

Just-likely intersections on Hilbert modular surfaces

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Abstract

In this paper, we prove an intersection-theoretic result pertaining to curves in certain Hilbert modular surfaces in positive characteristic. Specifically, we show that given two appropriate curves C, D parameterizing abelian surfaces with real multiplication, the set of points in $(x, y) \in C \times D$ with surfaces parameterized by x and y isogenous to each other is Zariski dense in $C \times D$, thereby proving a case of a just-likely intersection conjecture. We also compute the change in Faltings height under appropriate p -power isogenies of abelian surfaces with real multiplication over characteristic p global fields.

1 Introduction

The study of intersection theory in the context of Shimura varieties has produced several breakthroughs in arithmetic. For instance, Gross-Zagier [GZ86] prove their celebrated theorem on ranks of elliptic curves by extensively studying the arithmetic intersection theory of Heegner points on modular curves. This program has been extended to more complicated Shimura varieties in various directions (see [KRY06], [YZZ13], [YZ17], also [LL21]), and arithmetic intersection theory has been crucial to the development of this program.

In a different direction, the proof of the Andre-Oort conjecture heavily relies on the Average Colmez conjecture (interesting in its own right), whose proof ([AGHMP18], [YZ18]) relies on the Arakelov intersection theory of special cycles in orthogonal Shimura varieties.

Most relevant to our paper is the work of Charles [Cha18] proving that two elliptic curves over a number field are isogenous modulo infinitely many primes. This is achieved by studying the arithmetic intersection theory of the modular curve by an approach inspired by work of Chai-Oort, who prove the analogous result over function fields [CO06, Proposition 7.3]. This work has been generalized to the splitting of abelian surfaces over global fields [ST20], [SSTT22], [MST22b], (where the authors prove that abelian surfaces over global fields are isogenous to a product of elliptic curves modulo infinitely many primes) and to Picard-rank jumping results [SSTT22] [MST22a]. These theorems are obtained by proving that any fixed arithmetic curve intersects the Hecke translates of an appropriate “special divisor” at infinitely many points. These special divisors solve moduli problems, and (roughly speaking) parameterize abelian varieties with extra endomorphisms. This moduli-theoretic interpretation of special divisors is crucial to understanding and computing local and global intersections.

Despite the proofs relying heavily on this moduli theory of special divisors, these problems can be naturally phrased in the setting of arbitrary subvarieties of a Shimura variety. Specifically, let \mathcal{S} denote a Shimura variety, and let $C, D \subset \mathcal{S}$ be subschemes having complementary dimension. How does the set of points on C isogenous to some point of D distribute in C ? In a recent AIM workshop, the participants developed a framework that makes the following prediction.

Conjecture 1. *Let \mathcal{S} denote a simple Shimura variety over \mathbb{F}_q , and suppose that $C, D \subset \mathbb{F}_q$ are generically ordinary subvarieties having complementary dimension. Then, the set of points in C isogenous to some point of D is Zariski-dense in C . Further, the subset of $C \times D$ consisting of pairs of isogenous points is dense in $C \times D$.*

This conjecture is inspired both by the work in [Cha18, CO06, ST20, SSTT22, MST22b] as well as the Hecke orbit conjecture. There is also an arithmetic (i.e., number-field) analogue of this conjecture. These conjectures have several applications. For instance, they would imply that any abelian fourfold over a global field should be isogenous to a Jacobian modulo infinitely many primes! This is all the more striking as the existence of abelian fourfolds over global fields *not* isogenous to Jacobians has been established in [ST21] and [Tsi12].

In [G22], the first-named author proves this conjecture in the setting of $X(1)^n$. In this paper, we establish this conjecture for Hilbert modular surfaces (with some mild constraints).

Theorem 1.1. *Let F denote a real quadratic field, and let p denote a rational prime that splits in F . Let \mathcal{H} denote the mod p Hilbert modular surface associated to \mathcal{O}_F , and let $C, D \subset \mathcal{H}$ denote two generically ordinary, proper, curves in \mathcal{H} , at least one of which is ample. Then Conjecture 1 holds for $C, D \subset \mathcal{H}$.*

1.1 Outline of proof

Our proof is inspired by Chai-Oort’s work. Roughly speaking, we consider a sequence of Hecke operators T_n , and consider the intersections $(T_n(C).D)$. In order to prove our theorem, it suffices to prove that the local contributions of any finite set of points $x_1 \dots x_m \in D$ to the intersection $(T_n(C).D)$ is smaller than the global intersection number $(T_n(C).D)$ for some n . However, we encounter several additional difficulties that aren’t present in their work.

Firstly, their ambient space is a product of curves, and their proof makes crucial use of the product structure. In the hitherto solved cases where the ambient space is *not* a product variety, either C or D is special and the local intersection numbers at any fixed point is bounded using the moduli interpretation of the special divisor (see [SSTT22, Lemma 7.2], and [MST22b, Lemma 7.2.1]). We overcome this first difficulty by constructing a *local* product structure at every closed point (see Proposition 2.2) which is adapted to a very specific set of Hecke operators that exist only in positive characteristic. In fact, these Hecke operators endow \mathcal{H} with a partial Frobenius structure (see 2). This construction allows us to control the local intersection multiplicities (see Theorem 3.1).

The second difficulty is in estimating the global intersection number. The previously known cases (in the setting of special divisors) use earlier results establishing the modularity of sequences of special divisors ([Bor99]) to compute the global intersection numbers. In our setting, we have no such modularity results and we instead compute the intersections of $T_n(C)$ with the non-ordinary locus of \mathcal{H} , and then use the ampleness of D to control $(T_n(C).D)$. While doing so, we establish a result (Theorem 1.2) pertaining to the change of Faltings height under p -power isogeny which is of independent interest.

1.2 Change of Faltings height under p -power isogeny

The Faltings height of an abelian variety over a number field is defined to be the Arakelov degree of the Hodge bundle. The change of the Faltings height of an Abelian variety under isogeny was a crucial ingredient in the proof of the Mordell Conjecture by Faltings.

In the function field case, the definition of Faltings height for a proper curve is even simpler. Suppose \mathcal{A}/C is a family of abelian varieties over a proper curve C . Then we define the Faltings height of \mathcal{A}/C to just be the degree of the Hodge bundle. In our setting, this height only depends on the image of C in \mathcal{H} . The prime-to- p Hecke operators on \mathcal{H} are étale, and this makes it easy to compute the change in Faltings height under prime-to- p isogeny. The non-étaleness of p -power Hecke operators makes computing the change-in-height a formidable prospect. Further, p -power Hecke correspondences aren't even separable!

Indeed, the only prior work pertaining to this that we are aware of is [GP20]. They consider an isogeny $\varphi : E_1 \rightarrow E_2$ between two elliptic curves over $k(C)$ for C a curve over \mathbb{F}_q and [GP20, Theorem A] proves that

$$h(j(E_2)) = \frac{\deg_{\text{ins}}(\varphi)}{\deg_{\text{ins}}(\hat{\varphi})} h(j(E_1))$$

where h denotes the Weil height on $X(1)$ and $\deg_{\text{ins}}(\varphi)$ denotes the inseparable degree of φ . The proof there proceeds by an identification of the Weil height with an intersection number and since the Picard group of $X(1)$ is simply \mathbb{Z} , and the only p -power Hecke operators (mod p) are iterates of Frobenius and Verschiebung. This lets one calculate the effect of a Hecke translation on the Picard group simply as multiplication by an integer.

In our case, the Picard group of \mathcal{H} is complicated and the action of p -power Hecke correspondences (which are not étale!) on the Picard group is not a-priori easy to compute. Nevertheless we fully compute the change in Faltings height of an abelian surface with real multiplication by \mathcal{O}_F under p -power isogenies that respect the \mathcal{O}_F -action.

Suppose C is a proper curve and \mathcal{A}/C is a generically ordinary abelian surface with endomorphisms by \mathcal{O}_F (with F a real quadratic field as in Theorem 1.1) so that the image of C under the corresponding map $C \rightarrow \mathcal{H}$ is an ample divisor of \mathcal{H} . Suppose moreover that \mathcal{A}_n/C is another abelian surface, also having endomorphisms by \mathcal{O}_F , isogenous to \mathcal{A}/C by a purely inseparable isogeny $\varphi : \mathcal{A}' \rightarrow \mathcal{A}$ which respects the \mathcal{O}_F -actions, whose kernel is cyclic of size p^n (indeed, this is the hardest case to treat, and the general case can be deduced from this case). Let $h_F(\mathcal{A}/C)$ denote the Faltings height of the abelian variety \mathcal{A}/C , which is defined to be the degree of the Hodge bundle ω of \mathcal{H} restricted to C .

Theorem 1.2. *The height of \mathcal{A}_n/C grows exponentially in n .*

In fact, we prove an exact formula for the change of height under inseparable isogenies (Theorem 4.2) but for simplicity of notation, we will be content with the version above for the introduction. We prove this result by comparing the Hodge bundle to the class of the Hasse invariant, and using our computation of the intersection of Hecke orbits of C with the non-ordinary locus, and the product structure that we construct.

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2 Background material

We provide some background (and standardize notation) on Hilbert modular surfaces in this section, define the notion of a partial Frobenius structure and prove a theorem about curves parametrizing abelian varieties without extra endomorphisms.

Let F be a real quadratic field, \mathcal{O}_F its ring of integers, $p = \mathfrak{P}_1\mathfrak{P}_2$ a prime that splits in \mathcal{O}_F and \mathfrak{a} a fractional ideal of \mathcal{O}_F . It is a general result that there exists a scheme parametrizing Abelian surfaces with endomorphisms by \mathcal{O}_F . More precisely:

Definition 1. Let $\mathcal{H}_n^{\mathfrak{a}}$ be the moduli functor that associates to a \mathbb{F}_p scheme S the groupoid of tuples $(A, \iota, \lambda, \eta)$ where:

1. $A \rightarrow S$ is an abelian surface;
2. $\iota : \mathcal{O}_F \rightarrow \text{End}_S(A)$ is a ring homomorphism ;
3. $\lambda : \mathfrak{a} \rightarrow \text{Hom}_{\mathcal{O}_F}^{\text{Sym}}(A, A^V)$ is an \mathcal{O}_F -linear homomorphism such that $\lambda(a)$ is an \mathcal{O}_F -linear polarization of A for every totally positive $a \in \mathfrak{a}$, and the homomorphism $A \otimes_{\mathcal{O}_F} \mathfrak{a} \xrightarrow{\sim} A^V$ induced by λ is an isomorphism of abelian surfaces;
4. $\eta : (\mathcal{O}_F/n\mathcal{O}_F)_S^2 \rightarrow A[n]$ is an \mathcal{O}_F -linear isomorphism from the constant group scheme $(\mathcal{O}_F/n\mathcal{O}_F)_S^2$ to $A[n]$.

It is known that $\mathcal{H}_n^{\mathfrak{a}}$ is represented by a Deligne-Mumford stack over \mathbb{F}_p and when $n \geq 3$, it is even represented by a scheme (see [Gor02]). From now on, we fix some $n \geq 3$ that is prime to p , a polarization \mathfrak{a} and denote the representing scheme by \mathcal{H} .

We now set up some notation. Let \mathcal{A}/\mathcal{H} denote the universal Abelian surface over \mathcal{H} , and let \mathcal{G} denote its p -divisible group. We have that $\text{End}(\mathcal{A}) = \mathcal{O}_F$, and as p splits in F , we have that $\text{End}(\mathcal{G}) = \mathcal{O}_F \otimes \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p$. This implies that $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. At any point $x \in \mathcal{H}$, let \mathcal{A}_x and $\mathcal{G}_{i,x}$ denote the pullback of \mathcal{A} and \mathcal{G}_i to x . Note that both factors \mathcal{G}_i are one-dimensional and have height 2.

Lemma 2.1. *Let $x \in \mathcal{H}(\overline{\mathbb{F}}_p)$ and let the deformation space of \mathcal{G}_i be $\text{Spf } \overline{\mathbb{F}}_p[[t_i]]^1$. Then $\widehat{\mathcal{H}}_x$ is canonically isomorphic to $\text{Spf } \overline{\mathbb{F}}_p[[t_1]] \times \text{Spf } \overline{\mathbb{F}}_p[[t_2]]$.*

Proof. The formal neighbourhood $\widehat{\mathcal{H}}_x$ parameterizes formal deformations of \mathcal{A}_x which admit an action of \mathcal{O}_F (compatible with the action of \mathcal{O}_F on \mathcal{A}_x). By the Serre-Tate lifting theorem, we have that this is the same as formal deformations of \mathcal{G}_x which admit an action of $\mathcal{O}_F \otimes \mathbb{Z}_p$ – but this is the same as pairs of p -divisible groups which lift the pair $(\mathcal{G}_{1,x}, \mathcal{G}_{2,x})$. The lemma follows. \square

Henceforth, we will refer to the decomposition above as the product structure on $\widehat{\mathcal{H}}_x$.

2.1 Partial Frobenius structure and coordinates

For ease of exposition, we henceforth assume that \mathfrak{P}_1 (and therefore \mathfrak{P}_2) are trivial in the narrow class group. Otherwise, the analysis below will go through identically, except we work with p^a (and Frob_{p^a}) in place of p (and Frob_p), where a is the order of \mathfrak{P}_i in the narrow class group.

¹Implicit here is that the deformation space of a p -divisible group having height 2 and dimension 1 is one-dimensional and smooth.

Definition 2. We say a surface X over \mathbb{F}_p has partial Frobenius structure if there is a factorization

$$\text{Frob}_p = \pi_1 \pi_2,$$

where $\pi_i : X \rightarrow X$ are maps which satisfy the following condition:

For any q a power of p , locally around any $x \in X(\mathbb{F}_q)$, we can find coordinates t_1, t_2 such that:

$$(\pi_i^{q^*})(t_j) = \begin{cases} t_j^q & \text{if } i = j, \\ t_j & \text{otherwise.} \end{cases} \quad (1)$$

A product of curves over \mathbb{F}_p is one example of a variety that has a partial Frobenius structure (where the π_1, π_2 are induced from each factor independently), and there is an obvious choice of coordinates as in (1). In particular, the product of modular curves will be a very pertinent example.

The key assumption that p splits in \mathcal{O}_F results in a partial Frobenius structure on \mathcal{H} . In this case, we may define the maps π_i using our moduli interpretation. Suppose S is an arbitrary \mathbb{F}_p scheme and \mathcal{A}/S is an abelian surface corresponding to a point $x \in \mathcal{H}(S)$. Define $\pi_1(x)$ to be the point y corresponding to the abelian scheme \mathcal{A}/G_1 , where $G_1 \subset \mathcal{G}_{1,x}$ is the kernel of Frobenius. According to Section 2.2 of [Pap95], $\mathcal{A} \mapsto \mathcal{A}/G_1$ defines a morphism $\mathcal{H}^a \rightarrow \mathcal{H}^{\mathfrak{F}_1^a}$. As \mathfrak{F}_1 is trivial in the narrow class group, we have indeed obtain a morphism $\pi_1^a : \mathcal{H}^a \rightarrow \mathcal{H}^a$.

We define $\pi_2(x)$ analogously. Clearly, $\text{Frob}_p = \pi_1 \pi_2$. Given this description, we have for any $x \in \mathcal{H}(\mathbb{F}_q)$, π_1 induces the q^{th} -power map on the deformation space of $\mathcal{G}_{1,x}$ and leaves the deformation space of $\mathcal{G}_{2,x}$ unchanged. We therefore have the following proposition:

Proposition 2.2. *\mathcal{H} has a partial frobenius structure, with $\text{Frob} = \pi_1 \pi_2$. Further, at every point $x \in \mathcal{H}(\overline{\mathbb{F}_p})$, the coordinates t_1, t_2 are just the coordinates induced by the product structure as in Lemma 2.1.*

We will use the following lemma in the proof of the theorem immediately after.

Lemma 2.3. *Assume K is a global function field, and K_v is a completion of K at some place v . Let \mathcal{G}/K be an ordinary p -divisible group over K with height 2 and dimension 1, and \mathcal{G}_{K_v} be its base change to K_v . If the connected-étale exact sequence for \mathcal{G}_{K_v} splits then the connected-étale exact sequence for \mathcal{G} splits over K .*

Proof. Jiang proves a more general result in [Jia], but we include a different proof here for completeness. For any n the connected-étale exact sequence of $\mathcal{G}/K_v[p^n]$ splits over some finite extension L_n/K . We will prove that L_n can be chosen to be a subfield of K_v .

In order to show that we can choose $L_n \subset K_v$, we use the representability of $\text{Hom}_{\text{gp sch}}(H, G)$ for any finite flat group schemes H, G . In general, $\text{Hom}_S(X, Y)$ is representable if X, Y are finite type schemes over S and moreover, X is flat, projective and Y is quasi-projective over S [§4c, [Gro60]] and the conditions for being a homomorphism of groups defines a closed subscheme. Splittings of the connected étale sequence

$$0 \rightarrow G_{n,\text{conn}} \rightarrow G_n \xrightarrow{\pi} G_{n,\text{ét}} \rightarrow 0$$

can be identified with the closed subset of points of $f \in \text{Hom}_{\text{gp sch}}(G_{n,\text{ét}}, G_n)$ such that $f \circ \pi = \text{id}$. In particular, they are parametrized by a scheme which we call $\text{Split} / \text{Spec } K$.

We therefore have a splitting over K_v, L_n , i.e., K_v, L_n valued points of Split . We now consider $K_v, L_n \subset M$ for some algebraically closed field M . Let $\alpha \in \text{Split}(M)$ be defined over some subfield

L . Since (upto etale-twists) $G_{n,\text{conn}} \cong \mu_{p^n}$, $G_{n,\text{ét}} \cong p^{-n}\mathbb{Z}_p/\mathbb{Z}_p$, their automorphisms are defined over the ground field, we have that every element of $\text{Split}(M)$ is defined over L . Therefore,

$$\text{Split}(K_v) = \text{Split}(M) = \text{Split}(L_n) \implies \text{Split}(K_v) = \text{Split}(L_n \cap K_v).$$

Therefore, we can replace L_n by $L_n \cap K_v$ to assume that $L_n \subset K_v$.

We now claim that L_n/K is separable. Indeed, being a one-parameter function field, K admits a unique degree p inseparable extension necessarily containing all p th roots of K , and which therefore contains the p th root of a uniformizer of K at v , which therefore can't be contained in K_v . It follows that L_n/K is indeed separable.

The splitting behaviour of the connected-etale exact sequence for any p -divisible group is insensitive to separable extensions, and therefore the connected-etale exact sequence for $\mathcal{G}[p^n]$ splits over K for all n . The result follows. □

Theorem 2.4. *Let $D \subset \mathcal{H}$ be a generically ordinary, reduced, irreducible curve such that the generic abelian variety $A_{/K(D)}$ does not have an extra endomorphism². We identify $\widehat{\mathcal{O}}_{\mathcal{H},x}$ with $\overline{\mathbb{F}}_p[[t_1, t_2]]$ as in Lemma 2.1. Let $\widehat{\mathcal{O}}_{\mathcal{H},x}/(f_{D,x})$ be the formal completion of the local ring of D at x . Then t_i does not divide $f_{D,x}$ for any i .*

Remark 1. As an analogous situation to the above lemma, consider $X = X(1) \times X(1)$, then locally around x , we may choose t_i to be the corresponding coordinate of the i -th $X(1) \cong \mathbb{P}^1$.

Proof. Without loss of generality, we may replace D with its normalization, and pull back the universal abelian surface to D . Therefore, we may assume that D is smooth - however, the map $D \rightarrow \mathcal{H}$ need no longer be an embedding. Nevertheless, given any point $x \in D$, we obtain a map of formal schemes $\psi : \widehat{D}_x \rightarrow \widehat{\mathcal{H}}_x$ (we abuse notation by letting x denote both the point of D and its image in \mathcal{H}). Let $f_{D,x} \in \widehat{\mathcal{O}}_{\mathcal{H},x}$ denote the defining equation of the image of \widehat{D}_x .

It suffices to prove that if t_i divides $f_{D,x}$ for $i = 1$ or 2 , then either D is not generically ordinary, or $A/K(D)$ has extra endomorphisms. Without loss of generality, assume $t_1 \mid f_{D,x}$. If $\mathcal{G}_{x,1}$ is supersingular, then by the description of the non-ordinary locus in Section 4.1, we see that A/D must non-ordinary. Therefore, we suppose that $\mathcal{G}_{x,1}$ is ordinary. The inclusion of the divisor corresponds to

$$\begin{aligned} \widehat{D}_x &\cong \text{Spec } \overline{\mathbb{F}}_p[[u]] \rightarrow \widehat{\mathcal{H}}_{d,x} \cong \text{Spec } \widehat{\mathcal{O}}_{\mathcal{H},x}; \\ t_1 &\xrightarrow{\psi} 0, \\ t_2 &\xrightarrow{\psi} \bar{t}_2 \in \overline{\mathbb{F}}_p[[u]]. \end{aligned}$$

Here, \widehat{D}_x is the formal completion of D at x . As in Lemma 2.1, we know $\widehat{\mathcal{O}}_{\mathcal{H},x}$ is the product of the deformation spaces of $\mathcal{G}_{i,x}$. Notice that $\psi(t_1) = 0$ implies that $\mathcal{G}_{x,1}$ remains constant along \widehat{D}_x . In other words,

$$\mathcal{G}_{\widehat{D}_x} = (\mathcal{G}_{x,1} \times \widehat{D}_x) \times \mathcal{G}_{\widehat{D}_x,2}.$$

The connected-étale exact sequence of $\mathcal{G}_{x,1}$ splits since it is over a perfect field and hence, so does the connected-étale exact sequence of $\mathcal{A}_{\widehat{D}_x}[p^\infty]_1$. By Lemma 2.3, this implies that the connected-étale exact sequence for \mathcal{G}_1 already splits over $K(D)$. Consequently, $\text{End}(\mathcal{G}_{D,1}) = \mathbb{Z}_p \oplus \mathbb{Z}_p$, whence

²i.e., the ring of endomorphisms is \mathcal{O}_F .

$\text{End}(\mathcal{G}_D)$ is strictly larger than $\mathbb{Z}_p \otimes \mathcal{O}_F$. Applying the p -adic Tate conjecture for endomorphisms of abelian varieties implies \mathcal{A}_D has endomorphism ring larger than \mathcal{O}_F , as required. \square

3 Local intersection on varieties with a split Frobenius

In this section, we assume that X is a surface over \mathbb{F}_p with a partial Frobenius structure. Throughout this section, we fix such a system of coordinates around each point as in (1). As in Section 2, Hilbert modular surfaces where $p = \mathfrak{P}_1 \mathfrak{P}_2$ with \mathfrak{P}_i trivial in the narrow class group of \mathcal{O}_F are examples of such surfaces. As noted in Section 2, as long as p splits completely in \mathcal{O}_F , \mathcal{H} will admit a partial p^a -Frobenius structure where a is the order of \mathfrak{P}_1 in the narrow class group.

Note that the Frobenius (and hence the π_i) are universal homeomorphisms, i.e., they induce homeomorphisms on the underlying topological space. In particular, the preimage of any point under the π_i is also exactly one point.

The results in this section will be purely local around a point $x \in X(\mathbb{F}_q)$. Note that π_1 need not fix x but some power of it will. so we may assume that $X = \text{Spec } R$ with $R = \mathbb{F}_q[[t_1, t_2]]$ and x is the origin given by the vanishing of t_1, t_2 . In local co-ordinates (a power of) π_1 corresponds to (a power of) Frobenius along one of the co-ordinates.

Now, let $C, D \subset X$ be two curves and $x \in C \cap D$ and define $C_n = (\pi_1^n)^{-1}(C)$ to be the pullback of C under π_1^n , where n satisfies $\pi_1^n(x) = x$. In this section, we prove results about the intersection numbers $C_n \cdot D$. As is clear, Theorem 3.1 (1) is inspired by [CO06, Proposition 7.3].

Theorem 3.1. *Let $C = V(f), D = V(g)$ locally around $x \in \mathcal{H}$ and $n = nr(x) \rightarrow \infty$ be part of a sequence of integers divisible by $r(x)$ and increasing without bound. Suppose moreover that $t_1, t_2 \nmid f$ and $t_1 \nmid g$.*

1. *Suppose $t_2 \nmid g$. Then, the local intersection number $(C_n \cdot D)_x$ is bounded as $n \rightarrow \infty$.*
2. *Suppose $t_2 \mid g$. Then, the local intersection number $(C_n \cdot D)_x \rightarrow \infty$ as $n \rightarrow \infty$.*

The analogous result holds with the roles of π_1, π_2 (and t_1, t_2) reversed.

Proof. In order to compute intersection numbers, we can replace D by its normalization and consider each component separately [Ful13, Example A.3.1]. We therefore have $\hat{\mathcal{O}}_{D,x} = \overline{\mathbb{F}}_p[[u]]$ and we suppose that the morphism $D \rightarrow \mathcal{H}$ is locally around x given by

$$\begin{aligned} \overline{\mathbb{F}}_p[[t_1, t_2]] &\rightarrow \overline{\mathbb{F}}_p[[u]] \\ t_i &\rightarrow \alpha_i. \end{aligned}$$

For the first part of the theorem, $\alpha_1, \alpha_2 \neq 0$ since $t_1, t_2 \nmid g$. We therefore define $\alpha_i = a_i u^{k_i}$ with a_i a unit and $k_i \geq 1$. Since $x \in C$ and $r(x) \mid n$, we have $x \in C_n$. As C is defined by $f(t_1, t_2)$, C_n is defined by $f(t_1^{p^n}, t_2)$ so that

$$(C_n \cdot D)_x = \dim_{\overline{\mathbb{F}}_p} \frac{\overline{\mathbb{F}}_p[[u]]}{f(\alpha_1^{p^n}, \alpha_2)}.$$

By assumption, $t_1 \nmid f$ so that we can write $f(t_1, t_2) = t_2^e \tilde{f}(t_2) + t_1 h(t_1, t_2)$ for $e \geq 1$ and $\tilde{f}(t_2)$ a unit. Therefore,

$$\begin{aligned} f(\alpha_1^{p^n}, \alpha_2) &= \alpha_2^e \tilde{f}(\alpha_2) + \alpha_1^{p^n} h(\alpha_1^{p^n}, \alpha_2) \\ &= a_2^e u^{k_2 e} \tilde{f}(a_2 u^{k_2}) + a_1^{p^n} u^{p^n k_1} h(a_1^{p^n} u^{p^n k_1}, a_2 u^{k_2}). \end{aligned}$$

For n large enough, the u -adic valuation of the second term is larger than the u -adic valuation of the first term since \tilde{f} is a unit. Consequently (for n large enough), $f(\alpha_1^{p^n}, \alpha_2)$ is divisible by exactly u^{ek_2} which proves that $(C_n \cdot D)_x = ek_2$ which is independent of n .

In the second case, $\alpha_2 = 0$. Let $f(t_1, 0) = t_1^d w(t_1)$ with $d \geq 1$ and $w(t_1)$ a unit. We then have (with k_1 as in the first part)

$$(C_n \cdot D)_x = \dim_{\overline{\mathbb{F}}_p} \frac{\overline{\mathbb{F}}_p[[u]]}{f(\alpha_1^{p^n}, 0)} = dk_1 p^n \quad (2)$$

which proves that $(C_n \cdot D)_x \rightarrow \infty$ as $n \rightarrow \infty$. □

We have the following corollary.

Corollary 3.2. *Let C be as above, and suppose now that $Z_i \subset \widehat{\mathcal{H}}_x$ are the formal curves defined by $t_i = 0$ for $i = 1, 2$. Then, $(C_n \cdot Z_1)_x = (C \cdot Z_1)_x$ and $(C_n \cdot Z_2)_x = p^n (C \cdot Z_2)_x$.*

Proof. The first equality follows by inspection and the second from Equation (2). □

4 Change of height under p -power isogenies

In this section, we will describe some instances in which we can describe how the height of a generically ordinary proper curve $C \subset \mathcal{H}$ changes under isogenies induced by the π_i . The idea is to use the fact that the Faltings height equals (up to a scaling factor of $\frac{1}{p-1}$) the intersection of C with the non-ordinary locus.

4.1 The non-ordinary locus.

Let $x \in Z(\mathbb{F}_q)$ be a point, where $Z \subset \mathcal{H}$ is the non-ordinary locus. As in Section 2, the formal completion $\widehat{\mathcal{H}}_x = \text{Spf } \mathbb{F}_q[[t_1, t_2]]$, where t_i controls the deformation theory of $\mathcal{G}_{x,i}$ and leaves $\mathcal{G}_{x,i+1}$ constant (here, the indices are read modulo 2). Therefore, if $\mathcal{G}_{x,1}$ is ordinary and $\mathcal{G}_{x,2}$ is supersingular, the local equation of Z at x is just $t_1 = 0$; if $\mathcal{G}_{2,x}$ is ordinary and $\mathcal{G}_{1,x}$ is supersingular, then the local equation is $t_2 = 0$; and if x itself is supersingular, then the local equation is $t_1 t_2 = 0$.

We now recall the global geometry of $Z \subset \mathcal{H}$, following [BG99, §4, Theorem 4.2]. The non-ordinary locus is a union of smooth irreducible curves which intersect transversally at supersingular points. There are two ‘types’ of curves – curves of type 1 and curves of type 2. Every non-supersingular point x on curves of type 1 has the property that $\mathcal{G}_{1,x}$ is ordinary while every non-supersingular point y on curves of type 2 has the property that $\mathcal{G}_{2,y}$ is ordinary. Curves of type 1 never intersect (and neither do curves of type 2) while every supersingular point has exactly one curve of type 1 and one curve of type 2 passing through it. Finally, the local equations of these curves are precisely as described in the paragraph just above.

For brevity, we write $Z = Z_1 \cup Z_2$, where Z_i is the union of curves of type i . The main result of this section is the following:

Theorem 4.1. *Let $C/\mathbb{F}_q \subset \mathcal{H}$ denote a proper generically ordinary curve, and suppose that all the non-ordinary points of C are contained in $C(\mathbb{F}_q)$ for some $q = p^{n_0}$. Then as n goes to infinity, we have $((\pi_1^{n_0 n})^{-1}(C) \cdot Z) = q^n (C \cdot Z_2) + (C \cdot Z_1)$, and $((\pi_2^{n_0 n})^{-1}(C) \cdot Z) = q^n (C \cdot Z_1) + (C \cdot Z_2)$.*

Proof. This follows directly from the description of the non-ordinary locus as well as Corollary 3.2. \square

We keep the notation in the previous result. Recall that we define the Faltings height of the abelian variety \mathcal{A}_C to be the degree of the Hodge bundle ω restricted to C . As in [MST22a] and [MST22b], the class of the Hodge bundle can be expressed in terms of the non-ordinary locus. To be precise, we have that Z is the vanishing locus of a section of ω^{p-1} (see [Box15, Sections 1.4,1.5, and Theorem 6.23]). This, together with Theorem 4.1 gives the following result.

Theorem 4.2. *Notation as above. The Faltings height of $\mathcal{A}_{(\pi_1^{n_0 n})^{-1}(C)}$ is*

$$h_F(\mathcal{A}_{(\pi_1^{n_0 n})^{-1}(C)}) = \frac{q^n(C \cdot Z_2) + (C \cdot Z_1)}{p-1},$$

and $\mathcal{A}_{(\pi_2^{n_0 n})^{-1}(C)}$ is

$$h_F(\mathcal{A}_{(\pi_2^{n_0 n})^{-1}(C)}) = \frac{q^n(C \cdot Z_1) + (C \cdot Z_2)}{p-1}.$$

In particular, if C is also ample³, we have $h_F(\mathcal{A}_{(\pi_1^{n_0 n})^{-1}(C)}) \asymp q^n \asymp h_F(\mathcal{A}_{(\pi_2^{n_0 n})^{-1}(C)})$.

5 Just-likely intersections on Hilbert Modular Surfaces

Definition 3. Given two proper curves $C, D \subset \mathcal{H}$ defined over \mathbb{F}_q parametrizing Abelian surfaces $\mathcal{A}_C, \mathcal{A}_D$, we let

$$\mathcal{I}(C, D) = \{(x, y) \in \mathcal{H}^2 : \mathcal{A}_{C,x} \text{ is isogenous to } \mathcal{A}_{D,y} \text{ through a power of } \pi_i^n\}.$$

Moreover, let

$$\widetilde{\mathcal{I}}(C, D) = \{(x, y) \in \mathcal{H}^2 : \mathcal{A}_{C,x} \text{ is isogenous to } \mathcal{A}_{D,y} \text{ through a } p\text{-power isogeny}\}.$$

Note that $\mathcal{I}(C, D) \subset \widetilde{\mathcal{I}}(C, D)$.

We use the results of Section 3 to prove:

Theorem 5.1. *Let C, D be as above and suppose that $\mathcal{A}_C, \mathcal{A}_D$ parametrized respectively by C, D have no extra endomorphisms generically and moreover, suppose that D is ample.*

Then $\mathcal{I}(C, D)$ has infinitely many points not contained in a finite union of axes of the form $\bigcup_i \{x_i\} \times D \cup \bigcup_j C \times \{y_j\}$.

Remark 2. If one of the curves generically does have extra endomorphisms, then the theorem is still true by the results of [MST22b].

Theorem 5.2. *Suppose that $C, D \subset \mathcal{H}$ are two curves such that $\mathcal{I}(C, D)$ has infinitely many points not contained in a finite union of the axes as in the previous theorem. Then, $\widetilde{\mathcal{I}}(C, D)$ is dense in $C \times D$.*

³This would imply that $(C \cdot Z_1)$ and $(C \cdot Z_2)$ are both positive. Even if C is not ample, the curve $Z_1 \cup Z_2$ is ample, and so either $(C \cdot Z_1)$ or $(C \cdot Z_2)$ must be positive, whence we obtain the same asymptotic formula for the Faltings heights of either $\mathcal{A}_{(\pi_1^{n_0 n})^{-1}(C)}$ or $\mathcal{A}_{(\pi_2^{n_0 n})^{-1}(C)}$.

The strategy for the proof of Theorem 5.1 is as follows. We first prove that the local intersection numbers $d_n = ((\pi_1^n)^{-1}(C).D)_x$ are bounded for any $x \in D$. Next, we prove that the global intersection numbers $((\pi_1^n)^{-1}(C).D)$ are unbounded, by comparing these numbers with the quantities $((\pi_1^n)^{-1}(C).Z)$, using Theorem 4.1 and the ampleness of D . Finally, Theorem 5.2 follows by a "soft" argument. Note that Theorem 5.2 always holds for two curves which satisfy the conclusions of Theorem 5.1.

Throughout, we use the local coordinates t_1, t_2 of Lemma 2.1. Let C, D be as in Theorem 5.1. We prove Theorem 5.1 through a sequence of lemmas. Recall that for a point $x \in X$, $r(x)$ is the smallest value so that $\pi_1^{r(x)}(x) = \pi_2^{r(x)}(x) = x$.

Lemma 5.3. *For any point $x \in C$, the intersection number of $(\pi_i^n)^{-1}(C)$ with D at $(\pi_i^n)^{-1}(x) = x$ is bounded absolutely as n ranges through the multiples of $r(x)$.*

Proof. Around any point $x \in C, D$, the curves C, D don't contain the divisors $t_i = 0$ since otherwise they would either be generically non-ordinary or generically have extra endomorphisms (by Lemma 2.4).

Therefore, part (1) of Theorem 3.1 applies and shows that the intersection number of $((\pi_i^n)^{-1}(C).D)$ at $(\pi_i^n)^{-1}(x)$ is absolutely bounded as $n = mr(x) \rightarrow \infty$. \square

The next lemma deals with the global intersection number. We have that $(C.Z) = (C.Z_1) + (C.Z_2)$ is positive, so we assume without loss of generality that $(C.Z_2)$ is positive and define $C_n = (\pi_i^n)^{-1}(C)$.

Lemma 5.4. *The global intersection number $C_n \cdot Z \rightarrow \infty$ as $n \rightarrow \infty$ through the multiples of $r(x)$.*

Proof. This follows immediately from Theorem 4.1. \square

Lemma 5.5. *Let D now be an arbitrary ample, proper divisor. Then, the global intersection number $C_n \cdot D \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. We will reduce to the case where our divisor is Z , the Hasse locus.

Since D is ample, we can find a large enough m such that $mD - Z$ is also ample. Therefore:

$$C_n \cdot (mD - Z) \geq 0 \iff mC_n \cdot D \geq C_n \cdot Z$$

and since $C_n \cdot Z \rightarrow \infty$ by the previous Lemma 5.4, $C_n \cdot D$ is also unbounded. \square

We are now ready to prove the main results of this paper.

Proof of Theorem 5.1. Let $S_n \subset C_n = (\pi_1^n)^{-1}(C)$ be the set of points $x_{i,n}$ on C_n isogenous to some point $y_{i,n}$ on D . Note that there is a unique $x_i \in C$ so that $\pi_1^n(x_{i,n}) = x_i$ since π_1 is a universal homeomorphism. We first show that $|S_n| \rightarrow \infty$ as $n \rightarrow \infty$. For contradiction, suppose that $|S_n| \leq N$. Then, for $r \gg 0$ and n ranging through the multiples of the lcm of $r(x_1), \dots, r(x_N)$:

$$(C_{rn} \cdot D) = \sum_{i=1}^N (C_{rn} \cdot D)_{x_{i,n}} \leq NC_0$$

where the bound on the right hand side follows from Lemma 5.3 and C_0 is some constant. On the other hand, the left hand side goes to infinity by Lemma 5.5 which provides us with our contradiction.

We have thus shown that there is an infinite set of points $x_1, x_2, \dots \subset C$ isogenous to some point on D . We would like to show that the corresponding points y_1, y_2, \dots on D also form an infinite set. Suppose otherwise for contradiction.

Thus, we can find an infinite subset x_{i_1}, x_{i_2}, \dots isogenous to the same point y on D through the isogenies corresponding to $\pi_1^{n_1}, \pi_2^{n_2}, \dots$ with $n_i \rightarrow \infty$. That is:

$$\pi_1^{n_j}(y) = x_{i_j}.$$

On the other, the orbit of π_1 on any $\overline{\mathbb{F}}_q$ point of \mathcal{H} is finite since, in local co-ordinates, π_1 just corresponds to the Frobenius on one of the co-ordinates which certainly has a finite orbit. This forces us to identify some of the x_{i_j} which is contradictory to our assumption that the x_i are distinct.

We have thus shown that we can find two sequences $x_i, y_i \in C, D$ such that $x_i \neq x_j, y_i \neq y_j$ for $i \neq j$ and the x_i are isogenous to the y_i as required. \square

We now prove Theorem 5.2, and therefore finish the proof of Theorem 1.1.

Proof of Theorem 5.2. Suppose for contradiction that the closure of $\widetilde{\mathcal{S}}(C, D)$ is a proper subset of $C \times D \subset \mathcal{H}^2$. Let W be the complement of the axes inside the closure of $\mathcal{S}(C, D)$. By Theorem 5.1, W has positive dimension. We can therefore find an infinite sequence of points $(x_1, y_1), (x_2, y_2), \dots \in W$ such that the fields of definition of y_i individually go to infinity. We will prove that $N_i = \#\{y \in D : (x_i, y) \in W\} \rightarrow \infty$ as $i \rightarrow \infty$ which contradicts the fact that W is a closed subset of $C \times D$ and therefore has finite degree projections onto the first factor.

Indeed, if y_i has field of definition $\mathbb{F}_{q^{m_i}}$, then the size of the orbit of y_i under Frob_q has size m_i and moreover, each point in this orbit is p -power isogenous to x_i and lies on D (since D is defined over \mathbb{F}_q). Since W is defined over \mathbb{F}_q too, it is fixed by any Frobenius and the entire orbit is contained inside W proving that $N_i \geq m_i \rightarrow \infty$ as $i \rightarrow \infty$. \square

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