lambda, \Lambda(\mathbf{x}_1)) = d(\Lambda, \mathcal{L}(\mathbf{x}_1)) \leq d(\Lambda, \Lambda(L^\flat)) = d(\Lambda, \mathcal{L}(\mathbf{x}_2)).$$

Now by Proposition 9.6, we have

$$d(\Lambda, \mathcal{L}(\mathbf{x}_1)) \leq a, \ d(\Lambda, \mathcal{L}(\mathbf{x}_2)) \leq a.$$

Hence $d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a$. This shows that

$$\{\Lambda \in \mathcal{V} \mid d(\Lambda, \mathcal{L}(\pi^{-a}L^\flat)) \leq a\} = \mathcal{L}(\pi^a\mathbf{x}_1) \cap \mathcal{L}(\pi^a\mathbf{x}_2).$$

The general case of (1), (2) and (3) follows from the above equation and the case $a = 0$.

Notice that (1), (2) and (3) have covered all possibilities of $L^\flat$ due to the classification of Hermitian lattices. Notice that in every case $\mathcal{V}(L^\flat)$ is finite. This finishes the proof of the proposition. □

Definition 9.8. Assume that $L^\flat$ is an integral lattice of rank 2 in $\mathbb{V}$. Define $\mathcal{S}(L^\flat)$, the skeleton of $\mathcal{L}(L^\flat)$, as follows. If the fundamental invariant of $L^\flat$ is $(2a, b)$ $(b \geq 2a)$, define $\mathcal{S}(L^\flat) := \mathcal{L}(\pi^{-a}L^\flat)$. If the fundamental invariant of $L^\flat$ is $(2a+1, 2a+1)$, define $\mathcal{S}(L^\flat) := \emptyset$.

Remark 9.9. *The skeleton* $\mathcal{S}(L^\flat)$ *is isomorphic to a ball in the Bruhat-Tits tree of* $\mathcal{N}^{\mathrm{Pap}}_{2,\pm 1}$.

Corollary 9.10. *For each* $\Lambda_2 \in \mathcal{V}^2(L^\flat)$ *not on the skeleton* $\mathcal{S}(L^\flat)$, *one can find* $\Lambda_0 \in \mathcal{V}^0(L^\flat)$ *such that* Λ_2 *has the largest distance to the boundary* $\mathcal{B}(L^\flat)$ *of* $\mathcal{L}(L^\flat)$ *among all type* 2 *lattices in* $\mathcal{V}^2(L^\flat)$ *containing* Λ_0.

Proof. Assmue the fundamental invariant of $L^\flat$ is $(2a, b)$ or $(2a+1, 2a+1)$. Define $M^\flat := \pi^{-a}L^\flat$. Let b be the unique integer such that $\Lambda_2 \in \mathcal{L}(\pi^b M^\flat) \setminus \mathcal{L}(\pi^{b-1}M^\flat)$. Choose any $\Lambda_0 \in \mathcal{B}(\pi^b M^\flat)$ such that $\Lambda_0 \subset \Lambda$. Then by Proposition 9.7, Λ_0 satisfies the assumption of the corollary. □

9.4. **The Krämer model.** For $\Lambda \in \mathcal{V}^2$, let $\tilde{\mathcal{N}}_\Lambda$ be the strict transform of $\mathcal{N}_\Lambda$ under the blow-up $\mathcal{N}^{\mathrm{Kra}} \to \mathcal{N}^{\mathrm{Pap}}$. Since the strict transform of a regular curve along any of its closed point is an isomorphism, we know $\tilde{\mathcal{N}}_\Lambda \cong \mathbb{P}^1$.

Lemma 9.11. *For $\Lambda \neq \Lambda' \in \mathcal{V}^2$, $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect.*

Proof. If $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\Lambda'}$ do not intersect in $\mathcal{N}^{\mathrm{Pap}}$, then obviously $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect. Without loss of generality we can assume $\Lambda = \mathrm{Span}\{e_1, e_2, e_3\}$ and $\Lambda' = \mathrm{Span}\{\pi^{-1}e_1, \pi e_2, e_3\}$ where the Gram matrix of $\{e_1, e_2, e_3\}$ is $\mathrm{Diag}(\mathcal{H}, \epsilon)$. Take $\mathbf{x}_0 = e_3$. Then by Proposition 9.7, both $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ are in $\tilde{\mathcal{Z}}(\mathbf{x}_0) \cong \mathcal{N}_{2,1}^{\mathrm{Kra}}$. Now by [HSY, Lemma 5.3], $\tilde{\mathcal{N}}_\Lambda$ and $\tilde{\mathcal{N}}_{\Lambda'}$ do not intersect. □

Lemma 9.12. *Let $\Lambda \in \mathcal{V}^2$ and $\Lambda_0 \in \mathcal{V}^0$. When $\Lambda_0 \subset \Lambda$, $\tilde{\mathcal{N}}_\Lambda$ intersects properly with Exc_{Λ_0} and*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) = 1. \tag{9.4}$$

When Λ_0 is not contained in Λ, $\tilde{\mathcal{N}}_\Lambda$ does not intersect with Exc_{Λ_0}.

Proof. First assume $\Lambda_0 \subset \Lambda$. Since $\tilde{\mathcal{N}}_\Lambda$ is a strict transformation of a curve, it intersects the exceptional divisor properly. Let $\mathbf{x}_0$ be as in the proof of Lemma 9.11. Then $\tilde{\mathcal{N}}_\Lambda$ is in $\tilde{\mathcal{Z}}(\mathbf{x}_0) \cong \mathcal{N}_{2,1}^{\mathrm{Kra}}$.

$$\begin{aligned}\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) &= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)} \otimes \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}) \\ &= \chi(\tilde{\mathcal{Z}}(\mathbf{x}_0), \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\mathrm{Exc}'}).\end{aligned}$$

Here $\mathrm{Exc}' \cong \mathbb{P}^1/k$ is the exceptional divisor on $\tilde{\mathcal{Z}}(\mathbf{x}_0)$ corresponding to the rank 2 self-dual lattice

$$\Lambda' = \{v \in \Lambda_0 \mid v \perp \mathbf{x}_0\}.$$

By [HSY, Lemma 5.2(a)], we know $\chi(\tilde{\mathcal{Z}}(\mathbf{x}_0), \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes_{\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_0)}} \mathcal{O}_{\mathrm{Exc}'}) = 1$. When Λ_0 is not contained in Λ, the superspecial point $\mathcal{N}_{\Lambda_0}(k)$ is not contained in $\mathcal{N}_\Lambda$, hence $\tilde{\mathcal{N}}_\Lambda$ does not intersect with Exc_{Λ_0}. □

10. Intersection of vertical components and special divisors

In this section we study the intersection of $\tilde{\mathcal{N}}_\Lambda$ and special divisors. The main result is Theorem 10.2. To proceed we first study the decomposition of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ when $\mathrm{v}(L^\flat) = 0$. Since n is odd, we can without loss of generality assume that $\chi(\mathbb{V}) = \chi(C) = 1$. In the rest of the paper, we identify $\mathbb{V}$ with C by the isomorphism b defined in (2.3).

10.1. **Decomposition of ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$.** Let $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{V}$ are linearly independent and the Hermitian form restricted to L is non-degenerate.

Lemma 10.1. ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}] \in K_0(\mathcal{N}^{\mathrm{Kra}})$ *is in fact in* $\mathrm{F}^2 K_0(\mathcal{N}^{\mathrm{Kra}})$*. Moreover we have the decomposition*

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) = \mathcal{Z}^{\mathrm{Kra}}(L)_h + {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)_v. \tag{10.1}$$

where $\mathcal{Z}^{\mathrm{Kra}}(L)_h$ is described in Theorem 4.1 and ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L)_v \in F^2 K_0^{\mathcal{Z}^{\mathrm{Kra}}(L)_v}(\mathcal{N}^{\mathrm{Kra}})$.

Proof. By Lemma 2.10, $\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ is noetherian and has a decomposition

$$\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h \cup \mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v.$$

Expressing $\mathcal{Z}(\mathbf{x}_i)$ $(i = 1, 2)$ as in (3.1) and applying Propositions 3.2, 3.3 and Lemma 3.4, we have

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_2)}] + \sum_{\Lambda_0 \in \mathcal{V}^0(L^\flat)} (2m_{\Lambda_0}(\mathbf{x}_1)m_{\Lambda_0}(\mathbf{x}_2) + m_{\Lambda_0}(\mathbf{x}_1) + m_{\Lambda_0}(\mathbf{x}_2))H_{\Lambda_0}.$$

$\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_h$ is contained in $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \tilde{\mathcal{Z}}(\mathbf{x}_2)$ and has dimension 1 by Theorem 4.1. $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \tilde{\mathcal{Z}}(\mathbf{x}_2)_v$ also has dimension 1 as it is supported on the reduced locus of $\mathcal{N}^{\mathrm{Kra}}$ by Lemma 2.11 and does not contain any exceptional divisor Exc_{Λ_0}. Hence

$$[\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_2)}] = [\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \tilde{\mathcal{Z}}(\mathbf{x}_2)}] \in \mathrm{F}^2 K_0(\mathcal{N}^{\mathrm{Kra}}), \tag{10.2}$$

see for example [Zha21, Lemma B.2]. Hence we know that ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v \in \mathrm{F}^2 K_0(\mathcal{N}^{\mathrm{Kra}})$ and we have the desired decomposition. □

By Lemma 10.1 and (2.9) we know that ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v \in K_0'(Y)$ where we can take Y to be the reduced locus of $\mathcal{N}^{\mathrm{Kra}}$. By the Bruhat-Tits stratification of $\mathcal{N}^{\mathrm{Kra}}$ and the fact that $\mathrm{Gr}^1 K_0^{\mathrm{Exc}_{\Lambda_0}}(\mathcal{N}^{\mathrm{Kra}}) \cong \mathrm{CH}^1(\mathrm{Exc}_{\Lambda_0})$ is generated by H_{Λ_0}, we have the following decomposition in $\mathrm{Gr}^2 K_0(\mathcal{N}^{\mathrm{Kra}})$:

$$(10.3) \qquad {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v = \sum_{\Lambda_2 \in \mathcal{V}^2(L^\flat)} m(\Lambda_2, L^\flat)[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\Lambda_0 \in \mathcal{V}^0(L^\flat)} m(\Lambda_0, L^\flat) H_{\Lambda_0}.$$

We will determine the multiplicities $m(\Lambda_2, L^\flat)$ and $m(\Lambda_0, L^\flat)$ when $\mathrm{v}(L^\flat) = 0$ in this section and deal with the general case in Section 11.

Now assume that $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ has the Gram matrix $\mathrm{Diag}(u_1, u_2(-\pi_0)^n)$ with $u_1, u_2 \in \mathcal{O}_{F_0}^\times$. Applying Proposition 3.2 to $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1)$, we find

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{x}_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}] + \sum_{\Lambda_0 \in \mathcal{V}^0(\mathbf{x}_1)} [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)}].$$

By Proposition 2.6, we know the intersection $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cap \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_2)$ is proper and is isomorphic to $\mathcal{Z}^{\mathrm{Kra}}_{2,\chi(u_1)}(\mathbf{x}_2)$. Combining this with Corollary 3.5 we obtain

$$(10.4) \qquad {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = i_*({}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}_{2,\chi(u_1)}(\mathbf{x}_2)) - \sum_{\Lambda_0 \in \mathcal{V}^0(L^\flat)} H_{\Lambda_0}.$$

where i_* is the map $\mathrm{Gr}^1 K_0(\mathcal{N}^{\mathrm{Kra}}_{2,\chi(u_1)}) \to \mathrm{Gr}^2 K_0(\mathcal{N}^{\mathrm{Kra}}_{3,1})$ induced by the closed immersion $i : \mathcal{N}^{\mathrm{Kra}}_{2,\chi(u_1)} \to \mathcal{N}^{\mathrm{Kra}}_{3,1}$. Equation (10.4) reduces the problem of decomposing ${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)$ in this case to [Shi20, Theorem 4.5] and [HSY, Theorem 4.1]. We do not make the effort to write the complete result down, but instead look at two basic examples.

Let us begin by the case when $L^\flat$ is unimodular. By (10.4) and either [Shi20, Theorem 4.5] or [HSY, Theorem 4.1], we have

$$(10.5) \qquad {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{Z}}(L^\flat)^o}]$$

in the notation of Theorem 4.1.

Next consider $L^\flat = \mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ with Gram matrix $\mathrm{Diag}(1, -u\pi_0)$ where $u \in \mathcal{O}_{F_0}^\times$. Then $\mathrm{Span}\{\mathbf{x}_1\}^\perp$ is split and $\tilde{\mathcal{Z}}(\mathbf{x}_1) \cong \mathcal{N}^{\mathrm{Kra}}_{2,1}$. By Proposition 9.7 (3), $\mathcal{V}^2(L^\flat)$ consists of two adjacent lattices Λ and Λ'. Moreover by [HSY, Theorem 4.1] and (10.4), we have

$$(10.6) \qquad {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)_v = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda \cap \Lambda'}.$$

10.2. **The intersection number.** Assume $\Lambda \in \mathcal{V}^2$. For $\mathbf{x} \in \mathbb{V} \backslash \{0\}$, define

$$(10.7) \qquad \mathrm{Int}_\Lambda(\mathbf{x}) := \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}).$$

In this subsection we prove the following theorem.

Theorem 10.2. *Let $\Lambda \in \mathcal{V}^2$ and $\mathbf{x} \in \mathbb{V} \backslash \{0\}$. Then*

$$\mathrm{Int}_\Lambda(\mathbf{x}) = 1_\Lambda(\mathbf{x})$$

where 1_Λ is the characteristic function of $\Lambda \subset \mathbb{V}$.

Corollary 10.3. *Assume that $\Lambda_0 \in \mathcal{L}_0$ and $\Lambda \in \mathcal{L}_2$ such that $\Lambda_0 \subset \Lambda$. Then for any $y_0 \in \Lambda_0$ such that $y_0^\perp$ is nonsplit, we have*

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{N}}_\Lambda} \otimes^{\mathbb{L}} \mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}) = 0.$$

Proof. By Proposition 3.2, we know

$$\mathcal{Z}^{\mathrm{Kra}}(y_0) = \tilde{\mathcal{Z}}(y_0) + \mathrm{Exc}_{\Lambda_0}.$$

Now the corollary follows immediately from Theorem 10.2 and Lemma 9.12. □

Proof of Theorem 10.2: We consider three different cases. First if $x \notin \Lambda$ or $\mathrm{v}(\mathbf{x}) < 0$, then by Lemma 9.2, $\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cap \tilde{\mathcal{N}}_\Lambda = \emptyset$ hence $\mathrm{Int}_\Lambda(\mathbf{x}) = 0$. From now on we assume $x \in \Lambda$ and $\mathrm{v}(\mathbf{x}) \geq 0$. Write $\mathbf{x} = \mathbf{x}_0 \pi^n$ with $\mathbf{x}_0 \in \Lambda \backslash \pi\Lambda$ and $n \geq 0$.

Case 1: First we assume $\mathbf{x}_0 \in \Lambda \backslash \Lambda^\sharp$. Choose a basis $\{e_1', e_2', e_3'\}$ of Λ with Gram matrix $\mathcal{H}_1^{3,1}$ such that

$$\mathbf{x}_0 = x e_1' + y e_2' + z e_3'.$$

Then one of x and y is in $\mathcal{O}_F^\times$ as $\Lambda^\sharp = \mathrm{Span}\{\pi e_1', \pi e_2', e_3\}$. Apparently the equation

$$2u - v\bar{v} = h(\mathbf{x}_0, \mathbf{x}_0)$$

has a solution $(u, v) \in \mathcal{O}_{F_0}^2$ with $u \in \mathcal{O}_{F_0}^\times$. Now according to Lemma A.2, we can find a matrix $g \in \mathrm{U}(\mathcal{H}_1^{3,1})(\mathcal{O}_{F_0})$ such that

$$g \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \pi u \\ 1 \\ v \end{pmatrix}.$$

Now replace the basis $\{e_1', e_2', e_3'\}$ by $\{e_1, e_2, e_3\} = \{e_1', e_2', e_3'\}g^{-1}$, we have

$$\mathbf{x}_0 = \pi u e_1 + e_2 + v e_3$$

where $u \in \mathcal{O}_{F_0}^\times$, $v \in \mathcal{O}_F$.

Define

$$f_1 = \frac{1}{\pi} u^{-1} e_2, \ \ f_2 = \pi u e_1, \ \ f_3 = e_3. \tag{10.8}$$

Then $\{f_1, f_2, f_3\}$ has also Gram matrix $\mathcal{H}_1^{3,1}$ and $\Lambda' := \mathrm{Span}\{f_1, f_2, f_3\}$ is a type 2 lattice adjacent to Λ with $\Lambda_c = \Lambda \cap \Lambda' = \mathrm{Span}\{\pi e_1, e_2, e_3\}$ is a type 0 lattice. Now in terms of the basis $\{f_1, f_2, f_3\}$ we have

$$\mathbf{x} = \pi^n(\pi u f_1 + f_2 + v f_3).$$

Define $\theta \in \mathrm{U}(\mathbb{V})$ by taking the basis $\{e_1, e_2, e_3\}$ to $\{f_1, f_2, f_3\}$. Then

$$\theta(\mathbf{x}) = \mathbf{x}, \quad \theta(\Lambda) = \Lambda'.$$

In particular $\theta(\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = \mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})$ and

$$\mathrm{Int}_{\Lambda'}(\mathbf{x}) = \mathrm{Int}_\Lambda(\mathbf{x}). \tag{10.9}$$

Now let

$$\mathbf{y}_0 = e_3, \quad \mathbf{y}_1 = \pi(-\pi u e_1 + e_2),$$

$L^\flat = \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1\}$, and $L = \mathrm{Span}\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{x}\}$. Then by (10.6) and Theorem 4.1 we have

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda_c} + [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}]. \tag{10.10}$$

where $\tilde{\mathcal{Z}}(M^\flat)$ is the quasi canonical-lifting cycle of the lattice

$$M^\flat := \mathrm{Span}\{e_3, -\pi u e_1 + e_2\}.$$

Combining with (10.9), we have

$$\mathrm{Int}(L) = 2\mathrm{Int}_\Lambda(\mathbf{x}) + \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot H_{\Lambda_c}) + \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}]). \tag{10.11}$$

Let $\mathbf{x}' = \pi^n(\pi u e_1 + e_2) = \mathbf{x} - \pi^n v e_3$. Then we have

$$\begin{aligned}
\mathrm{Int}(L) &= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_0) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)) \\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_0) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}') \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)) \\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) \\
&\quad + \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)})
\end{aligned}$$

where we have used linear invariance ([How19, Corollary D]) and Proposition 3.2. Notice that the Gram matrix of $\{\mathbf{x}', \mathbf{y}_1\}$ is $\mathrm{Diag}(2u(-\pi_0)^n, -2u\pi_0)$. By Proposition 2.6 and [HSY, Theorem 1.1],

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = \begin{cases} 1 & \text{if } n = 0, \\ 1 + n - 2q & \text{if } n \geq 1. \end{cases}$$

By Corollary 3.6 and [HSY, Lemma 3.15], we know that

$$\sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = |\mathcal{V}^0(L)| = \begin{cases} 1 & \text{if } n = 0, \\ 2q + 1 & \text{if } n \geq 1. \end{cases}$$

Combining the above two equations we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})\cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) = n+2.$$

On the other hand, by Corollary 3.8,

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0}\cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = -1.$$

By [Gro86, Proposition 3.3]

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}) = n+1.$$

Hence we obtain by (10.11)

$$\mathrm{Int}_\Lambda(\mathbf{x}) = 1. \tag{10.12}$$

Case 2: Now we Assume $\mathbf{x}_0\in\Lambda^\sharp\backslash\pi\Lambda$. As in the proof of the previous case, we can find a basis $\{e_1,e_2,e_3\}$ of Λ with Gram matrix $\mathcal{H}_1^{3,1}$ by Lemma A.4 such that

$$\mathbf{x} = \pi^n(ue_3+\pi e_1).$$

where $u\in\mathcal{O}_{F_0}^\times$. Define

$$\Lambda' = \mathrm{Span}\{\pi e_1, \frac{1}{\pi}e_2, e_3\},\ \Lambda_c = \Lambda\cap\Lambda',$$

then $\mathbf{x}_0\in\Lambda'\backslash\Lambda'^\sharp$. Also define

$$\mathbf{y}_0 = e_3,\ \mathbf{y}_1 = \pi(-\pi e_1+e_2),$$

and $L^\flat := \mathrm{Span}\{\mathbf{y}_0,\mathbf{y}_1\}$. Then by Theorem 4.1 and (10.6) we have

$${}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) = [\mathcal{O}_{\tilde{\mathcal{N}}_\Lambda}] + [\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda'}}] + H_{\Lambda_c} + [\mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}], \tag{10.13}$$

where $\tilde{\mathcal{Z}}(M^\flat)$ is the quasi-canonical lifting cycle of the lattice

$$M^\flat := \mathrm{Span}\{e_3, -\pi e_1+e_2\}.$$

Let $\mathbf{x}' := \pi^{n+1}e_1 = \mathbf{x}-\pi^n ue_3$, then we have

$$\begin{aligned}
&\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})\cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat))\\
&= \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)})\\
&\quad + \sum_{\Lambda_0\in\mathcal{V}^0(L)}\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}).
\end{aligned}$$

Notice that the Gram matrix of $\{\mathbf{x}',\mathbf{y}_1\}$ is equivalent to $\mathcal{H}_1$ when $n=0$, and to $\mathrm{Diag}(u_1\pi_0^n, u_2\pi_0)$ for some $u_1,u_2\in\mathcal{O}_{F_0}^\times$ when $n\geq 1$. Hence by Proposition 2.6 and [HSY, Theorem 1.1], we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(\mathbf{y}_0)}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = \begin{cases} -(q-1) & \text{if } n=0,\\ 1+n-2q & \text{if } n\geq 1.\end{cases}$$

By Corollary 3.6 and Lemmas 3.15 and 3.16 of [HSY], we know that

$$\sum_{\Lambda_0\in\mathcal{V}^0(L)}\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}')}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{y}_1)}) = |\mathcal{V}^0(L)| = \begin{cases} q+1 & \text{if } n=0,\\ 2q+1 & \text{if } n\geq 1.\end{cases}$$

Hence we know that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})\cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) = n+2.$$

On the other hand, by Corollary 3.8,

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0}\cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})) = -1.$$

By [Gro86, Proposition 3.3]

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{O}_{\tilde{\mathcal{Z}}(M^\flat)}\otimes^{\mathbb{L}}\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x})}) = n+1.$$

Since $\mathbf{x}\in\Lambda'\backslash\Lambda'^\sharp$, by the previous case we also have

$$\mathrm{Int}_{\Lambda'}(\mathbf{x}) = 1.$$

Combining all above, we have by (10.13)

$$\mathrm{Int}_\Lambda(\mathbf{x}) = 1. \tag{10.14}$$

This finishes the proof of Theorem 10.2. □

11. Proof of the modified Kudla-Rapoport conjecture: three dimension case

In this section, we will prove Theorem 1.2. We need some preparation.

Proposition 11.1. *Assume that $L \subset \mathbb{V}$ has a Gram matrix $T = \mathrm{Diag}(u_1, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ with $u_i \in \mathcal{O}_{F_0}^\times$ and $0 \le b \le c$. Then*

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

Moreover, for every decomposition $L = L^\flat \oplus \mathrm{Span}\{\mathbf{x}\}$, we have

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}.$$

Proof. Fix a basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ of L such that the Gram matrix of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is $T = \mathrm{Diag}(u_1, T_2)$ where $u_1 \in \mathcal{O}_{F_0}^\times$ and $T_2 \in \mathrm{Herm}_2(\mathcal{O}_{F_0})$. Let $u_1^{-1} \cdot L$ be a lattice represented by $u_1^{-1} \cdot T$. Since $\mathrm{Int}(u_1^{-1} \cdot L) = \mathrm{Int}(L)$ and $\partial\mathrm{Den}(u_1^{-1} \cdot L) = \partial\mathrm{Den}(L)$, we may assume $u_1 = 1$. Let $L^\flat = \mathrm{Span}\{\mathbf{x}_2, \mathbf{x}_3\}$. According to Propositions 2.6, 3.2, and Corollary 3.6, we have

$$\begin{aligned}(11.1)\qquad &\mathrm{Int}(L) - \mathrm{Int}(L^\flat)\\ &= \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat)) - \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\tilde{\mathcal{Z}}(\mathbf{x}_1) \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat))\\ &= \sum_{\Lambda_0 \in \mathcal{V}^0(L)} \chi(\mathcal{N}^{\mathrm{Kra}}, [\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}] \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat))\\ &= |\{\mathcal{V}^0(L)\}|.\end{aligned}$$

Now the result we want follows by comparing (11.1) with (8.5), and the identity $\mathrm{Int}(L^\flat) = \partial\mathrm{Den}(L^\flat)$ proved in [Shi20, Theorem 1.3] and [HSY, Theorem 1.3]. (2) follows from (1) and Theorem 5.6 (2). □

Proof of Theorem 1.4: Under the assumption $\mathrm{v}(L^\flat) > 0$, we can decompose $\mathcal{D}(L^\flat)$ in $\mathrm{Gr}^2 K_0(\mathcal{N}^{\mathrm{Kra}})$ as

$$(11.2)\qquad \mathcal{D}(L^\flat) = \sum_{\Lambda_2 \in \mathcal{V}(L^\flat)} m(\mathcal{D}(L^\flat), \Lambda_2)[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}] + \sum_{\Lambda_0 \in \mathcal{V}(L^\flat)} m(\mathcal{D}(L^\flat), \Lambda_0) H_{\Lambda_0},$$

by (10.3) and Proposition 4.5.

Claim 1: $m(\mathcal{D}(L^\flat), \Lambda_0) = 0$ unless $L^\flat \subset \Lambda_0$. In such a case,

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \begin{cases} q+1 & \text{if } \Lambda_0 \in \mathcal{V}(L^\flat) \setminus \mathcal{B}(L^\flat),\\ 1 & \text{if } \Lambda_0 \in \mathcal{B}(L^\flat).\end{cases}$$

Indeed, since Λ_0 is of type 0, we may choose a $y_0 \in \mathbb{V} \setminus L_F^\flat$ such that $\mathrm{Span}\{y_0\}^\perp$ is non-split and $y_0 \in \Lambda_0 \setminus \pi\Lambda_0$. In this case, Proposition 3.2, Corollaries 3.7 and 3.8 imply that

$$\chi(\mathcal{N}^{\mathrm{Kra}}, H_{\Lambda_0} \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]) = 1.$$

So by (11.2), Corollaries 10.3 and 2.7, we have

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]).$$

Let $(2a, 2b)$ $(b > a)$ be the fundamental invariant of the projection of $L^\flat$ onto $\mathrm{Span}\{y_0\}^\perp$. Let φ be the natural quotient map $\Lambda_0 \to \Lambda_0/\pi\Lambda_0$ and define

$$m := \dim_{\mathbb{F}_q} \varphi(L^\flat) \le 2.$$

Equation (9.1) implies that $m = 0$ if and only if $\Lambda_0 \in \mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)$. First assume $m = 0$, in other words, $L^\flat \subset \pi\Lambda_0$ so $b \ge a \ge 1$. By the definition of $\mathcal{D}(L^\flat)$ and [Shi20, Theorem 1.2], we have

$$m(\mathcal{D}(L^\flat), \Lambda_0) = \mu(a, b) - q\mu(a-1, b) - \mu(a, b-1) + q\mu(a-1, b-1) = q+1,$$

as claimed where

$$(11.3)\qquad \mu(a, b) = \chi(\mathcal{N}^{\mathrm{Kra}}, {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L^\flat) \cdot [\mathcal{O}_{\tilde{\mathcal{Z}}(y_0)}]) = \begin{cases} 2\sum_{s=0}^{a} q^s(a+b+1-2s) - a - b - 2 & \text{if } a \ge 0\\ 0 & \text{if } a < 0.\end{cases}$$

Now assume $m=1$, then $\varphi(L^\flat)$ is a line ℓ and $b\geq 1$. By the assumption that $y_0\notin L^\flat_F$, we know ℓ is not in $\mathrm{Span}\{\varphi(y_0)\}$, hence the projection of ℓ onto $\varphi(y_0)^\perp$ is nonzero. Since $\varphi(y_0)^\perp$ is nonsplit, we must have $a=0$. Hence by the definition of $\mathcal{D}(L^\flat)$ and (11.3), we have

$$m(\mathcal{D}(L^\flat),\Lambda_0)=\mu(0,b)-q\mu(-1,b)-\mu(0,b-1)+q\mu(-1,b-1)=1$$

as claimed. Finally, $m=2$ is impossible since $v(L^\flat)>0$. This finshes the proof of Claim 1.

Claim 2: $m(\mathcal{D}(L^\flat),\Lambda_2)=2$ for any $\Lambda_2\in\mathcal{V}^2(L^\flat)$.

Indeed, according to Lemma 3.9, we have $\chi(\mathcal{N}^{\mathrm{Kra}},\mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}])=0$. On the other hand, Corollary 3.7 and Lemma 9.12 imply that

$$\chi(\mathcal{N}^{\mathrm{Kra}},\mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}])=\sum_{\Lambda_0\subset\Lambda_2}m(\mathcal{D}(L^\flat),\Lambda_2)-2m(\mathcal{D}(L^\flat),\Lambda_0). \tag{11.4}$$

Combining the above with Claim 1, we have

$$\begin{aligned}0&=\chi(\mathcal{N}^{\mathrm{Kra}},\mathcal{D}(L^\flat)\cdot[\mathcal{O}_{\mathrm{Exc}_{\Lambda_0}}])\\&=\begin{cases}\sum_{\Lambda_0\subset\Lambda_2}m(\mathcal{D}(L^\flat),\Lambda_2)-2(q+1) & \text{if } \Lambda_0\in\mathcal{L}(L^\flat)\setminus\mathcal{B}(L^\flat),\\ \sum_{\Lambda_0\subset\Lambda_2}m(\mathcal{D}(L^\flat),\Lambda_2)-2 & \text{if } \Lambda_0\in\mathcal{B}(L^\flat).\end{cases}\end{aligned} \tag{11.5}$$

Recall $\mathcal{S}(L^\flat)$ in Definition 9.8. First assume $\Lambda_2\in\mathcal{L}(L^\flat)\setminus\mathcal{S}(L^\flat)$. If $d(\Lambda_2,\mathcal{B}(L^\flat))=\frac{1}{2}$, choose $\Lambda_0\in\mathcal{B}(L^\flat)$ such that $\Lambda_0\subset\Lambda_2$, then Λ_2 is the unique lattice in $\mathcal{V}^2(L^\flat)$ that contains Λ_0. Hence (11.5) implies that $m(\mathcal{D}(L^\flat),\Lambda_2)=2$. Now Corollary 9.10 allows us to show $m(\mathcal{D}(L^\flat),\Lambda_2)=2$ by induction on the distance $d(\Lambda_2,\mathcal{B}(L^\flat))$ for any $\Lambda_2\in\mathcal{L}(L^\flat)\setminus\mathcal{S}(L^\flat)$.

Similarly for $\Lambda_2\in\mathcal{S}(L^\flat)$, we can show $m(\mathcal{D}(L^\flat),\Lambda_2)=2$ by induction on its distance to $\mathcal{S}(L^\flat)\cap\mathcal{B}(L^\flat)$. This finishes the proof of Claim 2.

Now the proposition follows from the fact that for $\Lambda_0\in\mathcal{V}(L^\flat)$

$$\sum_{\Lambda_2\in\mathcal{V}^2(L^\flat)}\sum_{\Lambda_0\subset\Lambda_2}1=\begin{cases}q+1 & \text{if } \Lambda_0\in\mathcal{V}(L^\flat)\setminus\mathcal{B}(L^\flat),\\ 1 & \text{if } \Lambda_0\in\mathcal{B}(L^\flat).\end{cases}$$

This finishes the proof of Theorem 1.4.

In the following discussion we freely use Theorem 10.2 and Corollary 3.8 without explicitly referring to them.

Proposition 11.2. *Assume $L=L^\flat\obot\mathrm{Span}\{\mathbf{x}\}$ with Gram matrix*

$$T=\mathrm{Diag}(\mathcal{H}_a,u_3(-\pi_0)^c)$$

where a is a positive odd integer, and $c\geq 0$. Then

$$\mathrm{Int}(L)^{(2)}=\partial\mathrm{Den}(L)^{(2)}=\begin{cases}1-q^a & \text{if } a\leq 2c,\\ 1-q^{2c+1} & \text{if } a>2c.\end{cases} \tag{11.6}$$

Proof. By Proposition 8.5, it suffices to prove the identity for $\mathrm{Int}(L)^{(2)}$.

Now we compute $\mathrm{Int}(L)^{(2)}$. We may take $L^\flat=\mathrm{Span}\{\pi^{\frac{a+1}{2}}e_1,\pi^{\frac{a+1}{2}}e_2\}$, where the Gram matrix of $\{e_1,e_2\}$ is $\mathcal{H}$. Let $e_3=\pi^{-c}\mathbf{x}$. Then $\mathcal{L}(L^\flat)$ is centered at $\mathrm{Span}\{e_1,e_2,e_3\}$ of radius $\frac{a}{2}$ by Proposition 9.7.

Assume $a\leq 2c$ first. In this case, $\mathcal{L}(L^\flat)\subset\mathcal{L}(\mathbf{x})$. As a result, we have $\mathrm{Int}_{\Lambda_2}(\mathbf{x})=1$ and $\mathrm{Int}_{\Lambda_0}(\mathbf{x})=-1$ for any $\Lambda_2\in\mathcal{L}^2(L^\flat)$ and $\Lambda_0\in\mathcal{L}^0(L^\flat)$. Hence by Theorem 1.4, we have

$$\begin{aligned}\mathrm{Int}(L)^{(2)}&=\sum_{\Lambda_2\in\mathcal{L}(L^\flat)}\chi(\mathcal{N}^{\mathrm{Kra}},(2[\mathcal{O}_{\tilde{\mathcal{N}}_{\Lambda_2}}]+\sum_{\Lambda_0\subset\Lambda_2}H_{\Lambda_0})\cdot\mathcal{Z}^{\mathrm{Kra}}(\mathbf{x}))\\&=(1-q)|\{\Lambda_2\mid\Lambda_2\in\mathcal{L}(L^\flat)\}|\\&=(1-q)(1+(1+q)q+(1+q)q^3+\cdots+(1+q)q^{a-2})\\&=1-q^a,\end{aligned} \tag{11.7}$$

as claimed.

Now we assume $a>2c$. We consider the case $c=0$ first. Recall that $\tilde{\mathcal{Z}}(e_3)\approx\mathcal{N}^{\mathrm{Kra}}_{2,1}$, hence $\mathcal{L}(L^\flat)\cap\mathcal{L}(e_3)$ is a ball of radius $\frac{a}{2}$ in the Bruhat-Tits tree $\mathcal{L}_{2,1}$ of $\mathcal{N}^{\mathrm{Pap}}_{2,1}$, within which a vertex lattice Λ_0 of type 0 is contained in two vertex lattices of type 2, and a vertex lattice Λ_2 of type 2 contains $q+1$ vertex lattice of type 0. Hence

$$|\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}| = 1 + q + (1+q)q + \cdots + (1+q)q^{\frac{a-3}{2}},$$

and

$$|\{\Lambda_0 \mid \Lambda_0 \in \mathcal{B}(L^\flat) \cap \mathcal{L}(e_3)\}| = (1+q)q^{\frac{a-1}{2}}.$$

Moreover, notice that if $e_3 \in \Lambda_0$, then $\mathrm{Int}_{\Lambda_2}(e_3) = 1$ for any Λ_2 such that $\Lambda_0 \subset \Lambda_2$. However, if $\Lambda_0 \in \mathcal{B}(L^\flat) \cap \mathcal{L}(e_3)$, there is a unique Λ_2 such that $m(\mathcal{D}(L^\flat), \Lambda_2) \neq 0$ among the $q+1$ vertex lattices of type 2 that contain Λ_0. As a result,

$$\begin{aligned}
&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot \mathcal{Z}^{\mathrm{Kra}}(e_3)) \\
&= 2(1 + q \cdot |\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}|) \\
&\quad - (q+1)|\{\Lambda_0 \mid \Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}| - |\{\Lambda_0 \mid \Lambda_0 \in \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)\}| \\
&= 2 + (q-1)(1 + q + (1+q)q + q + (1+q)q^2 + \cdots + (1+q)q^{\frac{a-3}{2}}) - (1+q)q^{\frac{a-1}{2}} \\
&= 1 - q,
\end{aligned}$$

which is compatible with (11.6).

Next we show

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi e_3) - \mathcal{Z}^{\mathrm{Kra}}(e_3))) = q - q^3.$$

According to Proposition 9.6, $\mathcal{L}(\pi e_3) = \{\Lambda \mid d(\Lambda, \mathcal{L}(e_3)) \leq 1\}$. Hence, around each $\Lambda_0 \in (\mathcal{L}(L^\flat) \setminus \mathcal{B}(L^\flat)) \cap \mathcal{L}(e_3)$, there will be $q(q-1)$ many new vertex lattices of type 0 in $\mathcal{L}(L^\flat) \cap \mathcal{L}(\pi e_3) \setminus \mathcal{L}(L^\flat) \cap \mathcal{L}(e_3)$. Hence,

$$\begin{aligned}
&\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi e_3) - \mathcal{Z}^{\mathrm{Kra}}(e_3))) \\
&= 2q \cdot q(q-1)(1 + q + (1+q)q + (1+q)q^2 + \cdots + (1+q)q^{\frac{a-1}{2}-2}) \\
&\quad - q(q-1)(q+1)(1 + q + (1+q)q + (1+q)q^2 + \cdots + (1+q)q^{\frac{a-1}{2}-2}) \\
&\quad - q(q-1)(1+q)q^{\frac{a-1}{2}-1} \\
&= q - q^3.
\end{aligned}$$

Continuing in this way, we can show

$$\chi(\mathcal{N}^{\mathrm{Kra}}, \mathcal{D}(L^\flat) \cdot (\mathcal{Z}^{\mathrm{Kra}}(\pi^i e_3) - \mathcal{Z}^{\mathrm{Kra}}(\pi^{i-1} e_3))) = q^{2i-1} - q^{2i+1}$$

for $2i < a$. So

$$\mathrm{Int}(L)^{(2)} = \mathcal{D}(L^\flat) \cdot \mathcal{Z}^{\mathrm{Kra}}(\pi^c e_3) = 1 - q^{2c+1} = \partial\mathrm{Den}^{(2)}(L)$$

as claimed. □

Proposition 11.3. *Assume* $L = L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$ *with Gram matrix*

$$T = \mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$$

where $0 < a \leq b \leq c$*, then*

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}.$$

Proof. By Proposition 8.4, it suffices to show

$$\mathrm{Int}(L)^{(2)} = 1 + \chi(-u_2u_3)q^a(q^a - q^b) - q^{a+b}. \tag{11.8}$$

If $\chi(-u_2u_3) = -1$, then $\chi(S)\chi(u_1) = 1$, and (11.8) specializes to

$$\partial\mathrm{Den}(T)^{(2)} = 1 - q^{2a}.$$

On the other hand, if $\chi(-u_2u_3) = -1$, then $\mathcal{L}(L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius a. Since $a < c$, $\mathcal{L}(L^\flat) \subset \mathcal{L}(\mathbf{x})$, and one can show $\mathrm{Int}(L)^{(2)} = 1 - q^{2a}$ exactly as in (11.7).

Now we assume $\chi(-u_2u_3) = 1$, hence $\chi(S)\chi(u_1) = -1$. In this case, (11.8) specializes to

$$\partial\mathrm{Den}(L)^{(2)} = 1 + q^{2a} - 2q^{a+b}.$$

Let $r = b - a$, and $L^\flat = \mathrm{Span}\{x_1, x_2\}$. Then $\mathcal{L}(\pi^{-a} L^\flat)$ is a ball centered at a vertex lattice of type 0 with radius r in the Bruhat-Tits tree $\mathcal{L}_{2,1}$. Hence,

$$\mathcal{L}(L^\flat) = \{\Lambda \mid \Lambda \in \mathcal{L}_3,\ d(\Lambda, \mathcal{L}(\pi^{-a} L^\flat)) \leq a\}.$$

When $a = 1$, $\mathcal{V}^0(\pi^{-1} L^\flat) = \mathcal{V}^0(L^\flat) \setminus \mathcal{B}(L^\flat)$. Then combining with Theorem 1.4, it is not hard to see

$$\begin{aligned}\mathrm{Int}(L)^{(2)} &= 2(q + 1 + q \cdot 2(q + q^2 + \cdots + q^r)) - (q+1)|\mathcal{V}^0(L^\flat) \setminus \mathcal{B}(L^\flat)| - |\mathcal{B}(L^\flat)| \\ &= 1 + q^2 - 2q^{b+1},\end{aligned}$$

where we use the fact

$$|\mathcal{V}^0(\pi^{-1} L^\flat)| = 1 + 2(q + q^2 + \cdots + q^r),$$

and

$$|\mathcal{B}(L^\flat)| = (q-1)q(1 + 2(q + q^2 + \cdots + q^{r-1})) + 2q^{r+2}.$$

Now assume $a > 1$. Let T be the Hermitian matrix associated with $L^\flat \obot \mathrm{Span}\{\mathbf{x}\}$, then

$$\begin{aligned}&\partial\mathrm{Den}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \partial\mathrm{Den}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} \\ &= 1 + q^{2a+2} - 2q^{r+2a+2} - (1 + q^{2a} - 2q^{r+2a}) \\ &= q^{2a}(q^2 - 1)(1 - q^{2r}) \\ &= q^2 \left(\partial\mathrm{Den}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \partial\mathrm{Den}(\pi^{-1} L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)}\right),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Int}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \mathrm{Int}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} \\ &= 2q|\mathcal{B}(L^\flat)| - q|\mathcal{B}(L^\flat)| - |\mathcal{B}(\pi L^\flat)| \\ &= (2q - q - q^2)|\mathcal{B}(L^\flat)| \\ &= q^2 \left(\mathrm{Int}(L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)} - \mathrm{Int}(\pi^{-1} L^\flat \obot \mathrm{Span}\{\mathbf{x}\})^{(2)}\right),\end{aligned}$$

where we use the fact $|\mathcal{B}(\pi L^\flat \obot \mathrm{Span}\{\mathbf{x}\})| = q^2 |\mathcal{B}(L^\flat)|$. Since r is arbitrary, an induction on a gives the result we want. □

Proof of Theorem 1.2: The case $\mathrm{v}(L) < 0$ follows from Proposition 8.1 and the fact that $\mathrm{Int}(L) = 0$ under this condition. Assume $v(L) \geq 0$. There are three cases.

Case 1: When L has a Gram matrix $\mathrm{Diag}(u_1, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ as in Proposition 11.1, it is proved by Proposition 11.1.

Case 2: When L has a basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ with Gram matrix $T = \mathrm{Diag}(\mathcal{H}_a, u_3(-\pi_0)^c)$, take $L^\flat = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$ and $\mathbf{x} = \mathbf{x}_3$. By Propositions 11.1, 11.2, and 11.3, we have

$$\mathrm{Int}(L^{\flat,\prime} \oplus \mathrm{Span}\{\mathbf{x}\})^{(2)} = \partial\mathrm{Den}(L^{\flat,\prime} \oplus \mathrm{Span}\{\mathbf{x}\})^{(2)}$$

for any $L^\flat \subset L^{\flat,\prime} \subset L^\flat_F$ (direct sums in the above identity are actually orthogonal direct sums). Thus we have by Theorem 5.6 (1)

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

Case 3: When L has a Gram matrix $\mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b, u_3(-\pi_0)^c)$ with $0 \leq a \leq b \leq c)$, the same argument as Case 2 gives $\mathrm{Int}(L) = \partial\mathrm{Den}(L)$. This finishes the proof of the theorem.

Theorem 1.2 and Theorem 5.6 imply the following corollary.

Corollary 11.4. *For any lattice $L = L^\flat \oplus \mathcal{O}_F \mathbf{x} \subset \mathbb{V}$ of rank 3, we have*

$$\mathrm{Int}(L)^{(2)} = \partial\mathrm{Den}(L)^{(2)}.$$

Appendix A. Calculation of primitive local density

In this appendix, we provide the proof of Propositions 5.9 and 5.10. Throughout this section, M is unimodular of rank $m \geq 2$ unless clearly stated otherwise. Let $\{v_1, \cdots, v_{2k}, v_{2k+1}, \cdots, v_{2k+m}\}$ be a basis of $M_k = \mathcal{H}^k \oplus M$ with Gram matrix $\mathrm{Diag}(\mathcal{H}^k, S)$ and $S = \mathrm{Diag}(I_{m-1}, \nu)$ Let L be a Hermitian lattice of rank n with Gram matrix T. An isometric embedding $\varphi : L \to M$ is called primitive if its image in $M/\pi M$ has dimension $\mathrm{rank}_{\mathcal{O}_F}(L)$. We call a vector v primitive in M if $\pi^{-1}v \notin M$, or equivalently the natural embedding $\varphi : \mathrm{Span}_{\mathcal{O}_F}\{v\} \hookrightarrow M$ is primitive. For a $v \in M_k$, we let $\mathrm{Pr}_{\mathcal{H}^k}(w_i)$ be the projection of w_i to $\mathcal{H}^k$.

A.1. **Proof of Proposition 5.9.** The main purpose of this subsection is to prove the first four parts of Proposition 5.9. Part (5) of this proposition follows from Proposition 5.7 and Corollaries A.10 and A.12.

Proof. For (1), Choose $M(1) = \frac{t\pi}{2}v_1 + v_2 \in M_k$ with $q(M(1)) = t$. Then

$$M(1)^\perp = \mathrm{Span}_{\mathcal{O}_F}\{\frac{-t\pi}{2}v_1 + v_2, v_3, ..., v_{2k}, v_{2k+1}, \cdots, v_{2k+m}\} \cong \langle -t\rangle \oplus \mathcal{H}^{k-1} \oplus M, \tag{A.1}$$

which is represented by $\mathrm{Diag}(-t, \mathcal{H}^{k-1}, S)$. It is easy to check

$$|M_k : M(1) \oplus M(1)^\perp|^{-1}|M(1)^\vee : M(1)| = |t\pi|_F |t\pi|_F^{-1} = 1.$$

For (2) and (3), assume first that M is isotropic (and unimodular). In this case, we may choose a basis $\{v'_{2k+1}, \ldots, v'_{2k+m}\}$ of M with Gram matrix $\mathrm{Diag}(\mathcal{H}_0, 1, \ldots, 1, -\nu)$. Choose $M(0) = \frac{t}{2}v'_{2k+1} + v'_{2k+2}$ with $q(M(0)) = t$. Then

$$\begin{aligned} M(0)^\perp &= \mathrm{Span}\{v_1, \cdots, v_{2k}, -\frac{t}{2}v'_{2k+1} + v'_{2k+2}, v'_{2k+3}, \cdots, v'_{2k+m}\} \\ &\cong \mathcal{H}^k \oplus \mathrm{Span}\{v'_{2k+3}, \cdots, v'_{2k+m}\} \oplus \langle -t\rangle. \end{aligned}$$

as claimed. Moreover

$$|M_k : M(0) \perp M(0)^\perp|^{-1}|M(0)^\vee : M(0)| = |t|_F |t\pi|_F^{-1} = q$$

Next, assume that M is anisotropic. In this case, M has rank 2 and has Gram matrix $S = \mathrm{Diag}(1, \nu)$ with $\chi(M) = \chi(-\nu) = -1$. In this case, $E = F_0(\sqrt{-\nu})$ is a unramified quadratic field extension of F_0, and $N_{E/F_0}\mathcal{O}_E^\times = \mathcal{O}_{F_0}^\times$. When $\mathrm{v}(t) = 0$, $t \in N_{E/F_0}\mathcal{O}_E^\times$, i.e., $t = a\bar{a} + b\bar{b}\nu$. Take $M(0) = av_{2k+1} + bv_{2k+2}$. Then $q(M(0)) = t$, and

$$M(0)^\perp = \mathrm{Span}\{v_1, \cdots, v_{2k}, -\nu\bar{b}v_{2k+1} + \bar{a}v_{2k+2}\} = \mathcal{H}^k \oplus \langle t\nu\rangle,$$

and

$$|M_k : M(0) \perp M(0)^\perp|^{-1}|M(0)^\vee : M(0)| = |\pi|_F^{-1} = q.$$

When $\mathrm{v}(t) > 0$, $t \notin N_{E/F_0}\mathcal{O}_E^\times$. So there is no primitive $M(0) \in M$ with $q(M(0)) = t$. This proves (1)—(3) of Proposition 5.9.

The proof of (4) follows from the following 4 lemmas.

Lemma A.1. *For primitive vectors $w_1,\ w_2 \in \mathcal{H}_i$ with $q(w_1) = q(w_2)$, we can find an element $g \in \mathrm{U}(\mathcal{H}_i)$ such that $g(w_1) = w_2$.*

Proof. We treat the case i is odd first. Assume $v = a_1v_1 + a_2v_2$. v is primitive implies that a_1 or a_2 is a unit. Without loss of generality, we assume a_2 is a unit and we can further assume $a_2 = 1$ by the action of $\begin{pmatrix} \bar{a}_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix}$. Now notice that $q(v) = (v, v) = (a_1 - \bar{a}_1)\pi^i$. Hence we can write $a_1 = \alpha + \frac{q(v)\pi^{-i}}{2}$, where $\alpha \in \mathcal{O}_{F_0}$. Now let $g = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$, and it is straightforward to check that $g \in \mathrm{U}(\mathcal{H}_i)$ and $g(v) = \frac{q(v)\pi^{-i}}{2}v_1 + v_2$.

Now we deal with the case i is even. Again, we can assume $v = a_1v_1 + v_2$. Then $q(v) = (a_1 + \bar{a}_1)\pi^i$. Hence we can write $a_1 = \frac{q(v)}{2}\pi^{-i} + \beta\pi$, where $\beta \in \mathcal{O}_{F_0}$. Now let $g = \begin{pmatrix} 1 & -\beta\pi \\ 0 & 1 \end{pmatrix}$, and it is straightforward to check that $g \in \mathrm{U}(\mathcal{H}_i)$ and $g(v) = \frac{q(v)\pi^{-i}}{2}v_1 + v_2$. $\square$

Lemma A.2. *Assume M is any lattice such that $\mathrm{v}(M) \geq i$. For $w_1,\ w_2 \in \mathcal{H}_i^k \oplus M$, if $\mathrm{Pr}_{\mathcal{H}_i^k}(w_1)$ and $\mathrm{Pr}_{\mathcal{H}_i^k}(w_2)$ are primitive and $q(w_1) = q(w_2)$, then there exists $g \in \mathrm{U}(\mathcal{H}_i^k \oplus M)$ with $g(w_1) = w_2$.*

Proof. Choose a basis $\{v_1, ..., v_{2k}\}$ of $\mathcal{H}_i^k$ such that the associated Gram matrix is $\mathcal{H}_i^k = \mathrm{Diag}(\mathcal{H}_i, ..., \mathcal{H}_i)$. We also choose a basis $\{v_{2k+1}, ..., v_{2k+m}\}$ of M. Write $w_1 = \sum_{i=1}^{2k+m} a_i v_i$. Since $\mathrm{Pr}_{\mathcal{H}_i^k}(w_1)$ is primitive, a_i is a unit for some $i \in \{1, ..., 2k\}$. Without loss of generality, we may assume $a_1 = 1$. Let $w' = w_1 + \frac{(-1)^{i+1} q(w_1)\pi^{-i}}{2} v_2$, then

$$q(w') = q(w_1) + (w_1, \frac{(-1)^{i+1} q(w_1)\pi^{-i}}{2} v_2) + (\frac{(-1)^{i+1} q(w_1)\pi^{-i}}{2} v_2, w_1) = 0,$$

and $(w', v_2) = (v_1, v_2)$. As a result, $M_1 = \mathrm{Span}_{\mathcal{O}_F}\{w_1, v_2\} = \mathrm{Span}_{\mathcal{O}_F}\{w', v_2\}$ is isometric to $\mathcal{H}_i$. Notice that $\mathrm{val}_\pi(q(w_1)) \geq i$ is guaranteed by the assumption $\mathrm{v}(M) \geq i$.

Similarly, we can show $w_2 \in M_2$ for some M_2 that is isometric to $\mathcal{H}_i$. However, the assumption $\mathrm{v}(M) \geq i$ and [Jac62, Proposition 4.2] imply that there exist $g \in \mathrm{U}(\mathcal{H}_i^k \oplus M)$ such that $g(M_1) = M_2$. In particular, $g(w_1) \in M_2$. Since both $g(w_1)$ and w_2 are in M_2, the problem is reduced to Lemma A.1. □

Lemma A.3. *For primitive vectors $w_1,\ w_2 \in M$ with $q(w_1) = q(w_2)$, we can find an element $g \in \mathrm{U}(M)$ such that $g(w_1) = w_2$.*

Proof. Since M is unimodular, we can decompose

$$M = \mathcal{H}_0^k \oplus M',$$

where $M' = 0$ or an anisotropic unimodular Hermiatian lattice of rank 1 or 2. If $\mathrm{Pr}_{\mathcal{H}_0^k}(w_1)$ and $\mathrm{Pr}_{\mathcal{H}_0^k}(w_2)$ are primitive, this is Lemma A.2. If $\mathrm{Pr}_{\mathcal{H}_0^k}(w_1)$ is not primitive, then $\mathrm{Pr}_{M'}(w_1)$ is primitive and thus $q(\mathrm{Pr}_{M'}(w_1)) \in \mathcal{O}_F^\times$. This implies that $q(w_2) = q(w_1)$ is a unit, and $M = \mathcal{O}_F w_i \oplus (\mathcal{O}_F w_i)^\perp$. Therefore there is some $g \in \mathrm{U}(M)$ with $g(w_1) = w_2$. □

Lemma A.4. *Assume that $w_1,\ w_2 \in M_k$ are primitive and that $\mathrm{Pr}_{\mathcal{H}^k}(w_1)$ and $\mathrm{Pr}_{\mathcal{H}^k}(w_2)$ are not primitive. Then we can find $g \in \mathrm{U}(M_k)$ such that $g(w_1) = w_2$.*

Proof. Let $\{v_1, ..., v_{2k+m}\}$ be a basis of $\mathcal{H}^k \oplus M$, whose Gram matrix is $\mathcal{H}^k \oplus \mathrm{Diag}(1, ..., \nu)$ where ν is a unit. Assume $v \in M_k$ is primitive and $\mathrm{Pr}_{\mathcal{H}^k}(v)$ is not primitive, then we can write $v = \sum_{i=1}^{2k} \pi a_i v_i + \sum_{j=2k+1}^{2k+m} a_j v_j$, where some a_j is a unit for $2k+1 \leq j \leq 2k+m$. Again, without loss of generality, we may assume $a_{2k+m} = 1$. For $i \leq k$, we set

$$v'_{2i-1} = v_{2i-1} + \frac{\bar{a}_{2i}}{\nu} v_{2k+m}, \quad v'_{2i} = v_{2i} + \frac{-\bar{a}_{2i-1}}{\nu} v_{2k+m}$$

Let $M_v = \mathrm{Span}_{\mathcal{O}_F}\{v'_1, ..., v'_{2k}\}$. Then it is easy to check that M_v is perpendicular to v. Moreover, M_v is isometric to $\mathcal{H}^k$ since $\mathrm{val}_\pi((v'_{2i-1}, v'_{2i})) = -1$ and $0 \leq \mathrm{val}_\pi((v'_i, v'_j))$ for other $1 \leq i, j \leq 2k$. Hence we can find $g_v \in \mathrm{U}(M_k)$ such that $g_v(M_v) = \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{v_1, ..., v_{2k}\}$, and $g_v(v) \in \mathrm{Span}_{\mathcal{O}_\mathrm{F}}\{v_{2k+1}, ..., v_{2k+m}\} = M$.

Applying the above to w_1 and w_2, we can find $g_{w_1}, g_{w_2} \in \mathrm{U}(M_k)$ such that $g_{w_1}(w_1),\ g_{w_2}(w_2) \in M$. Now the problem is reduced to Lemma A.3, and the lemma is proved. □

According to Lemma A.2 and Lemma A.4, a primitive vector $v \in M_k$ is either in the same orbit of a vector $M(1) \in \mathcal{H}^k$ or a vector $M(0) \in M$. Lemma A.1 implies that primitive vectors $M(1), M'(1) \in \mathcal{H}^k$ with $q(M(1)) = q(M'(1))$ lie in the same orbit. Lemma A.3 implies the similar result for primitive $M(0), M'(0) \in M$ with $q(M(0)) = q(M'(0))$. A combination of the above proves Part (4) of Proposition 5.9. □

A.2. **Proof of Proposition 5.10.** In this subsection, we prove the first part of Proposition 5.10, which we restate as follows for the convenience of the reader.

Proposition A.5. *Let L be Hermitian $\mathcal{O}_F$-lattice of rank 2 and $\mathrm{v}(L) > 0$. Let $\varphi : L \to M_k$ be a primitive isometric embedding. Let $d(\varphi)$ be the dimension of the image of the map*

$$\mathrm{Pr}_{\mathcal{H}^k} \circ \varphi : L \to \mathcal{H}^k$$

in $\mathcal{H}^k/\pi\mathcal{H}^k$. Then

$$\varphi(L)^\perp \cong (-L) \oplus \mathcal{H}^{k-d(\varphi)} \oplus M^{(d(\varphi))}$$

where $M^{(d(\varphi))}$ is unimodular of rank equal to $(\mathrm{rank}(M) - 2(2 - d(\varphi)))$ and $\det M^{(d(\varphi))} = (-1)^{d(\varphi)} \det M$. In particular, if $d(\varphi) = 1$ then $\mathrm{rank}(M) \geq 2$, and if $d(\varphi) = 0$ then $\mathrm{rank}(M) \geq 4$.

Proof. This proposition follows from Lemmas A.6 and A.7 below. □

Lemma A.6. *Let the notation be as in Proposition A.5. If* $\mathrm{rank}(M_k) \leq 4$*, then*

$$\varphi(L)^{\perp} \approx -L.$$

In particular, such an φ *does not exist if* $\chi(M_k) = -1$ *or* $\mathrm{rank}(M_k) < 4$*.*

Proof. First, assume $M_k = \mathcal{H}^2$ and $L \approx \mathcal{H}_i$ where $i > 0$. Let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$ such that the Gram matrix of w_1, w_2 is $\mathcal{H}_i$. By Lemma A.1, we may assume $w_1 = v_1$. Then we may write $w_2 = a_1 v_1 + \pi^{i+1} v_2 + a_3 v_3 + a_4 v_4$, and $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$ by assumption. Without loss of generality, we may assume $a_3 = 1$. Now a direct calculation shows that

$$\varphi(L)^{\perp} = \mathrm{Span}_{\mathcal{O}_F}\{v_1 + (-\pi)^{i+1} v_4, v_3 + \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & (-\pi)^i \\ \pi^i & (a_4 - \bar{a}_4)\pi^{-1} \end{pmatrix} = \begin{pmatrix} 0 & (-\pi)^i \\ \pi^i & -a_1(-\pi)^i - \bar{a}_1 \pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix},$$

hence

$$\varphi(L)^{\perp} \approx -L.$$

Now we treat the case $M_k = \mathcal{H}^2$ and $L \approx \mathrm{Diag}(u_1(-\pi_0^a), u_2(-\pi_0)^b)$ where $0 < a \leq b$. Again, let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$ such that the Gram matrix of w_1, w_2 is $\mathrm{Diag}(u_1(-\pi_0^a), u_2(-\pi_0)^b)$, and we can assume $w_1 = v_1 - \frac{q(w_1)\pi}{2} v_2$ without loss of generality. Then we may write $w_2 = a_1(v_1 + \frac{q(w_1)\pi}{2} v_2) + a_3 v_3 + a_4 v_4$, hence $\min\{v_\pi(a_3), v_\pi(a_4)\} = 0$ by assumption again. We may assume $a_3 = 1$ and a direct calculation shows that

$$\varphi(L)^{\perp} = \mathrm{Span}_{\mathcal{O}_F}\{v_1 + \frac{q(w_1)\pi}{2} v_2 - \bar{a}_1 q(w_1)\pi v_4, v_3 + \bar{a}_4 v_4\}.$$

Set $v_3' = v_1 + \frac{q(w_1)\pi}{2} v_2 - \bar{a}_1 q(w_1)\pi v_4$ and $v_4' = a_1 v_3' + v_3 + \bar{a}_4 v_4$. Then $\varphi(L)^{\perp} = \mathrm{Span}_{\mathcal{O}_F}\{v_3', v_4'\}$ and the Gram matrix of $\{v_3', v_4'\}$ is

$$\begin{pmatrix} -q(w_1) & 0 \\ 0 & a_1\bar{a}_1 q(w_1) - (a_3\bar{a}_4 - \bar{a}_3 a_4)\pi^{-1} \end{pmatrix} = \begin{pmatrix} -q(w_1) & 0 \\ 0 & -q(w_2) \end{pmatrix}.$$

Now let $M_k = \mathcal{H} \oplus M$, where M is unimodular of rank 2. We only treat the case $L \approx \mathcal{H}_i$ in detail, and the argument for L represented by a diagonal matrix is similar. We assume that M_k has a basis $\{v_1, \ldots, v_4\}$ with Gram matrix $\mathcal{H} \oplus \mathrm{Diag}(1, \nu)$ where ν is a unit. Let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$ where the Gram matrix of $\{w_1, w_2\}$ is $\mathcal{H}_i$. Then one can check that at least one of w_1 and w_2 is primitive in $\mathcal{H}$. By Lemma A.1, we can assume that

$$w_1 := \varphi(m_1) = v_1, w_2 := \varphi(m_2) = a_1 v_1 + \pi^{i+1} v_2 + a_3 v_3 + a_4 v_4$$

and

$$(w_2, w_2) = a_1 \pi^i - \bar{a}_1 \pi^i + a_3 \bar{a}_3 + a_4 \bar{a}_4 \nu = 0. \tag{A.2}$$

By our assumption we know that $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$. Since we assume $i \geq 1$, (A.2) implies that both a_3 and a_4 are in $\mathcal{O}_{F_0}^\times$. This in turn implies that $-\nu \in \mathrm{Nm}_{F/F_0^\times}(\mathcal{O}_F^\times) = \mathcal{O}_{F_0}^2$. Hence $M_k \approx \mathcal{H} \oplus \mathcal{H}_0$ and we can instead assume that $\{v_1, v_2, v_3, v_4\}$ has Gram matrix $\mathcal{H} \oplus \mathcal{H}_0$. We can furthur assume that

$$w_1 = v_1, w_2 = a_1 v_1 + \pi^{i+1} v_2 + v_3 + a_4 v_4$$

with

$$(w_2, w_2) = a_1 \pi^i - \bar{a}_1 \pi^i + a_4 + \bar{a}_4 = 0.$$

By direct calculation, it is easy to see that

$$\varphi(L)^{\perp} = \mathrm{Span}_{\mathcal{O}_F}\{v_1 - (-\pi)^i v_4, v_3 - \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & -a_4 - \bar{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & a_1 \pi^i - \bar{a}_1 \pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix}.$$

Finally, let M_k be unimodular of rank 4. We treat the case $L \approx \mathcal{H}_i$ in detail, and the other cases follow from a similar argument. Let $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$ such that the Gram matrix of $\{w_1, w_2\}$ is

$\mathcal{H}_i$. Apparently M_k contains a $\mathcal{H}_0$. We can assume that M_k has a basis $\{v_1, v_2, v_3, v_4\}$ with Gram matrix $\mathcal{H}_0 \oplus \mathrm{diag}\{1, \epsilon\}$ where $\epsilon \in \mathcal{O}_{F_0}^\times$. By Lemma A.3 we can assume that $w_1 = v_1$. Then we have

$$w_2 = a_1 v_1 + \pi^i v_2 + \sum_{j=3}^{4} a_j v_j,$$

and

$$(w_2, w_2) = a_1(-\pi)^i + \bar{a}_1 \pi^i + a_3 \bar{a}_3 + a_4 \bar{a}_4 \epsilon = 0. \tag{A.3}$$

By our assumption we know that $\min\{\mathrm{v}_\pi(a_3), \mathrm{v}_\pi(a_4)\} = 0$. Since we assume $i \geq 1$, (A.3) implies that both a_3 and a_4 are in $\mathcal{O}_{F_0}^\times$. This in turn implies that $-\epsilon \in \mathrm{Nm}_{F/F_0^\times}(\mathcal{O}_F^\times) = \mathcal{O}_{F_0}^2$. Hence $M_k = \mathcal{H}_0^2$ and we can instead assume that $\{v_1, v_2, v_3, v_4\}$ has Gram matrix $\mathcal{H}_0 \oplus \mathcal{H}_0$. We can furthur assume that

$$w_1 = v_1, w_2 = a_1 v_1 + \pi^i v_2 + v_3 + a_4 v_4$$

with

$$(w_2, w_2) = a_1(-\pi)^i + \bar{a}_1 \pi^i + a_4 + \bar{a}_4 = 0.$$

By a direct calculation, it is easy to see that

$$\varphi(L)^\perp = \mathrm{Span}_{\mathcal{O}_F}\{v_1 - (-\pi)^i v_4, v_3 - \bar{a}_4 v_4\}.$$

Its Gram matrix is

$$\begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & -a_4 - \bar{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & a_1(-\pi)^i + \bar{a}_1 \pi^i \end{pmatrix} \approx \begin{pmatrix} 0 & -(-\pi)^i \\ -\pi^i & 0 \end{pmatrix}.$$

Notice that, as a byproduct of the above argument, we actually also proved that if $\mathrm{rank}(M_k) < 4$ or M is not split, then no such φ exists. The lemma is proved.

□

Lemma A.7. *Assume* $\mathrm{v}(L) \geq 0$. *Let* $\varphi : L \to M_k$ *be a primitive isometric embedding. Let* $d(\varphi)$ *be the dimension of* $\mathrm{Pr}_{\mathcal{H}^k}(\varphi(L)) \otimes_{\mathcal{O}_F} \mathbb{F}_q$ *in* $\mathcal{H}^k / \pi \mathcal{H}^k$. *Then there exist a* $g \in \mathrm{U}(M_k)$ *such that*

$$g(\varphi(L)) \subset \mathcal{H}^{d(\varphi)} \oplus I_{4-2d(\varphi)} \subset M_k,$$

where $I_{4-2d(\varphi)}$ *is a unimodular sublattice of* M_k *with rank* $4 - 2d(\varphi)$.

Proof. We prove the case for $L \approx \mathcal{H}_i$ in detail, and the other cases are similar. Let $\{v_1, \ldots, v_{2k+m}\}$ be a basis of M_k whose Gram matrix is $\mathcal{H}^k \oplus \mathrm{diag}\{1, \ldots, 1, \nu\}$ where ν is a unit. Set $\varphi(L) = \mathrm{Span}_{\mathcal{O}_F}\{w_1, w_2\}$. Assume $d(\varphi) = 2$. If $i = -1$, then there is nothing to prove. Therefore, we may assume $i > -1$. By Lemma A.2, without loss of generality, we can assume that $w_1 = v_1$. Then

$$w_2 = a_1 v_1 + \pi^{i+1} v_2 + \sum_{j=3}^{2k+m} a_j v_j.$$

By the assumption that $d(\varphi) = 2$, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 3 \leq j \leq 2k\} = 0.$$

Hence applying Lemma A.2 to $\mathcal{H}^{k-1} \oplus M$, we can find a $g \in \mathrm{U}(M_k)$ such that

$$g w_1 = v_1, \quad g w_2 \in \mathcal{H}^2.$$

where $\mathcal{H}^2$ refers to the first direct summand in the decomposition $\mathcal{H}^k \oplus M = \mathcal{H}^2 \oplus \mathcal{H}^{k-2} \oplus M$.

When $d(\varphi) = 1$, without loss of generality, we can assume $\mathrm{Pr}_{\mathcal{H}^k}(w_1)$ is primitive. By Lemma A.2, we can assume that $w_1 = v_1$. Then

$$w_2 = a_1 v_1 + \pi^{i+1} v_2 + \sum_{j=3}^{2k+m} a_j v_j.$$

By the assumption that $d(\varphi) = 1$, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 3 \leq j \leq 2k\} \geq 1.$$

Since we assume φ is primitive, we know that

$$\min\{\mathrm{v}_\pi(a_j) \mid 2k+1 \leq j \leq 2k+r\} = 0.$$

Then we are done by applying Lemma A.4 to $\mathcal{H}^{k-1} \oplus M$.

When $d(\varphi) = 0$, without loss of generality, we may assume $w_1 = v_{2k+1} + v_{2k+2}$ by Lemma A.4. Here, we pick v_{2k+i} so that the corresponding Gram matrix is $\mathrm{Diag}(1, -1, 1, \ldots, -\nu)$ (this is possible since we assume $m \geq 4$). Since φ is primitive with $d(\varphi) = 0$, then

$$w_2 = \sum_{i=1}^{2k} \pi a_i v_i + \sum_{i=2k+1}^{2k+m} a_i v_i,$$

and

$$\min\{\mathrm{v}_\pi(a_j) \mid 2k+3 \leq j \leq 2k+r\} = 0.$$

We are done by applying Lemma A.4 to $\mathcal{H}^k \oplus \mathrm{Span}_{\mathcal{O}_F}\{v_{2k+3}, \ldots, v_{2k+m}\}$. □

A.3. **Calculation of primitive local density.** In this subsection, we compute primitive local density polynomials and prove the formulas in Propositions 5.9 and 5.10. Assume L is represented by a nonsingular Hermitian matrix T of rank $n \leq 2$. We let $\bar{v}$ denote the image of v in $M_k \otimes_{\mathcal{O}_F} \mathbb{F}_q$. Let

$$M_k^n(i) := \{(v_j) \in M_k^{n,(n)} | \mathrm{Span}_{\mathbb{F}_q}\{\mathrm{Pr}_{\mathcal{H}^k}(\bar{v}_j),\ 1 \leq j \leq n\} \text{ has rank } i\}$$

where $M^{n,(n)}$ is as in (5.5), and

$$\beta_i(M, L, X) := \int_{\mathrm{Herm}_n(F)} dY \int_{M_k^n(i)} \psi(\langle Y, T(\mathbf{x}) - T\rangle) d\mathbf{x}. \tag{A.4}$$

Notice that

$$\sum_{i=0}^{n} \beta_i(M, L, X) = \beta(M, L, X)^{(n)} \tag{A.5}$$

is the primitive local density defined earlier, and we will shorten it as $\beta(M, L, X)$.

Lemma A.8. *Assume L is of rank $n \leq 2$, then*

$$\beta_n(M, L, X) = \begin{cases} \beta(\mathcal{H}^k, L) & \text{if } k \geq n \text{ or } L = \mathcal{H}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that $T(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$ is the moment matrix of $\mathbf{x} \in M_k^n$. Notice that

$$\begin{aligned} \beta_n(M, L, X) &= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T({}^t(\mathbf{x}_1, \mathbf{x}_2)) - T\rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\ &= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) + T(\mathbf{x}_2) - T\rangle) d\mathbf{x}_1 d\mathbf{x}_2. \end{aligned}$$

When $k \geq n$ or $T = \mathcal{H}$, there exists a $g \in \mathrm{U}(M_k)$ by Lemma A.2 or Lemma A.7 such that

$$g(\mathbf{x}_2) = 0, \text{ and } T(\mathbf{x}_1) + T(\mathbf{x}_2) = T(g(\mathbf{x}_1)).$$

So

$$\begin{aligned} &\int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) + T(\mathbf{x}_2) - T\rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\ &= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(g(\mathbf{x}_1)) - T\rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\ &= \int_{\mathrm{Herm}_n(F)} dY \int_{M^n} \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T\rangle) d\mathbf{x}_1 d\mathbf{x}_2 \\ &= \int_{\mathrm{Herm}_n(F)} dY \int_{(\mathcal{H}^k)^{n,(n)}} \psi(\langle Y, T(\mathbf{x}_1) - T\rangle) d\mathbf{x}_1 \\ &= \beta(\mathcal{H}^k, L). \end{aligned}$$

When $k < n$ and $L \neq \mathcal{H}$, we can check $\beta_n(M, L, X) = 0$ by the argument in Lemma A.7. □

By a variant of [CY20], Chao Li and Yifeng Liu obtained the following formula of $\beta(\mathcal{H}^k, L)$.

Lemma A.9 (Lemma 2.16 of [LL21])**.** *Let $b_1 \leq \cdots \leq b_n$ be the unique integers such that $L^\vee/L \approx \mathcal{O}_F/(\pi_1^b) \oplus \cdots \oplus \mathcal{O}_F/(\pi^{b_n})$. Let $t_o(L)$ be the number of nonzero entries in $(b_1, \cdots, b_n)$. Then*

$$\beta(\mathcal{H}^k, L) = \prod_{k-\frac{n+t_o(L)}{2}<i\leq k} (1-q^{-2i}).$$

Combining the above two lemmas, we have the following.

Corollary A.10. *Assume L is of rank 1, we have*

$$\beta_1(M, L, X) = 1 - X.$$

Assume L is of rank 2, we have

$$\beta_2(M, L, X) = \begin{cases} (1-X) & \text{if } L = \mathcal{H}_{-1}, \\ (1-X)(1-q^2X) & \text{otherwise.} \end{cases}$$

Lemma A.11. *For an $\mathcal{O}_F$ Hermitian lattice, let $\bar{L} = L/\pi L$ be its reduction modulo π with resulting quadratic form. Let $r(\overline{M}, \overline{L})$ to be the number of isometries from $\overline{L}$ to $\overline{M}$. Then*

$$\beta_0(M, L, X) = X^n \beta(M, L) = q^{-mn+n^2} r(\overline{M}, \overline{L}) X^n.$$

Proof. For the first identity, we may repeat the same calculation in the proof of Lemma A.8 by replacing $M^n \obot (\mathcal{H}^k)^{n,(n)}$ with $M^{n,(n)} \obot (\pi\mathcal{H}^k)^n$. Notice that the factor X^n shows up because $\mathrm{vol}((\pi\mathcal{H}^k)^n) = X^n$. The second identity follows from the same proof of [CY20, Theorem 3.12] and we leave the detail to the reader. □

Notice that [LZ21, Lemma 3.2.1] provides a uniform formula for $|r(\overline{M}, \overline{L})|$. As a result, we obtain the following corollaries.

Corollary A.12. *Assume $L = \mathcal{O}_F\mathbf{x}$ is of rank 1 (we allow $q(\mathbf{x}) = 0$).*

(1) If $\mathrm{v}(L) = 0$, then

$$\beta_0(M, L, X) = \begin{cases} (1 + \chi(M)\chi(L)q^{-\frac{m-1}{2}})X & \text{if } m \text{ is odd,} \\ (1 - \chi(M)q^{-\frac{m}{2}})X & \text{if } m \text{ is even.} \end{cases}$$

(2) If $\mathrm{v}(L) > 0$, then

$$\beta_0(M, L, X) = \begin{cases} (1 - q^{1-m})X & \text{if } m \text{ is odd,} \\ \left(1 - q^{1-m} + \chi(M)(q-1)q^{-\frac{m}{2}}\right) X & \text{if } m \text{ is even.} \end{cases}$$

Corollary A.13. *Assume L is of rank 2. When $t(L) = 1$, we assume that L has gram matrix $T = \mathrm{Diag}(u_1, u_2(-\pi_0)^b)$ with $b > 0$.*

(1) If m is odd, then

$$\beta_0(M, L, X) = \begin{cases} q(1-q^{1-m})X^2 & \text{if } t(L) = 0, \\ q(1 + \chi(M)\chi(u_1)q^{\frac{3-m}{2}})(1-q^{1-m})X^2 & \text{if } t(L) = 1, \\ q(1-q^{1-m})(1-q^{3-m})X^2 & \text{if } t(L) = 2. \end{cases}$$

(2) If m is even, then

$$\beta_0(M, L, X) = \begin{cases} q\left(1 - \chi(L)q^{1-m} + \chi(L)\chi(M)(q - \chi(L))q^{-\frac{m}{2}}\right) X^2 & \text{if } t(L) = 0, \\ q(1 - \chi(M)q^{-\frac{m}{2}})(1-q^{2-m})X^2 & \text{if } t(L) = 1, \\ q\left((1-q^{2-m}) + \chi(M)(q^2-1)q^{-\frac{m}{2}}\right)(1-q^{2-m})X^2 & \text{if } t(L) = 2. \end{cases}$$

Finally, we calculate $\beta_1(M, L, X)$.

Proposition A.14. *Assume L is as in Corollary A.13. Let $\delta_e(m) = 1$ or 0 depending on whether m is even or odd.*

(1) If $t(L) = 2$, then

$$\beta_1(M, L, X) = q(q+1)\left((1-q^{1-m}) + \delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}}\right) X(1-X).$$

(2) If $t(L)=1$, then

$$\beta_1(M,L,X)=\begin{cases} q(1+q-q^{1-m}+\chi(M)\chi(u_1)q^{\frac{3-m}{2}})X(1-X) & \text{if } m \text{ is odd,}\\ q(1+q-q^{1-m}-\chi(M)q^{-\frac{m}{2}})X(1-X) & \text{if } m \text{ is even.}\end{cases}$$

(3) If $t(L)=0$ and $\chi(L)=1$, i.e. $L\cong\mathcal{H}_0$, then

$$\beta_1(M,L,X)=q\left(q+1-2q^{1-m}+\delta_e(m)\chi(M)(q-1)q^{-\frac{m}{2}}\right)X(1-X),$$

(4) If $t(L)=0$ and $\chi(L)=-1$, then

$$\beta_1(M,L,X)=\beta_1(M,\mathcal{H}_0,X)+2q\left(q^{1-m}-\chi(M)q^{-\lfloor\frac{m-1}{2}\rfloor}\right)X(1-X).$$

Proof. We first assume $L=\mathcal{H}_i$. We claim that

$$\beta_1(M,\mathcal{H}_i,X)=\begin{cases} q(1-X)\left(2\beta_0(M,0,X)+\sum_{\alpha\in(\mathcal{O}_{F_0}/(\pi_0))^\times}\beta_0(M,\langle-2\alpha\rangle,X)\right) & \text{if } i=0\\ q(q+1)(1-X)\beta_0(M,0,X) & \text{if } i\ge 1.\end{cases}$$

Here $\alpha(M,0,X)=\alpha(M,\mathcal{O}_F\mathbf{x},X)$ with $q(\mathbf{x})=0$ and $\mathbf{x}\neq 0$. Now the proposition for $L=\mathcal{H}_i$ follows from Corollary A.12.

Recall

$$I(M_k,L,d)=\{\phi\in\mathrm{Hom}_{\mathcal{O}_F}(L/\pi_0^dL,M/\pi_0^dM)\mid(\phi(x),\phi(y))=(x,y)\in\pi_0^d\cdot\mathrm{Herm}_n(\mathcal{O}_F)^\vee,\ \forall x,y\in L\}.$$

Let

$$J(M_k,L,d):=\{\phi\in I(M_k,L,d)\mid\mathrm{rank}_{\mathbb{F}_q}\overline{\mathrm{Pr}_{\mathcal{H}^k}(\phi(L))}=1,\mathrm{rank}_{\mathbb{F}_q}\overline{\mathrm{Pr}_M(\phi(L))}=1\}.$$

Then

$$\beta_1(S,T,X)=\lim_{d\to\infty}q^{-(4(2k+m)-4)d}|J(M_k,L,d)|.$$

Let $\{l_1,l_2\}$ be a basis of L with Gram matrix $\mathcal{H}_i$. For $\phi\in J(M_k,L,d)$, it will be determined by $w_i=\phi(l_i)$. Let $w_{i,\mathcal{H}}=\mathrm{Pr}_{\mathcal{H}^k}(w_i)$, and $w_{i,M}=\mathrm{Pr}_M(w_i)$. Since $\mathrm{rank}_{\mathbb{F}_q}\overline{\mathrm{Pr}_{\mathcal{H}^k}(\phi(L))}=1$, $\mathrm{rank}_{\mathbb{F}_q}\overline{\mathrm{Pr}_{\mathcal{H}^k}(w_i)}=1$ for $i=1$ or 2.

Now we define a partition of $J(M_k,L,d)$. Assume $\alpha\in\mathbb{F}_q$. Let

$$J_\alpha(M_k,L,d):=\{\phi\in I(M_k,L,d)\mid\mathrm{rank}_{\mathbb{F}_q}\overline{w}_{1,\mathcal{H}}=1,\ \overline{w}_{2,\mathcal{H}}=\alpha\overline{w}_{1,\mathcal{H}}\},\text{ and}$$
$$J_\infty(M_k,L,d):=\{\phi\in I(M_k,L,d)\mid\mathrm{rank}_{\mathbb{F}_q}\overline{w}_{2,\mathcal{H}}=1,\ \overline{w}_{1,\mathcal{H}}=0\}.$$

Then it is easy to verify

$$J(M_k,L,d)=\bigcup_{\alpha\in\mathbb{F}_q}J_\alpha(M_k,L,d)\cup J_\infty(M_k,L,d).$$

Now we compute $|J_\alpha(M_k,L,d)|$. To determine a $\phi\in J_\alpha(M_k,L,d)$, we choose $w_1=\phi(l_1)$ first. By definition, we have

(A.6)
$$\lim_{d\to\infty}q^{(2(2k+m)-1)d}\sharp\{w_1\in M_k/\pi_0^dM_k\mid w_1\text{ is primitive, and }q(w_1)=0\mod\pi_0^d\}=\beta_1(S_k,0)=1-X.$$

Given such a w_1, now we find the number of $w_2=\phi(l_2)$ such that ϕ lies in $J_\alpha(M_k,L,d)$. By Lemma A.2, we may assume $w_{1,S}=0$. Let $w_2=w_{2,M}+\alpha w_1+\pi w_{\mathcal{H}}$, where $w_{\mathcal{H}}\in\mathcal{H}^k$. Then the corresponding ϕ lies in $J_\alpha(M_k,L,d)$ if and only if

$$(w_1,w_2)=(w_1,\pi w_{\mathcal{H}})=\pi^i\mod\pi_0^d$$

and

$$0=q(w_2)=\mathrm{tr}((\alpha w_1,\pi w_{\mathcal{H}}))-\pi_0q(w_{\mathcal{H}})+q(w_{2,M})=\alpha\mathrm{tr}(\pi^i)-\pi_0q(w_{\mathcal{H}})+q(w_{2,M})\mod\pi_0^d.$$

First,

$$\lim_{d\to\infty}q^{-2d(2k-1)}\sharp\{\pi w_{\mathcal{H}}\in\mathcal{H}^k/\pi_0^d\mathcal{H}^k|(w_1,\pi w_{\mathcal{H}})=\pi^i\mod\pi^{2d-1}\}=q^{1-2k}.\tag{A.7}$$

Second, for each fixed $\pi w_{\mathcal{H}}$ we have

$$\begin{aligned}\lim_{d\to\infty} q^{(-2m+1)d}|\{w_{2,M}\in M/\pi_0^d M|\, w_{2,M} \text{ primitive, } q(w_{2,M}) = -\alpha\mathrm{tr}(\pi^i)+\pi_0 q(w_{\mathcal{H}}) \mod \pi_0^d\}| \\ = \beta(M, \langle -\alpha\mathrm{tr}(\pi^i)+\pi_0 q(w_{\mathcal{H}})\rangle), \\ = \begin{cases}\beta(M,\langle -2\alpha\rangle) & \text{if } i=0,\\ \beta(M,0) & \text{if } i>0.\end{cases}\end{aligned} \tag{A.8}$$

By symmetry, $|J_\infty(M_k,L,d)| = |J_0(M_k,L,d)|$. Now a combination of (A.6), (A.7) and (A.8) implies that

$$\begin{aligned}&\beta_1(M,\mathcal{H}_i,X)\\ &= \lim_{d\to\infty} q^{(-4(2k+m)+4)d}\left(\sum_\alpha |J_\alpha(M_k,L,d)| + |J_\infty(M_k,L,d)|\right)\\ &= \begin{cases} q(1-X)\left(2\beta_0(M,0,X)+\sum_{\alpha\in\mathcal{O}_{F_0}^\times/(\pi_0)}\beta_0(M,-2\alpha,X)\right) & \text{if } i=0,\\ q(q+1)(1-X)\beta_0(M,0,X) & \text{if } i\geq 1,\end{cases}\end{aligned}$$

as claimed.

Next, we assume L has a basis $\{l_1,l_2\}$ with Gram matrix $T=\mathrm{Diag}(u_1(-\pi_0)^a, u_2(-\pi_0)^b)$ with $0\le a\le b$. Let $w_i=\phi(l_i)$ as before. Then the number of possible choices for w_1 is given by

$$q^{(2(2k+m)-1)d}\beta_1(M,\langle u_1(-\pi_0)^a\rangle, X)$$

for sufficiently large d. We may assume $w_1 = w_{1,\mathcal{H}}$ without loss of generality. Let $w_2 = w_{2,M}+\alpha w_1+\pi w_{\mathcal{H}}$ as before. Then ϕ lies in $J_\alpha(M_k,L,d)$ if and only if

$$(w_1,w_2) = (w_1,\pi w_{\mathcal{H}}) = 0 \mod \pi_0^d$$

and

$$\begin{aligned}u_2(-\pi_0)^b = q(w_2) &= (w_{2,M}+\alpha w_1+\pi w_{\mathcal{H}}, w_2)\\ &= q(w_{2,M}) + (\pi w_{\mathcal{H}}, w_2) \mod \pi_0^d.\end{aligned}$$

Now

$$\lim_{d\to\infty} q^{(-4k+2)d}\sharp\{\pi w_{\mathcal{H}}\in\mathcal{H}^k/\pi_0^d\mathcal{H}^k | (w_1,\pi w_{\mathcal{H}})=0 \mod \pi^{2d-1}\} = q^{1-2k},$$

and for a fixed $\pi w_{\mathcal{H}}$ we have

$$\begin{aligned}&\lim_{d\to\infty} q^{(-2m+1)d}\sharp\{w_{2,M}\in L_S/\pi_0^d L_S | w_{2,M} \text{ primitive, } q(w_{2,M}) = u_2(-\pi_0)^b - (\pi w_{\mathcal{H}}, w_2) \mod \pi_0^d\}\\ &= \beta(M, \langle u_2(-\pi_0)^b - (\pi w_{\mathcal{H}}, w_2)\rangle).\end{aligned}$$

Now this proposition follows from a similar argument as before, and we leave the details to the reader. □

Finally, we record a lemma that we use frequently.

Lemma A.15. [Kat99, Proposition 2.5 (1)] *Suppose M' is a Hermtitian lattice represented by S'. Let $i(T)$ be the least integer ℓ such that $\pi^\ell T^{-1}$ is integral. Assume that $\mathrm{v}(S') > i(T)$. Then we have*

$$\alpha(M\oplus M', L) = \alpha(M,L).$$

Proof. This follows from a similar argument as in the proof of Lemma A.8 and a Hermitian analogue of [Kit99, Corollary 5.4.4]. We leave the details to the reader.

□

References

[Bro89] Kenneth S Brown. *Buildings*. Springer, 1989.

[CY20] Sungmun Cho and Takuya Yamauchi. A reformulation of the siegel series and intersection numbers. *Mathematische Annalen*, 377(3):1757–1826, 2020.

[Gro86] Benedict Gross. On canonical and quasi-canonical liftings. *Inventiones mathematicae*, 84(2):321–326, 1986.

[GS19] Luis E. Garcia and Siddarth Sankaran. Green forms and the arithmetic Siegel-Weil formula. *Invent. Math.*, 215(3):863–975, 2019.

[GY00] Wee Teck Gan and Jiu-Kang Yu. Group schemes and local densities. *Duke Mathematical Journal*, 105(3):497–524, 2000.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[How19] Benjamin Howard. Linear invariance of intersections on unitary Rapoport–Zink spaces. In *Forum Mathematicum*, volume 31, pages 1265–1281, 2019.

[HSY] Qiao He, Yousheng Shi, and Tonghai Yang. The Kudla-Rapoport conjecture at a ramified prime for $U(1,1)$. *Transactions of the AMS, to appear.*

[Jac62] Ronald Jacobowitz. Hermitian forms over local fields. *American Journal of Mathematics*, 84(3):441–465, 1962.

[Kat99] Hidenori Katsurada. An explicit formula for siegel series. *American Journal of Mathematics*, 121(2):415–452, 1999.

[Kit83] Yoshiyuki Kitaoka. A note on local densities of quadratic forms. *Nagoya Mathematical Journal*, 92:145–152, 1983.

[Kit99] Yoshiyuki Kitaoka. *Arithmetic of quadratic forms*, volume 106. Cambridge University Press, 1999.

[KR11] Stephen Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties I. unramified local theory. *Inventiones mathematicae*, 184(3):629–682, 2011.

[KR14a] Stephen Kudla and Michael Rapoport. An alternative description of the Drinfeld p-adic half-plane. In *Annales de l'Institut Fourier*, volume 64, pages 1203–1228, 2014.

[KR14b] Stephen Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties II: Global theory. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2014(697):91–157, 2014.

[Krä03] Nicole Krämer. Local models for ramified unitary groups. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 73, pages 67–80. Springer, 2003.

[Kud97] Stephen Kudla. Central derivatives of eisenstein series and height pairings. *Annals of mathematics*, 146(3):545–646, 1997.

[Liu11] Yifeng Liu. Arithmetic theta lifting and L-derivatives for unitary groups, I. *Algebra Number Theory*, 5(7):849–921, 2011.

[LL21] Chao Li and Yifeng Liu. Chow groups and L-derivatives of automorphic motives for unitary groups, II. *arXiv preprint arXiv:2101.09485*, 2021.

[LZ] Chao Li and Wei Zhang. Kudla–Rapoport cycles and derivatives of local densities. *J. Amer. Math. Soc. to appear.*

[LZ21] Chao Li and Wei Zhang. On the arithmetic Siegel–Weil formula for GSpin Shimura varieties. *arXiv preprint arXiv:2106.15038*, 2021.

[RSZ17] Michael Rapoport, Brian Smithling, and Wei Zhang. On the arithmetic transfer conjecture for exotic smooth formal moduli spaces. *Duke Mathematical Journal*, 166(12):2183–2336, 2017.

[RSZ18] Michael Rapoport, Brian Smithling, and Wei Zhang. Regular formal moduli spaces and arithmetic transfer conjectures. *Mathematische Annalen*, 370(3-4):1079–1175, 2018.

[RTW14] Michael Rapoport, Ulrich Terstiege, and Sean Wilson. The supersingular locus of the Shimura variety for $\mathrm{GU}(1,n-1)$ over a ramified prime. *Mathematische Zeitschrift*, 276(3-4):1165–1188, 2014.

[RZ96] M Rapoport and Thomas Zink. *Period spaces for p-divisible groups*. Number 141. Princeton University Press, 1996.

[SABK94] Christophe Soulé, Dan Abramovich, JF Burnol, and JK Kramer. *Lectures on Arakelov geometry*. Number 33 in Cambridge Studies in Advanced Mathematics. Cambridge university press, 1994.

[Shi18] Yousheng Shi. Special cycles on unitary Shimura varieties at ramified primes I. *arXiv preprint arXiv:1811.11227*, 2018.

[Shi20] Yousheng Shi. Special cycles on unitary Shimura curves at ramified primes. *arXiv preprint arXiv:2004.07158*, 2020.

[Tam63] Tsuneo Tamagawa. On the ζ-functions of a division algebra. *Annals of Mathematics*, 77(2):387–405, 1963.

[Ter10] Ulrich Terstiege. Intersections of special cycles on the Shimura variety for $\mathrm{GU}(1,2)$. *Journal fur die Reine und Angewandte Mathematik*, 2013, 06 2010.

[Zha21] Wei Zhang. Weil representation and arithmetic fundamental lemma. *Annals of Mathematics*, 193(3):863–978, 2021.

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