DE BRANGES FUNCTIONS OF SCHROEDINGER EQUATIONS

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Abstract. We characterize the Hermite-Biehler (de Branges) functions E which correspond to Shroedinger operators with L^2 potential on the finite interval. From this characterization one can easily deduce a recent theorem by Horvath. We also obtain a result about location of resonances.

1. Introduction

The Krein–de Branges theory of Hilbert spaces of entire functions was created in the mid-twentieth century to treat spectral problems for second order differential equations. The central object of the theory is a Canonical System of differential equations on the real line. The main result of the theory states that there exists a one-to-one correspondence between such systems and de Branges spaces of entire functions. Each de Branges space is generated by a single de Branges entire function E which encodes full information about the space and the differential operator. Via this result, spectral problems for differential operators translate into uniqueness and interpolation problems for spaces of entire functions. After such a translation, they can be viewed in a systematic way and treated using powerful tools of complex analysis. Since its creation, the theory has exceeded its original purpose and now extends to many fields of mathematics. Among them are complex function theory and functional model theory, spectral theory of Jacobi matrices and the theory of orthogonal polynomials, number theory and intriguing relations with the Riemann Hypothesis, see for instance [4, 9, 10, 18, 14].

The Krein–de Branges theory is a classical, yet still developing area of analysis with many important open questions. Among them is a number of 'characterization problems' where a description of de Branges functions corresponding to various important sub-classes of Canonical Systems is required (see e.g., [1, Theorems 1.4, 1.5, 6.1]). Through a standard procedure, see Section 2, many second order equations and systems can be rewritten as Canonical Systems. Among them are Schroedinger operators on an interval or half-line, Dirak systems, Krein strings, etc. Via the main theorem of the Krein–de Branges theory mentioned above, these classes of differential operators can be uniquely identified with corresponding classes of de Branges entire functions E. Descriptions (characterizations) of such classes of entire functions present a whole set of interesting and challenging problems. Most of them are still open. However, the set of de Branges functions which correspond to Schroedinger operators with L^2-potentials on the interval can be fully described using the work of E. Korotyaev [8], see Section 3.

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One of the strongest recent results in the inverse spectral theory is the theorem of M. Horvath [6] which establishes the equivalence between the defining sets for the Weyl functions with the uniquenes sets for the Paley–Wiener spaces. As was demonstrated in [6], this result implies a number of classical and recent results such as Ambarzumian's theorem, Borg's two-spectra theorem, the results by Hoschtadt–Lieberman, Gesztesy–Simon, del Rio–Simon and several others.

Achieving an understanding of connections between Horvath' methods with the Krein–de Branges theory served as a motivation for this paper. Using the result of Korotyaev together with its inversion (established in the present paper) we give a new simple proof of Horvath' theorem.

As another application of the de Branges space techniques we prove that some logarithmic strip is free of zeros of E. For physicists zeros of E are known as *resonances* (poles of the scattering matrix). There are many results of this type on the localization of the resonances (see [19, 7, 5]), but our methods allow to show the sharpness of the obtained bound.

Organization of the paper. The paper is organized as follows. In Section 2 we remind the reader the basics of the theory and give further references. In Section 3 we formulate the main results of the paper. Our main tool (an inverse characterization theorem) is proved in Section 4. Section 5 is devoted to the proof of Horvath' theorem, while the results about resonances are proved in Section 6.

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2. Preliminaries

2.1. **De Branges spaces.** Consider an entire function $E(z)$ satisfying the inequality

$$|E(z)| > |E(\bar{z})|, \qquad z \in \mathbb{C}_+,$$

and such that $E \neq 0$ on $\mathbb{R}$. Such functions are usually called *de Branges functions* (or *Hermite–Biehler functions*). The *de Branges space* $\mathcal{H}(E)$ associated with E is defined to be the space of entire functions F satisfying

$$\frac{F(z)}{E(z)} \in H^2(\mathbb{C}_+), \qquad \frac{F^\#(z)}{E(z)} \in H^2(\mathbb{C}_+),$$

where $F^\#(z) = \overline{F(\overline{z})}$, $H^2(\mathbb{C}_+)$ is a Hardy class in $\mathbb{C}_+$. It is a Hilbert space equipped with the norm $\|F\|_E = \|F/E\|_{L^2(\mathbb{R})}$. If $E(z)$ is of exponential type then all the functions in the de Branges space $\mathcal{H}(E)$ are of exponential type not greater then the type of E (see, for example, the last part in the proof of Lemma 3.5 in [4]). A de Branges space is called *short* (or *regular*) if together with every function $F(z)$ it contains $(F(z) - F(a))/(z - a)$ for any $a \in \mathbb{C}$.

One of the most important features of de Branges spaces is that they admit a second, axiomatic, definition. Let $\mathcal{H}$ be a Hilbert space of entire functions that satisfies the following axioms:

- (A1) For any $\lambda \in \mathbb{C}$, point evaluation at λ is a non-zero bounded linear functional on $\mathcal{H}$;
- (A2) If $F \in \mathcal{H},\ F(\lambda) = 0$, then $F(z)\frac{z-\bar{\lambda}}{z-\lambda} \in \mathcal{H}$ with the same norm;
- (A3) If $F \in \mathcal{H}$ then $F^{\#} \in \mathcal{H}$ with the same norm.

Then $\mathcal{H} = \mathcal{H}(E)$ for a suitable de Branges function E. This is Theorem 23 in [2].

2.2. **Canonical Systems.** Let Ω be a symplectic matrix in $\mathbb{R}^2$:

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A *canonical system* of differential equations (CS) is the system

$$\Omega X'(t) = zH(t)X(t), \tag{2.1}$$

where H is a real, locally summable, 2×2-matrix valued function on an interval $(a, b) \subset \mathbb{R}$, called a Hamiltonian of the system, and X is an unknown vector valued function $X = \begin{pmatrix} u \\ v \end{pmatrix}$. Such systems were considered by M. Krein as a general form of a second order linear differential operator. As was mentioned in the introduction, many standard classes of second order equations can be equivalently rewritten as canonical systems.

Canonical systems and de Branges spaces together constitute the so-called Krein–de Branges theory. The connection between the two is as follows. Let $X = \begin{pmatrix} A(z,t) \\ B(z,t) \end{pmatrix}$ be a solution to (2.1) satisfying some initial condition at a, for instance $X(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then for any fixed $t \in (a, b)$, the function $E_t(z) = A(z,t)+iB(z,t)$ is a de Branges function. Under some minor technical restrictions on the Hamiltonian, the corresponding spaces $\mathcal{H}(E_t)$ are nested, i.e., $\mathcal{H}(E_t)$ is isometrically embedded into $\mathcal{H}(E_s)$ for any $t < s$. In the opposite direction, any de Branges function E can be obtained this way from a canonical system (2.1), see Theorem 40 in [2] or Theorems 16-18 [17]. The solution X can be used as a kernel of an integral operator (Weyl transform) which identifies the space of vector-valued square-summable functions on (a, t) with $\mathcal{H}(E_t)$. For more on Krein–de Branges theory see [4, 2, 17].

2.3. **Schroedinger operators.** This paper is devoted to a particular case of a canonical system, the Schroedinger equation on an interval (a, b),

$$-u'' + qu = zu. \tag{2.2}$$

Since this equation can be rewritten as a canonical system (see for instance [14, 17]), it corresponds to a chain of de Branges functions/spaces as described above. Let us present a shorter way to establish this connection, without transforming the equation to the canonical form (2.1) (see [16, 14]).

Assume that the potential $q(t)$ is integrable on a finite interval (a, b), i.e. that the operator we consider is regular. We make this restriction for the sake of simplicity, although the theory can be extended to general non-regular cases, see [14]. Let $u_z(t)$

be the solution of (2.1) satisfying the boundary conditions $u_z(a) = 0$ and $u'_z(a) = 1$. The Weyl m-function is defined as

$$\mathbf{m}(z) = \frac{u_z(b)}{u'_z(b)}.$$

It is well known that the Weyl function is a Herglotz function, i.e., a meromorphic function in $\mathbb{C}$ with a positive imaginary part in $\mathbb{C}_+$.

Now let us assume that the operator with Neumann boundary conditions at a and Dirichlet boundary conditions at b is positive (otherwise one can add a large positive constant to q). In this case one may consider the Weyl function after the 'square root transform', i.e. after a change of variables $m(z) = z\mathbf{m}(z^2)$. If the operator is positive then the modified Weyl function $m(z)$ is also a Herglotz function.

The entire function

$$E(z) = zu_{z^2}(b) + iu'_{z^2}(b) \tag{2.3}$$

is the de Branges function for the corresponding canonical system. In particular, if $q \equiv 0$, then $E(z) = ie^{-iz}$, and we get the classical Paley–Wiener space PW_1 as the corresponding de Branges space $\mathcal{H}(E)$. Recall that PW_1 is the space of all entire functions of exponential type at most 1 which are in $L^2(\mathbb{R})$ or, equivalently, the Fourier image of the space $L^2(-1, 1)$.

Closely related analytic (meromorphic) function is the Weyl inner function

$$\Theta = \frac{m - i}{m + i} \quad \textit{or} \quad \Theta = \frac{E^\#}{E},$$

where $E^\#(z) = \overline{E(\bar{z})}$. Such functions will also be used in our discussions below. Each of the functions, $E(z)$, $m(z)$, $\mathbf{m}(z)$ or $\Theta(z)$, determines the operator uniquely, as follows from classical results of Marchenko [12, 13].

3. Main results

3.1. Spectra of the Schroedinger operators. Every de Branges function comes from a canonical system, as follows from the main result of Krein–de Branges theory. But which of the de Branges functions can be obtained from Schroedinger operators via the procedure described above? How the properties of the potential translate into the properties of E?

Once again, let us consider the Schroedinger equation

$$-u'' + qu = zu \tag{3.1}$$

on the interval $[0, 1]$. Denote by σ_{DD} the spectrum of L with Dirichlet boundary conditions at the endpoints, i.e., $\sigma_{DD} = \{\lambda_n^2\}$ such that for each λ_n^2 there exists a solution of (3.1) with $z = \lambda_n^2$ satisfying $u(0) = u(1) = 0$. Similarly, $\sigma_{ND} = \{\mu_n^2\}$ will denote the spectrum for the mixed Neumann/Dirichlet conditions, $u'(0) = u(1) = 0$.

We denote by $\pm\sqrt{\sigma_{DD}}$ the sequence of points $\{\lambda_n | \ \lambda_n^2 \in \sigma_{DD}\}$. Similarly, $\{\mu_n\} = \pm\sqrt{\sigma_{ND}}$. We assume that $q \in L^1([0, 1])$ is such that the operator with ND boundary conditions is positive (otherwise add a positive constant to q). In that case both $\{\lambda_n\}$ and $\{\mu_n\}$ are real.

As defined in Subsections 2.2 and 2.3, $E(z)$ and $m(z)$ will denote the de Branges and Weyl functions of the Schroedinger operator. Another standard object associated with $E(z)$ is the phase function $\varphi(x) = -\arg E(x)$, a continuous branch of the argument of E on the real line.

Note, that in terms of φ the spectra of the operator can be identified as

$$\pm\sqrt{\sigma_{DD}} \cup \{0\} = \{x : \varphi(x) = n\pi\}, \qquad \pm\sqrt{\sigma_{ND}} = \left\{x : \varphi(x) = n\pi + \frac{\pi}{2}\right\},$$

Since the phase function is always a growing function on $\mathbb{R}$, we see that the spectra σ_{DD}, σ_{ND} are alternating sequences, which is a well-known fact of spectral theory. If E is a de Branges function, it can be represented as $E(z) = A(z) + iB(z)$ where A and B are entire functions which are real on the real line. These functions have alternating zero sequences on $\mathbb{R}$ and can be viewed as analogs of sine and cosine. In terms of these functions, the spectra are seen as the zero sets (note that by the constructions A is odd and B is even see (2.3)):

$$\pm\sqrt{\sigma_{DD}} \cup \{0\} = \{x : A(x) = 0\}, \qquad \pm\sqrt{\sigma_{ND}} = \{x : B(x) = 0\}.$$

The following characterization of the spectra of regular Schroedinger operators is well known [15] (note that, essentially, it is contained in the work of Marchenko [13]). Two alternating sequences $\{\lambda_n^2\}$ and $\{\mu_n^2\}$ on $\mathbb{R}$ are equal to the spectra, σ_{DD} and σ_{ND} correspondingly, for some Schroedinger operator on the interval $[0,1]$ with $q \in L^2([0,1])$ if and only if they satisfy the asymptotics

$$\lambda_n^2 = \pi^2 n^2 + C + a_n, \qquad \mu_n^2 = \pi^2\left(n - \frac{1}{2}\right)^2 + C + b_n, \qquad n \in \mathbb{N}, \tag{3.2}$$

for some real C and some $\{a_n\}, \{b_n\} \in l^2$. Note that the constant C is equal to $\int_0^1 q(x)dx$.

A simple proof of this characterization of spectra as well as its generalization to the case of L^p potentials can be found in [3].

Applying the square root transform to (3.2) we get

$$\lambda_n = \pi n + \frac{C}{n} + \frac{a_n}{n}, \qquad \mu_n = \pi n - \frac{\pi}{2} + \frac{C}{n} + \frac{b_n}{n}, \qquad n \geq 1, \tag{3.3}$$

for some real C and $\{a_n\}$, $\{b_n\} \in \ell^2$. Put $\lambda_{-n} = -\lambda_n$, $\mu_{-n} = -\mu_n$, $n \geq 1$ and $\lambda_0 = 0$. Thus, a function $E = A + iB$ is a de Branges function corresponding to a Schroedinger equation with L^2 potential if and only if the zeros of A and B satisfy (3.3) and $A(iy)/B(iy) \sim \sin iy/\cos iy$ (the latter condition is necessary, e.g., by [13, Theorem 2.2.1]).

3.2. **Characterization theorem and its inversion.** We say that a de Branges entire function $E(z)$ corresponds to a Schroedinger equation (2.1) if it can be obtained from it following the procedure described in Subsection 2.3. As before, to avoid inessential technicalities we will assume that q with ND boundary conditions generates positive operators.

The following theorem characterizes de Branges functions corresponding to Schroedinger euqations with L^2 potentials. Its first statement is a reformulation of a result of Korotyaev [8, Theorem 1.1(i)]. Moreover, in [8] the case of L^1 potentials is considered as well. We prove a certain inverse theorem (Statement 2) which will play the key role in the proof of Horvath' theorem.

Theorem 1. *1. Assume that $\tilde{E} = \tilde{A} + i\tilde{B}$ corresponds to a Schroedinger equation on $[0,1]$ with $q \in L^2([0,1])$ and let $E = A + iB$ be a de Branges entire function of exponential type 1 and of Cartwright class. Then E corresponds to a Schroedinger equation on $[0,1]$ with a square-integrable potential if and only if*

$$z(A(z)\tilde{B}(z) - \tilde{A}(z)B(z)) = f(z) + C \tag{3.4}$$

for some real constant C and some even real-valued function $f \in L^2(\mathbb{R})$.

2. There exists $\varepsilon > 0$ such that for any even function $f \in PW_2$ which is real on $\mathbb{R}$ with $\|f\|_2 < \varepsilon$ there exists a de Branges function $E = A + iB$ which corresponds to a Schroedinger equation on $[0,1]$ with $q \in L^2([0,1])$ such that

$$z(A(z)\cos z - B(z)\sin z) = f(z) - f(0). \tag{3.5}$$

Remark 1. It is well-known that any entire function $\tilde{E}$ corresponding to a Schroedinger equation on $[0,1]$ with $q \in L^2([0,1])$ is of exponential type 1. Hence the function f in Statement 1 is actually a function from the Paley–Wiener space PW_2.

Remark 2. The property (3.4) may be rewritten as $\mathrm{Re}\,(zE\tilde{E}) \in Const + L^2(\mathbb{R})$, where we denote by $Const$ the class of constant functions.

3.3. **Horvath' theorem.** As before, if Λ is a sequence of real points, we denote by $\pm\sqrt{\Lambda}$ the set $\{z | z^2 \in \Lambda\}$. The notation $\pm\sqrt{\Lambda} \cup \{*, *\}$ stands for the set obtained from $\pm\sqrt{\Lambda}$ by addition of any two real numbers (not from Λ).

We say that $\Lambda \in \mathbb{R}$ is a defining set in the class $Schr(L^2, D)$ of Schroedinger operators on $[0,1]$ with L^2-potential and Dirichlet boundary condition at 0 if there do not exist two different operators $L, \tilde{L}$ from this class whose Weyl functions $\mathbf{m}$ and $\tilde{\mathbf{m}}$ are equal on Λ.

Then the theorem of Horvath [6] for the class $Schr(L^2, D)$ can be stated as follows.

Theorem 2. ([6, Theorem 1.1]) *A set $\Lambda \in \mathbb{R}$ is a defining set in the class $Schr(L^2, D)$ if and only if $\pm\sqrt{\Lambda} \cup \{*, *\}$ is a uniqueness set in the Paley–Wiener space PW_2.*

A version of the above theorem is proved in [6] for all $1 \le p \le \infty$. In this paper we treat only the case $p = 2$, although a similar argument can be applied to other p.

3.4. **Localization of resonances.** In this subsection we discuss the distribution of zeros of an entire function E (resonances) corresponding to the Shroedinger equation. Recall that all zeros of a de Branges function belong to $\mathbb{C}_-$. We will show that all zeros of E lie outside some logarithmic strip. There exist many results of this type in the literature (see [19, 7, 5]).

Our result provides a sharp logarithmic bound for resonances in the class $Schr(L^2, D)$. More precisely, the bound with the constant $1/2$ in front of the logarithm is true for any L^1 potential (we are grateful to the referee who pointed out this fact to us). Using the Characterization Theorem 1 we can show that this constant can not be improved for L^2 potentials.

Theorem 3. *Let E be a de Branges function which corresponds to a Shroedinger equation with L^1 potential. Then there exists $C > 0$ such that the logarithmic strip*

$$\Big\{z \in \mathbb{C}_- : -\frac{1}{2}\log(|\mathrm{Re}\, z| + 2) + C \le \mathrm{Im}\, z\Big\}$$

contains no zeros of E.

Conversely, for any $\varepsilon > 0$, there exists a Hermite–Biehler function $E = A + iB$ corresponding to the Schroedinger operator with L^2 potential such that E has infinitely many zeros in the strip

$$\Big\{ -\frac{1}{2}\log(|\mathrm{Re}\, z| + 2) - \Big(\frac{1}{2} + \varepsilon\Big)\log\log(|\mathrm{Re}\, z| + 2) < \mathrm{Im}\, z < 0\Big\}.$$

M. Hitrik [5] obtained similar results on the localization of resonances for Schroedinger operators with L^1-potentials. In [5] the constant in front of $\log|x|$ in the definition the the resonance-free logarithmic strip depends on the norm $\|q\|_{L^1}$ and can be better than our bound for small potentials. Also, considering potentials $q(x) = (1 - x)^{-\alpha}$, $\alpha < 1$, one can show that the constant $1/2$ is sharp in the class of L^1 potentials, however, this class of examples fails to provide the sharp bound for L^2 potentials.

4. Proof of Theorem 1

4.1. **Preliminary estimates.** In the proof of Theorem 1 we will use the following simple lemma about canonical products with zeros satisfying the asymptotics (3.3).

Lemma 1. *Let A and B be Cartwright class entire functions which are real on $\mathbb{R}$ and whose zeros λ_n and μ_n satisfy the asymptotics (3.3). Then*

$$\left|\frac{A(z)}{\sin z}\right| \asymp \frac{\mathrm{dist}(z, \{\lambda_n\})}{\mathrm{dist}\big(z, \mathbb{Z}\big)}, \qquad \left|\frac{B(z)}{\cos z}\right| \asymp \frac{\mathrm{dist}(z, \{\mu_n\})}{\mathrm{dist}\big(z, \mathbb{Z} + \frac{1}{2}\big)}. \tag{4.1}$$

Proof. We prove the formulas (4.1) for the function A. The case of the function B is similar. Since A is a Cartwright class functions real on $\mathbb{R}$, it may be represented as a principal value product (with obvious modification if $0 \in \{\lambda_n\}$):

$$A(z) = Kz \lim_{R\to\infty} \prod_{0<|\lambda_n|\le R} \Big(1 - \frac{z}{\lambda_n}\Big), \tag{4.2}$$

where $K \in \mathbb{R}$. The formula for B is analogous.

Let $|z|$ be sufficiently large and let $n = n(z)$ be the closest integer to z. Then we have

$$\frac{A(z)}{\sin z} = K\frac{z - \lambda_n}{z - \pi n} \cdot \prod_{k\ne 0} \frac{\pi k}{\lambda_k} \cdot \prod_{k\ne n} \Big(1 - \frac{\lambda_k - \pi k}{z - \pi k}\Big). \tag{4.3}$$

Clearly, $\sum_{k\ne n(z)} \left|\frac{\lambda_k - \pi k}{z - \pi k}\right| \to 0$ as $|z| \to \infty$, which implies the first estimate in (4.1). ∎

4.2. **Proof of Theorem 1.** Statement 1 is a straightforward corollary of [8, Theorem 1.1(i)]. Thus, we need to prove only Statement 2.

Let $f \in PW_2$ be given with the norm $\|f\|_2 < \varepsilon$ (where the choice of ε will be specified later). We need to find the functions A, B such that (3.5) is satisfied. Comparing the values at πn, and $\pi n + \frac{1}{2}$ we get

$$A(\pi n) = (-1)^n \frac{f(\pi n) - f(0)}{\pi n}, \quad n \neq 0, \qquad A(0) = f'(0),$$
$$B\Big(\pi n - \frac{\pi}{2}\Big) = (-1)^{n+1} \frac{f(\pi n - \pi/2) - f(0)}{\pi n - \pi/2}.$$

There exist unique functions $g, h \in PW_1$ (which are real on $\mathbb{R}$) such that

$$g(0) = f'(0), \qquad g(\pi n) = (-1)^n \frac{f(\pi n) - f(0)}{\pi n}, \quad n \neq 0,$$
$$h\Big(\pi n + \frac{\pi}{2}\Big) = (-1)^{n+1} \frac{f(\pi n - \pi/2) - f(0)}{\pi n - \pi/2}.$$

Now put $A = \sin z + g$, $B = \cos z + h$. It is clear that the function

$$Q(z) = z(A(z)\cos z - B(z)\sin z) = z(g(z)\cos z - h(z)\sin z)$$

coincides with $f - f(0)$ at the points $\big\{\frac{\pi m}{2}\big\}_{m\in\mathbb{Z}}$. Hence, $Q(z) = f(z) - f(0) + P(z)\sin 2z$ for some entire function P. Since $f \in PW_2$ and $Q \in zPW_2$, a standard Phragmén–Lindelöf principle shows that P is at most constant. Since additionally $Q'(0) = f'(0)$ we conclude that $P \equiv 0$.

Thus, the constructed functions A and B satisfy equality (3.5). It remains to show that the zeros of A, B satisfy (3.3) and are interlacing.

Let us study the zero asymptotics for $A = \sin z + g$, the case of $B = \cos z + h$ is analogous. Since $\big\{\frac{\sin z}{z - \pi n}\big\}_{n\in\mathbb{Z}}$ is an orthogonal basis in PW_1, we have

$$g(z) = f'(0)\frac{\sin z}{z} + \sum_{n\neq 0} \frac{\sin z}{z - \pi n}\bigg(\frac{C}{n} + \frac{s_n}{n}\bigg),$$

where $C = f(0)/\pi$, $s_n = -f(\pi n)/\pi$, $\{s_n\} \in \ell^2$.

Since the coefficients in the definition of g depend on f linearly, we can choose ε to be so small that $|g(z)| < 1/4$ whenever $|\operatorname{Im} z| \le \pi/6$ and $\|f\|_2 < \varepsilon$. Note that $\inf_{|z-\pi n|=\pi/6} |\sin z| = 1/2$ for any $n \in \mathbb{Z}$. Hence, by the standard Rouché theorem, the function $A = \sin z + g$ has a unique zero in the disc $|z - \pi n| < \pi/6$. Moreover, since g is real on $\mathbb{R}$, this zero is real.

Denote the zero of A in $(\pi n - \pi/6, \pi n + \pi/6)$ by λ_n. Assuming that $n \neq 0$ and $\lambda_n \neq \pi n$, we have

$$1 + \frac{f'(0)}{\lambda_n} + \sum_{k\in\mathbb{Z}, k\neq 0} \frac{1}{\lambda_n - \pi k}\bigg(\frac{C}{k} + \frac{s_k}{k}\bigg) = 0, \tag{4.4}$$

whence

$$\lambda_n = \pi n - \bigg(\frac{C}{n} + \frac{s_n}{n}\bigg)\cdot\bigg(1 + \frac{f'(0)}{\lambda_n} + \sum_{k\neq 0,n} \frac{1}{\lambda_n - \pi k}\cdot\bigg(\frac{C}{k} + \frac{s_k}{k}\bigg)\bigg)^{-1}.$$

It is easy to see that

$$\left\{\frac{f'(0)}{\lambda_n}+\sum_{k\neq n}\frac{1}{\lambda_n-\pi k}\cdot\left(\frac{C}{k}+\frac{s_k}{k}\right)\right\}_{n\in\mathbb{Z}}\in\ell^2$$

for any choice of $\lambda_n\in(\pi n-\pi/6,\pi n+\pi/6)$. One can either refer to the boundedness in ℓ^2 of the discrete Hilbert transform or estimate the sum directly.

We conclude that

$$\lambda_n=\pi n-\frac{C}{n}+\frac{\alpha_n}{n}, \tag{4.5}$$

where $\{\alpha_n\}\in\ell^2$.

Thus, we have shown that the zeros λ_n of A have the required asymptotics. Note also that A has no other zeros. Indeed, if we denote by A_0 the Cartwright class canonical product with zeros λ_n and write $A=A_0P$ for some entire function P, then the estimate (4.1) implies that P is a constant.

Analogously, one shows that for sufficiently small $\varepsilon>0$ the zeros μ_n of the function $B=\cos z+h$ satisfy $\mu_n\in(\pi n-2\pi/3,\pi n-\pi/3)$ and $\mu_n=\pi n-\frac{\pi}{2}+\frac{C}{n}+\frac{\beta_n}{n}$, $\{\beta_n\}\in\ell^2$.

In particular, λ_n and μ_n are interlacing. It is well known that if A and B are principal value canonical products of the form (4.2) (with the constants $K>0$) then $E=A+iB$ is a de Branges function if and only if the zeros of A and B are interlacing (see, e.g., [11, Ch. VII, Th. 3]). Thus, $E=A+iB$ is a de Branges function. Since the zeros of A and B have the asymptotics (3.3) and $A(iy)/B(iy)\to 1$ as $y\to\infty$, the function E corresponds to some Schroedinger equation with an L^2 potential. ☐

5. Proof of Horvath' theorem

In this section we use the Characterization Theorem 1 to give a short proof of Horvath' theorem.

Proof. A simple proof of the 'if' part was given in [14]. Here we present a version of it for reader's convenience. The 'only if' part follows from Statement 2 of Theorem 1.

If: Suppose that $\mathbf{m}$ and $\tilde{\mathbf{m}}$ are equal on Λ for some $L,\tilde{L}$. Once again, without loss of generality we can assume that both operators are positive. Otherwise, we may add a large positive constant a to both potentials, and using the transformation

$$F(z)\mapsto F(\sqrt{z^2+a^2})$$

for even entire functions we observe that $\pm\sqrt{\Lambda}$ is a uniqueness set if $\pm\sqrt{\Lambda+a}$ is.

Then, after the square root transform, m and $\tilde{m}$ are equal on the set $\pm\sqrt{\Lambda}$. Also, by our definitions, $m(0)=\tilde{m}(0)$. Hence $\tilde{\Theta}=\Theta$ on $\pm\sqrt{\Lambda}\cup\{0\}$, i.e. the function $(z-a)(\tilde{\Theta}-\Theta)=0$ on $\pm\sqrt{\Lambda}\cup\{0,a\}$, where a is any point not in $\pm\sqrt{\Lambda}\cup\{0\}$. By the definition of Weyl inner functions, the last equation translates into $(z-a)(E^\#\tilde{E}-\tilde{E}^\#E)=0$ or equivalently $(z-a)(A\tilde{B}-\tilde{A}B)=0$. Since by Statement 1 of Theorem 1 the function $(z-a)(A\tilde{B}-\tilde{A}B)$ belongs to PW_2, we obtain a contradiction.

Only if: Without loss of generality, $0,1\notin\pm\sqrt{\Lambda}$. Suppose that $\pm\sqrt{\Lambda}\cup\{0,1\}$ is not a uniqueness set for PW_2 and let $f\in PW_2$ be a non-trivial function which vanishes on that set and real on $\mathbb{R}$. At least one of the functions $f(z)$ and $\frac{f(z)}{z-1}$ is not odd. Assume that f is not odd. Put $\tilde{f}(z) = f(z) + f(-z)$. Clearly $\tilde{f}$ is a non-trivial even function.

By Statement 2 of Theorem 1, $\tilde{f} = z(A\cos z - B\sin z)$ for some $E = A + iB$ corresponding to a Schroedinger operator L. It is left to notice that then $\mathbf{m} = \mathbf{m}_0$ on Λ, where $\mathbf{m}$ corresponds to L and $\mathbf{m}_0$ corresponds to the free operator. □

Remark 3. In the second part of our proof we could obtain the following more precise statement. If $\pm\sqrt{\Lambda}\cup\{*,*\}$ is not a uniqueness set in the Paley–Wiener class PW_2, then for any operator from $Schr(L^2, D)$ there exists another operator from the same class whose $\mathbf{m}$-function takes the same values on Λ.

6. Distribution of zeros of E. Proof of Theorem 3

Proof of Statement 1 of Theorem 3. The following argument (which is essentially simpler than our initial proof) was suggested by the referee. It is well known that the zeros of the de Branges function E corresponding to a Schroedinger equation on $(0,1)$ with an L^1 potential q coincide with the zeros of the Jost solution $w(0,k)$, that is, the solution of the Volterra integral equation

$$w(x,k) = e^{ikx} - \int_x^1 \frac{\sin k(x-y)}{k} q(y)w(y,k)dy$$

(see, e.g., [8]). Denote by T the integral operator $(Tf)(x) = \int_x^1 \frac{\sin k(x-y)}{k} q(y)f(y)dy$ acting in $C[0,1]$. Let $k = u + iv$ where, for some $C > 1$, we have $|u| > e^{2C}$ and $-\frac{1}{2}\log|u| + C < v < 0$. Then, obviously, $|v| \le \frac{1}{2}\log|u| - C$ and

$$|(Tf)(x)| \le \|f\|_{C[0,1]} \int_x^1 \frac{e^{|v|}}{|k|}|q(y)|dy \le \frac{e^{-C}}{|u|^{1/2}}\|q\|_1\|f\|_{C[0,1]}, \qquad x \in [0,1].$$

Choosing C sufficiently large (depending on $\|q\|_1$) we have, for k as above, $\|T\| < 1/2$ and $\|(I+T)^{-1} - I\| \le 2e^{-C}|u|^{-1/2}\|q\|_1$.

On the other hand, $|e^{ikx}| = e^{x|v|} \le e^{-C}|u|^{1/2}$, $x \in [0,1]$. Therefore, if we put $g(x) = e^{ikx}$, we have

$$\|(I+T)^{-1}g - g\|_{C[0,1]} \le 2e^{-2C} < 1,$$

whence

$$|w(x,k)| = |((I+T)^{-1}g)(x)| > |g(x)| - 1 = 0.$$

In particular, $E(k) = w(0,k)$ does not vanish when $|\operatorname{Re} k| > e^{2C}$, $-\frac{1}{2}\log|\operatorname{Re} k| + C < \operatorname{Im} k < 0$, and C is sufficiently large.

Proof of Statement 2 of Theorem 3. Choose an increasing sequnce $\{n_k\}_{k\ge1}$ of positive integers such that:

(*i*) $\{n_k\}_{k\ge1}$ is lacunary, i.e., $\inf_k \frac{n_{k+1}}{n_k} > 1$;

(*ii*) the sequence $a_k = \frac{1}{(\log n_k)^{\varepsilon/2}}$ satisfies $\sum_k a_k < \infty$, $\sup_k a_k < \frac{1}{100}$.

Put $z_k = n_k + \frac{a_k}{n_k}$ and consider the meromorphic function

$$S(z) = \prod_k \frac{z^2 - \pi^2 z_k^2}{z^2 - \pi^2 n_k^2}.$$

Then clearly the functions $A(z) = S(z)\sin z$ and $B(z) = \cos z$ have the required asymptotics of the zeros and so E corresponds to some Schroedinger operator.

Since

$$\frac{z^2 - \pi^2 z_l^2}{z^2 - \pi^2 n_l^2} - 1 = -\frac{2\pi^2 a_l}{z^2 - \pi^2 n_l^2} - \frac{\pi^2 a_l^2}{n_l^2(z^2 - \pi^2 n_l^2)},$$

it is easy to see that for $|z - z_k| \lesssim n_k^{1/2}$,

$$\prod_{l \neq k} \frac{z^2 - \pi^2 z_l^2}{z^2 - \pi^2 n_l^2} - 1 = O\Big(\frac{1}{|z|^2}\Big)$$

(use the lacunarity of $\{n_k\}$ and the fact that $\sum_k a_k < \infty$). Assuming additionally that $|\mathrm{Im}\, z| \geq 1$, we get

$$S(z) - 1 = \frac{z^2 - \pi^2 z_k^2}{z^2 - \pi^2 n_k^2} - 1 + O\Big(\frac{1}{|z|^2}\Big) = -\frac{2\pi^2 a_k}{z^2 - \pi^2 n_k^2} + O\Big(\frac{1}{|z|^2}\Big).$$

The equation

$$E(z) = S(z)\sin z + i\cos z = 0, \tag{6.1}$$

is equivalent to

$$e^{2iz} = \frac{S(z)+1}{S(z)-1}.$$

If there is a root z of this equation such that $|z - z_k| \lesssim n_k^{1/2}$ and $|\mathrm{Im}\, z| \geq 1$, then we have

$$e^{2iz} = -\frac{z^2 - \pi^2 n_k^2}{\pi^2 a_k} + O\left(\frac{|z^2 - \pi^2 n_k^2|^2}{a_k^2 n_k^2}\right).$$

Writing $z = \pi n_k + w$ we can rewrite this as

$$e^{2iw} = g(w), \tag{6.2}$$

where

$$\begin{aligned} g(w) &= -\frac{w(w + 2\pi n_k)}{\pi^2 a_k} + O\left(\frac{|w|^2 |w + 2\pi n_k|^2}{a_k^2 n_k^2}\right) \\ &= -\frac{2n_k w}{\pi a_k} + O\Big(\frac{|w|^2}{a_k^2}\Big) = -\frac{2n_k w}{\pi a_k} + o\Big(\frac{n_k |w|}{a_k}\Big). \end{aligned}$$

We used the fact that $|w| \geq 1$ and $a_k n_k >> |w|$. The approximate equation

$$e^{2iw} = -\frac{2n_k w}{\pi a_k}$$

has a solution

$$\begin{aligned} w_k &= -\frac{i}{2}\log\frac{2n_k}{\pi a_k} - \frac{i}{2}\log\log\frac{2n_k}{\pi a_k} + O(1) \\ &= -\frac{i}{2}\log n_k - i\Big(\frac{1}{2} + \frac{\varepsilon}{2}\Big)\log\log n_k + O(1). \end{aligned}$$

Let us show that the equation (6.2) has a unique solution close to w_k when k is sufficiently large. Indeed, put $h(w) = e^{2iw} + \frac{2n_k w}{\pi a_k}$ and let $|v| = 1$. Then

$$h(w_k + v) = e^{2iw_k + 2iv} + \frac{2n_k(w_k + v)}{\pi a_k} = \frac{2n_k w_k}{\pi a_k}(1 - e^{2iv}) + \frac{n_k v}{\pi a_k},$$

whence $|h(w_k + v)| \asymp \frac{n_k |w_k|}{a_k}$. By the above estimates, $e^{2iw} - g(w) = h(w) + o(h(w))$, $|w - w_k| = 1$, and so, by Rouché's theorem, the equation (6.2) has a solution w such that $|w - w_k| < 1$. Thus, the equation (6.1) has the solution of the form

$$z = n_k - \frac{i}{2}\log n_k - i\Big(\frac{1}{2} + \frac{\varepsilon}{2}\Big)\log\log n_k + O(1).$$

$\square$

References

[1] E. Abakumov, A. Baranov, Y. Belov, *Localization of zeros for Cauchy transforms,* Int. Math. Res. Notices, 2015, (2015), 6699–6733

[2] L. de Branges, *Hilbert spaces of entire functions*, Prentice–Hall, Englewood Cliffs, NJ, 1968.

[3] D. Chelkak, *An application of the fixed point theorem to the inverse Sturm–Liouville problem,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 370 (2009); English transl. in J. Math. Sci. 166 (2010), 1, 118–126.

[4] H. Dym, H.P. McKean, *Gaussian processes, function theory and the inverse spectral problem*, Academic Press, New York, 1976.

[5] M. Hitrik, *Bounds on scattering poles in one dimension*, Comm. Math. Phys. 208 (1999), 2, 381–411.

[6] M. Horváth, *Inverse spectral problems and closed exponential systems*, Ann. Math. 162 (2005), 2, 885–918.

[7] A. Iantchenko, E. Korotyaev, *Resonances for Dirac operators on the half-line,* J. Math. Anal. Appl. 420 (2014), no. 1, 279–313.

[8] E. Korotyaev, *Inverse resonance scattering on the half line*, Asymptotic Anal. 37 (2004), 3-4, 215–226.

[9] J. Lagarias, *Zero spacing distributions for differenced L-functions,* Acta Arithmetica 120 (2005), 2, 159–184.

[10] J. Lagarias, *The Schrödinger operator with Morse potential on the right half line*, Comm. Number Theory Phys. 3 (2009), 2, 323–361.

[11] B.Ya. Levin, *Distribution of Zeros of Entire Functions*, Amer. Math. Soc., Providence, 1964; revised edition: Amer. Math. Soc., 1980.

[12] V.A. Marchenko, *Certain problems in the theory of second-order differential operators*, Dokl. Akad. Nauk SSSR 72 (1950), 457–460 (Russian).

[13] V.A. Marchenko, *Some questions in the theory of one-dimensional linear differential operators of the second order.* I, Trudy Moskov. Mat. Ob. 1 (1952), 327–420; English transl. in Amer. Math. Soc. Transl. (2) 101 (1973), 1–104.

[14] N. Makarov, A. Poltoratski, *Meromorphic inner functions, Toeplitz kernels, and the uncertainty principle,* in *Perspectives in Analysis*, Carleson's Special Volume, Springer Verlag, Berlin, 2005, 185–252.

[15] J. Poshel, E. Trubowitz, *Inverse Spectral Theory*, Academic Press, New-York, 1987.

[16] C. Remling, *Schrödinger operators and de Branges spaces*, J. Funct. Anal. 196 (2002), 323–394.

[17] R. Romanov, *Canonical systems and de Branges spaces,* http://arxiv.org/abs/1408.6022.

[18] R. Romanov, *Order problem for canonical systems and a conjecture of Valent,* http://arxiv.org/abs/1502.04402, to appear in Trans. Amer. Math. Soc.

[19] B. Simon, *Resonances in one dimension and Fredholm determinants,* J. Funct. Anal. 178 (2) (2000) 396–420.

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