# RESTRICTED INTERPOLATION BY MEROMORPHIC INNER FUNCTIONS

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## 1    Introduction

Meromorphic Inner Functions or MIFs form a subclass of inner functions in the upper half-plane consisting of those inner functions that can be extended meromorphically into the whole complex plane. Such functions play an important role in many applications to functional and spectral analysis. Via the Nagy-Foias functional model theory, MIFs lead to the study of one-point spectrum and Volterra operators. One of the important function-theoretic properties of MIFs, the fact that any MIF corresponds to a so-called Hermit-Biehler entire function, connects this class to the celebrated Krein-de Branges theory. From there, MIFs enter a large and important area of applications to the spectral problems for differential equations.

In the 1920s, Weyl studied differential equations from the perspective of complex analysis and discovered what is now called the Weyl m-functions, arising from certain $2^{nd}$ order differential equations. The Weyl m-functions are in one-to-one correspondence with MIFs. Properties of second order differential operators are carried over to the related MIF. These connections were later systematized by Krein who studied canonical systems of differential equations, a generalization of Schrödinger, Dirac, string and many other classes of second order differential operators. The complex analytic part of the theory was further developed by de Branges in 1960s. A key object of the Krein-de Branges theory is the so-called Hermit-Biehler entire function and the Hilbert space of entire functions it generates. Every such function can be easily modified into an MIF, which connects the theory to the Nagy-Foias model and a number of new applications.

The connections mentioned above caused considerable interst to function theoretic properties of MIFs in recent years. Uniqueness and existence results for functions from this class,

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related restricted interpolation problems and applications to functional and spectral analysis were recently studied in a number of papers. The class of MIFs serves as a natural object in the so called Toeplitz approach to the Uncertainty Principle, see ([23], [22], [31]). It was used to obtain an extension of the Beurling-Malliavin theory ([23], [22]) and to study classical problems of Fourier analysis ([26], [27], [30], [29]). Some of the function theoretic questions arising from such applications were treated in [33].

The goal of this paper is to survey recent results on uniqueness, existence and interpolation for MIFs and show their relations to spectral problems. The last section contains statements of new results whose proofs will be published elsewhere. We outline possibilities for further research in this direction.

## 2 Preliminaries

One may **define meromorphic inner functions** in the following way.

Let $\Theta : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function in $\mathbb{C}$ which is bounded and holomorphic in the upper half plane $\mathbb{C}_+$ and such that $|\Theta| = 1$ on $\mathbb{R}$. Then $\Theta$ is called a meromorphic inner function on the upper half plane (MIF).

MIFs can be easily described via the standard Blaschke/singular factorization. All MIFs have the following form:

$$\Theta(z) = C e^{iaz} \prod_{n=0}^{\infty} c_n \frac{z - w_n}{z - \bar{w}_n},$$

where $a$ is a non-negative constant, $w_n$ is a sequence of points in $\mathbb{C}_+$ satisfying the Blaschke condition

$$\sum_{n=0}^{\infty} \frac{\Im w_n}{1 + |w_n|^2} < \infty$$

and tending to infinity as $n \to \infty$, $C$ is a unimodular constant and $c_n = \frac{i + \bar{w}_n}{i + w_n}$.

### 2.1 Clark Theory

It is important and useful to study MIFs through the lens of Clark theory. This theory is about a family of measures arising out of self-maps of the unit disk. Clark first studied this family of measures in 1972, see ([12], [32]) in the case of inner functions. In the late 80s and 90s, they were studied in great depth by Alekasandrov (see [1]-[5], [32]) who extended the theory to non-inner case. We will call such measures AC measures.

We recall the basic definitions, see for instance ([32], [11]). If $\Phi$ is a holomorphic self map of the disk (a holomorphic function in the disk whose absolute value is bounded by one), and $\alpha$ is a point on $\mathbb{T}$, then $\frac{\alpha+\Phi}{\alpha-\Phi}$ is holomorphic on $\mathbb{D}$, with a positive real part. Thus, its real part $\frac{1-|\Phi|^2}{|\alpha-\Phi|^2}$ is the Poisson integral of a positive measure $\mu_\alpha$ on the unit circle $\mathbb{T}$, i.e.,

$$\frac{1-|\Phi(z)|^2}{|\alpha - \Phi(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\tau - z|^2} d\mu_\alpha(\tau). \tag{1}$$

We call $\mu_\alpha$ an AC-measure of $\Phi$. Conversely, if $\mu$ is a positive Borel measure on $\mathbb{T}$ and $H\mu$ is the Herglotz transform of $\mu$,

$$H\mu(z) = \int_{\mathbb{T}} \frac{\tau + z}{\tau - z} d\mu(\tau), \tag{2}$$

then $\Re H\mu > 0$. Hence,

$$\Phi(z) := \frac{H\mu(z) - 1}{H\mu(z) + 1}$$

is an analytic self-map of the disk. Moreover,

$$P\mu(z) = \Re H\mu(z) = \frac{1-|\Phi(z)|^2}{|1-\Phi(z)|^2}.$$

Thus, $\mu$ is an AC measure for $\Phi$ ($\mu = \mu_1$). If $\mu$ is singular, then

$$\lim_{r\to 1^-} \frac{1-|\Phi(r\tau)|^2}{|1-\Phi(r\tau)|^2} = \lim_{r\to 1^-} P\mu(r\tau) = \frac{d\mu}{dm}(\tau) = 0 \text{ m- a.e.}.$$

Since $\lim_{r\to 1^-} |1-\Phi(r\tau)| > 0$ for almost all $\tau \in \mathbb{T}$, it must be the case that

$$\lim_{r\to 1^-} (1 - |\phi(r\tau)|) = 0$$

for a.e. $\tau$, that is $\Phi$ is an inner function. Thus, every positive, singular Borel measure is a Clark measure to some inner function. The support of the measure $\mu$ is exactly the set $\{\Phi = 1\}$. In general, for an inner $\Phi$ the measure $\mu_\alpha$ is supported by the set $\{\Phi = \alpha\}$.

In the case of the upper half plane, the construction is similar. A measure $\mu$ on $\hat{\mathbb{R}}$ is Poisson finite if $\mu = \mu_{\mathbb{R}} + c\delta_\infty$ with

$$\int_{\mathbb{R}} \frac{d\mu_{\mathbb{R}}(x)}{1 + x^2} < \infty.$$

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We recall that the Herglotz transform of a Poisson finite measure $\nu$ on $\hat{\mathbb{R}}$ is given by

$$K\nu(z) = \frac{1}{\pi i}cz + \frac{1}{\pi i}\int_{\mathbb{R}}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\nu(t),$$

where $c$ is the size of the pointmass of $\nu$ at infinity. Let $\Theta$ be an inner function in the upper half plane. To simplify the formulas slightly we will assume that $\Theta(i) = 0$. In this case for each $\alpha \in \mathbb{T}$ there exists a unique Poisson finite measure on $\hat{\mathbb{R}}$ such that

$$K\mu_\alpha = \frac{\alpha + \Theta}{\alpha - \Theta}.$$

Conversely, for any Poisson finite singular measure $\mu$ on $\hat{\mathbb{R}}$ one can consider an inner function $\Theta$ defined by

$$\Theta(z) = \frac{K\mu(z) - 1}{K\mu(z) + 1}.$$

Then $\mu$ becomes the Clark measure $\mu_1$ for $\Theta$.

If the inner function is meromorphic, the spectrum $\{\Theta = 1\}$ (and the support of any of the Clark measures $\{\Theta = \alpha\}$) is discrete, i.e., it is a sequence of points in $\hat{\mathbb{R}}$ without finite accumulation points. Given a discrete sequence of points in $\hat{\mathbb{R}}$ it is not difficult to construct an MIF with such spectrum.

Indeed, let $\{\lambda_n\}_{-\infty}^{\infty}$ be a discrete sequence on $\hat{\mathbb{R}}$. Let $\mu$ be a Poisson finite, positive measure on $\hat{\mathbb{R}}$ with point masses at the $\lambda_n$, i.e.,

$$\mu = w_\infty \delta_\infty + \sum_{n=-\infty}^{\infty} w_n \delta_{\lambda_n} \tag{3}$$

for some $w_n > 0$ such that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{w_n}{1+\lambda_n^2} < \infty$ and some $w_\infty \geq 0$. This will be the Clark measure of the MIF we will construct. Applying the Herglotz transform to the measure $\mu$ just defined,

$$K\mu(z) = \frac{w_\infty}{\pi i}z + \frac{1}{\pi i}\sum_{n=-\infty}^{\infty}\frac{w_n}{\lambda_n - z} - \frac{w_n\lambda_n}{1+\lambda_n^2},$$

we have that $K\mu$ is an analytic funtion from the upper half plane $\mathbb{C}_+$ to the right half plane. We compose $K\mu$ with a fractional linear transformation that maps the right half plane into

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the unit disk to get $\Theta : \mathbb{C}_+ \to \mathbb{D}$ as follows,

$$\Theta(z) = \frac{K\mu(z) - 1}{K\mu(z) + 1}.$$

Observe that $\Theta$ is a meromorphic inner function on $\mathbb{C}_+$. Morever, we notice that $\Theta$ would take the value 1 exactly at the singularities of $K\mu$, i.e. at the $\lambda_n$s.

A natural question to ask is if the inner function with spectrum $\{\lambda_n\}$ is unique. A look at (3) assures us that that is quite not the case, for the $w_n$ are almost arbitrarily chosen. We can obtain restrictions on the $w_n$ by imposing additional conditions on the MIF.

## 2.2  Spectral Theory

Consider the Schrödinger equation

$$-u'' + qu = \lambda u \tag{4}$$

on some interval $(a, b)$ and assume that the potential $q(t)$ is locally integrable and $a$ is a *regular* point i.e., a is finite and $q$ is in $L^1$ at $a$. Let us fix the following boundary condition at $b$.

$$\cos(\beta)u(b) + \sin(\beta)u'(b) = 0. \tag{5}$$

Then for each $\lambda \in \mathbb{C}$, there is a solution $u_\lambda$ to (4) satisfying (5) such that $u_\lambda(t)$ is an entire function of $\lambda$ for each fixed $t \in (a, b)$ [21]. This family of solutions $\{u_\lambda\}_\lambda$ gives rise to a function called the *Weyl-Titchmarsh m* function, defined as

$$m(\lambda) = \frac{u'_\lambda(a)}{u_\lambda(a)}.$$

Here we only deal with the compact resolvent case, i.e. when $m$ extends to a meromorphic function in $\mathbb{C}$. We can then define a meromorphic inner function as

$$\Theta(z) = \frac{m(z) - i}{m(z) + i}.$$

It is called the *Weyl inner function* corresponding to the potential $q$ and the boundary condition at $b$.

Now consider the related Schrödinger operator

$$u \rightarrow -u'' + qu,$$

defined on the space $L^2(a,b)$, along with the boundary conditions

$$\cos(\alpha)u(a) + \sin(\alpha)u'(a) = 0$$
$$\cos(\beta)u(b) + \sin(\beta)u'(b) = 0.$$

This operator has a discrete spectrum on $\mathbb{R}$, which we denote by $\sigma(q,\alpha,\beta)$. Suppose we fix the Dirichlet boundary condition at $a$, i.e., $\alpha = 0$ and denote the resulting spectrum thus obtained as $\sigma(q,D,\beta)$. Then,

$$\sigma(\Theta) = \sigma(q,D,\beta),$$

where $\Theta$ is the Weyl inner function obtained by fixing the boundary condition $\beta$ at $b$. We can also obtain the spectrum of the operator with the original boundary conditions, i.e. $\alpha$ and $\beta$ at $a$ and $b$ respectively by considering a rotation of $\Theta$ in the following way. Let us fix the boundary condition $\alpha$ at $a$, i.e.,

$$\cos(\alpha)u(a) + \sin(\alpha)u'(a) = 0.$$

denoting the resulting spectrum by $\sigma(q,\alpha,\beta)$, we have the relationship

$$\sigma(e^{-i\alpha}\Theta) = \sigma(q,\alpha,\beta),$$

where $\Theta$ is, as before, the Weyl inner function obtained by fixing the Dirichlet boundary condition at $a$.

## 3   Inverse Spectral Theory

Inverse spectral theory concerns the reconstruction of the operator from spectral data. One of the classical results in this area dates back to 1929, when Ambarzumian [6] proved that a Schrödinger oparator with spectrum at $\{n^2\}_{n\in\mathbb{N}}$ and Neumann boundary condition at both end points is a free operator, i.e., its potential must be 0 a.e. In the years 1950-1952, Borg and Marchenko independently proved several results of what are now called the Borg-Marchenko uniqueness type theorems ([10],[24]). Along with Levinson, all three are credited for proving that two spectra are enough to recover the potential [20]. In recent

times, Simon, Gesztezy and del Rio have proved results on mixed spectral data (conditions under which data coming from multiple spectra may be enough to recover the potential) ([14], [16], [17],[15]). An important contribution is by Hórvath [19], who proved in 2005 the following result.

**Theorem 1.** *Let* $1 \leq p \leq \infty$, $q \in L^p(0, \pi)$, $0 \leq a < \pi$ *and let* $\lambda_n \in \sigma(q, \alpha_n, 0)$ *be real numbers with* $\lambda_n \not\to -\infty$. *Then,* $\beta = 0$, $q$ *on* $(0, a)$ *and* $\lambda_n$ *determine* $q$ *in* $L^p$ *if and only if*

$$e(\Lambda) = \{e^{\pm 2i\sqrt{\lambda_n}x}, e^{\pm 2i\mu} : n \geq 1\} \tag{6}$$

*is closed in* $L^p(a - \pi, \pi - a)$ *for some (any)* $\mu \neq \pm\sqrt{\lambda_n}$.

Recall that a family of functions is closed (complete) if their finite linear combinations are dense in the corresponding space.

Our approach is to use complex analytic tools, namely Weyl functions, model spaces and Toeplitz kernels to treat problems arising from spectral settings similar to those described above. Let us describe the fundamental result this theory is based on. Let

$$u \to -u'' + q_i u$$

be two Schrödinger operators on $(a, b)$ with the same boundary condition at $b$. The related Weyl m-functions are given by $m_1$ and $m_2$ respectively. Then, Marchenko [24] proved the following

**Theorem 2.** *Suppose* $m_1(z) = m_2(z)$, *for all* $z \in \mathbb{C} \setminus \mathbb{R}$, *then* $q_1(x) = q_2(x)$ *for almost all* $x \in (a, b)$.

Since the Weyl m-functions and the Weyl inner functions are in 1-1 correspondence, one deduces that knowing the Weyl inner function of a Schrödinger operator is enough to recover the potential. As a result, questions about recovery of the potential of a Schrödinger operator from its spectral data can be reduced to the recovery of the related Weyl meromorphic inner function from its spectral data. This question actually has two parts: firstly, given some spectral data, does there exist an MIF corresponding to this data? and secondly, is this MIF unique? In this section, we address the uniqueness question and in the last section, we discuss the existence problem.

### 3.0.1 Krein's formula & Two spectra Interpolation

A natural question to ask is if one spectrum is enough to recover an MIF uniquely. We focus on the uniqueness part here. As we saw before, the answer is a resounding no for the natural

reason that there exist infinitely many Poisson finite measures with the same support. The next natural question is, is two spectra enough for uniqueness. The answer in this case is given by Krein with a beautiful formula that captures all the MIFs with fixed sets $\{\Theta = \pm 1\}$.

For the sake of brevity, we consider the $\{\pm 1\}$ as the two spectra in question. Let us state the precise result here. *Let $\Phi$ be a meromorphic inner function. Then a meromorphic inner function $\tilde{\Phi}$ satisfies $\{\tilde{\Phi} = 1\} = \{\Phi = 1\}$ and $\{\tilde{\Phi} = -1\} = \{\Phi = -1\}$ iff*

$$\tilde{\Phi} = \frac{\Phi - c}{1 - c\Phi}, \qquad c \in (-1, 1). \tag{7}$$

The easiest way to see this is to use Krein's shift construction: since

$$\Re \left[ \frac{1}{\pi i} \log \frac{\tilde{\Phi} + 1}{\tilde{\Phi} - 1} \right] = \chi_e \qquad \text{on} \quad \mathbb{R},$$

where $e = \{\Im\Phi > 0\}$, we have

$$\frac{1}{\pi i} \log \frac{\tilde{\Phi} + 1}{\tilde{\Phi} - 1} = \mathcal{S}\chi_e + \text{const}, \qquad \square$$

where $\mathcal{S}\chi_e$ is the Schwarz transform of the characteristic function $\chi_e$.

### 3.0.2 Mixed Spectral Data Interpolation

A natural generalization of the two spectra result is the case of mixed spectral data, having the *same amount* of data as two spectra, in a sense to be made clear. The following is an example of this generalization.

**Lemma 1** ([28]). *Let $\Lambda_1$, $\Lambda_2$ and $\Lambda$ be three pairwise disjoint sequences such that $\Lambda_1 \cup \Lambda_2$ interlaces $\Lambda$. Then, there exist meromorphic inner functions $\Theta_i$ such that $\Lambda_i = \sigma(\Theta_i)$, $i = 1, 2$ and $\Lambda = \sigma(\Theta_1\Theta_2)$, that are unique up to a Möebius transformation.*

In this case the union of the two spectra, $\Lambda_1 \cup \Lambda_2$, plays the role of the second spectrum of $\Theta$. Another way to generalize the two spectra result is as follows. Suppose we have a discrete sequence on $\mathbb{R}$ such that each point corresponds to different spectrum $\{\Theta = \alpha\}$ of the MIF, then can we say that there must be at least as much data as two spectra *in some sense* in order to recover the MIF uniquely? The following results describe the situation when the MIFs correspond to a Schrödinger operator.

### 3.0.3 Schrödinger Operator case

Specifically, for the Schrödinger operator case, we have the following results. Let $\Phi$ be a Weyl inner function that corresponds to a Schrödinger operator with potential $q \in L^2(0, a)$. Recall that $\Phi$ is an MIF and therefore its argument $\arg \Phi$ can be defined as a real analytic function on $\mathbb{R}$. We will say that $\Lambda$ is **S-defining** for $\Phi$ if the following is true: For $\Phi_2$ a Weyl inner function of a Schrödinger operator with a potential $q_2 \in L^2(0, a)$

$$\text{if } \arg \Phi = \arg \Phi_2 \text{ on } \Lambda \text{ then } \Phi \equiv \Phi_2.$$

We recall that a model space $K_\Phi$ corresponding to an MIF $\Phi$ is the space $K_\Phi = \mathcal{H}^2 \ominus \Phi \mathcal{H}^2$.

**Lemma 2** ([28])**.** *The sequence $\Lambda$ is not S-defining for $\Phi$, if and only if there is an $F \in K_\Phi$ such that $F = \overline{F}$ on $\Lambda$ and $F(a) = 0$ for some $a \in \mathbb{R} \setminus \Lambda$.*

This lemma harks back to a classical problem in complex analysis. Namely, the problem of completeness of a family of exponentials in some $L^2(\mu)$ space, where $\mu$ is a positive, finite Borel measure. To state a particular case precisely, suppose $\Lambda = \{\lambda_n\}_n$ is a sequence of separated points on $\mathbb{R}$. Is the family $\{e^{i\lambda_n x}\}_n$ of exponentials complete in the space $L^2(-a, a)$? This problem had long remained unsolved. In the first half of the 20th century several prominent analysts, including Paley, Wiener, Kahane, Koosis and Levinson studied this problem. It was solved in the early 60s by Beurling and Malliavin ([8], [9]). More about this problem can be found in [31].

In [23] and [22] the results of Beurling and Malliavin are extended using the language of Toeplitz kernels and model spaces.

## 4 Restrictions on gap size and derivative

An important object of study in complex function theory and spectral theory is a class of Hilbert spaces of entire introduced in a seminal book [13] by Louis de Branges published in 1968. Meromorphic inner functions are in one-to-one correspondence with de Branges functions. Often, it is easier to state results in de Branges theory in the language of MIFs.

In his book [13], de Branges claimed a property about de Branges functions (and hence MIFs) that influenced future work in this area. It was only much later that it was discovered to be false. One of the lemmas of [13] claimed that given any sequence $\Lambda$ of separated points on $\mathbb{R}$, there exists an MIF $\Theta$ such that $\sigma(\Theta) = \Lambda$ and $\sup_{x \in \mathbb{R}} |\Theta'(x)| < \infty$. In [13] this lemma was used to solve a version of the gap problem: for what sequences $\Lambda \subset \mathbb{R}$ does there exist a

measure $\mu$ such that $\mu$ is supported on $\Lambda$ and its Fourier transfomr $\hat{\mu}$ vanishes on an interval of non-zero length. In fact, de Branges proved that a measure with a non-trivial spectral gap can be supported on any of the sequences of a particular kind, the so-called a *regular* sequences. A sequence $\Lambda$ is *a*-reglrar if its counting function $n_\Lambda$ satisfies

$$\int_{\mathbb{R}} \frac{|n_\Lambda(x) - ax|}{1 + x^2} dx < \infty.$$

This statement holds true despite the use of the erroneous lemma above, which was recently disproved by a counterexample of A. Baranov.

Baranov's counterexample says that if an MIF $\Theta$ has $\mathbb{N}$ as spectrum, then $\sup_{x \in \mathbb{R}} |\Theta'(x)| = \infty$. It leads us to ask the natural question: if we have $\mathbb{N}$ on $\mathbb{R}_+$, how sparse can the sequence be on $\mathbb{R}_-$ in order to have an MIF, with bounded derivative, that has this sequence as its spectrum? Simple computations tell us that on the other side, the gaps may be at most geometrically increasing, i.e., $\lambda_n \lesssim -e^{c|n|}$, for some $c > 0$ and negative $n$. Here and below we assume our discrete sequences to be enumerated from $-\infty$ to $\infty$ in the natural increasing order. To put it precisely,

**Observation 1.** *Let $\Theta$ be an MIF on $\mathbb{C}_+$ with uniformly bounded derivative on $\mathbb{R}$ and let $\Lambda$ be the spectrum of $\Theta$. If $\Lambda_\pm = \Lambda \cap \mathbb{R}_\pm$ and $\Lambda_+ = \mathbb{N}$ , then $\Lambda_-$ is infinite and there exists a $c \geq 0$ such that $|\lambda_n| \lesssim e^{c|n|}$ for $\lambda_n \in \Lambda_-$.*

Another result in the same direction is contained in [22].

**Lemma 3** ([23])**.** *If $\sigma'(x) \asymp |x|^\kappa$, then there is a meromorphic inner function $\Theta = e^{i\theta}$, such that $\theta - \sigma \in L^\infty(\mathbb{R})$, $\theta'(x) \asymp |x|^\kappa$.*

The techniques used in the proof of the above result have been used to construct MIFs with similar conditions [33]. It is useful to describe the sequence on $\mathbb{R}$ in terms of the gaps between successive points. Here,

$$\Delta_n := \begin{cases} \lambda_{n+1} - \lambda_n & \forall n > 0 \\ \lambda_n - \lambda_{n-1} & \forall n \leq 0. \end{cases} \tag{8}$$

**Theorem 3** ([33])**.** *Let $\{\lambda_n\}$ be a separated sequence on $\mathbb{R}$ satisfying one of the conditions below,*

1. *$\Delta_{n+1} \asymp \Delta_n$ and $\frac{\ln |\lambda_n|}{\ln \ln |\Delta_n|} \lesssim \Delta_n \lesssim \ln |\lambda_n|$ OR*

2. $\Delta_{n+1} \asymp \Delta_n$ *and* $\Delta_n \gtrsim (\ln |\lambda_n|)^2$ *OR*

3. *there is a $d > 0$ such that the sequence can be partitioned into clusters $\Lambda = \cup C_n$, $C_n = \{\lambda_n^j\}_{j=1}^{k_n}$, with number of points in clusters being uniformly bounded ($k_n < C$, $n \in \mathbb{Z}$) and such that $\frac{\lambda_n^1}{\lambda_n^{k_n}} \to 1$ as $n \to \infty$; between successive clusters $\frac{\lambda_{n+1}^1}{\lambda_n^1} - 1 > d > 0$.*

*then there exists a meromorphic inner function with spectrum $\{\lambda_n\}$ with uniformly bounded derivative on $\mathbb{R}$.*

Some examples of sequences that satisfy one of the criteria above are : $\lambda_n = (\text{sgn}\, n)|n|^k$, where $k > 0$ and $\lambda_n = (\text{sgn}\, n)r^{|n|}$, where $r > 1$. Sparser sequences also fall into this class $a_n = (\text{sgn}\, n)e^{e^{|n|}}$. Most spectra that arise naturally, i.e., through the study of differential operators, fall into one of these categories. For example, if we consider the Schrödinger operator with the $0$ potential, defined on the interval $(0, 1)$, then the resulting spectrum is $\sqrt{n}\pi$. One the other hand, we have the following generalization of Baranov's counterexample.

**Proposition 1.** *([7], [33]) Suppose $\{s_n\}$ is a separated sequence on the real line and $D > 0$ is a constant such that given any $N > 0$, there is a subset $\{t_n\}_{n=1}^N$ such that $(t_1, t_N) \bigcap \{s_m\} = \{t_n\}_{n=1}^N$ for which $t_2 - t_1 > ND$ and $t_{n+1} - t_n < D$ for all $2 \leq n \leq N - 1$ and let $\Theta$ be an MIF with this spectrum $\{s_n\}$. Then given any $\delta > 0$, there is a zero $z_n = x_n + iy_n$ of $\Theta$ such that $0 < y_n < \delta$. Hence, $|\Theta'|$ is unbounded on $\mathbb{R}$.*

It is an interesting problem to 'close the gap' between the conditions in the last two statements.

Baranov remarks in [7] that since the placement of 'other points' does not affect the calculations, we can make such clusters and gaps along a very rare subsequence of $\mathbb{N}$, without affecting the regularity. In fact, this is easy to see using the result above. For example, let us consider the following sequence $\Lambda = \mathbb{N} \setminus A$, where $A = \{2^{n_k} + m\}$ for $m = 1, 2, ..., k$, where $n_k$ is a rare subsequence of $\mathbb{N}$, say the sequence $n_k = 3^k$. Then, we have gaps of length $k$, which is unbounded, followed by clusters with gaps of size 1, the size of the clusters $\geq 2^{n_k+1}$. This sequence is *a-regular*, where $a = 1$. For,

$$\int_{\mathbb{R}} \frac{|n_\Lambda(x) - x|}{1 + x^2} dx \asymp \sum_k \frac{k}{1 + (2^{n_k})^2} < \infty.$$

Thus, even for regular sequences, there may not exist any MIF with bounded derivative.

# 5 Uncertainty Principle

In this section, we return to the problem of spectral theory of differential operators. We recall the relationship of a Schrödinger operator with MIFs as given in section 2.2. In the section 3, we were concerned mostly with the uniqueness of potential, with respect to spectral data. Here, we explore the existence of potential with respect to spectral data. As usual, our problem is expressed in terms of MIFs. In general one can ask the following question: Let $\Lambda = \{\lambda_n\}_n$ be a discrete sequence on $\mathbb{R}$ and $\{\alpha_n \in \mathbb{T}\}_n$. Does there exist an MIF, $\Theta$, corresponding to a Schrödinger operator such that $\Theta(\lambda_n) = \alpha_n$, for all $n$? Does there exist a non-Schrödinger MIF solving the problem? For instance, if $\alpha_n = 1$ for all $n$, this is the one-spectrum problem, which we know has a solution. Next, suppose that $\alpha_{2n} = 1$ and $\alpha_{2n+1} = -1$, then this corresponds to the 2 spectra problem, which we also know to have a solution. In fact, this solution is unique, when we restrict MIFs to those corresponding to Schrödinger operators.

The set of points $\lambda_n$ in the two spectra case is maximal in the sense that adding any more points would over determine the problem. A natural question to ask is the following: Suppose the number of points that we have 'corresponds' to the 2 spectra case but the choice of values $\alpha_n$ is different, will there still exist a corresponding MIF? For instance, in the 2 spectra case, where all $\alpha_n$ are $\pm 1$, suppose that we replace one of $\alpha_n$ with a different value. Will $\Theta$ still exist? The answer is 'not always' as shown by the following statement.

**Proposition 2.** *Let $\Lambda = \{\lambda_n\}$ be a discrete sequence of real points, $\Lambda_1 = \{\lambda_{2n}\}_{n \in \mathbf{Z}}$, $\Lambda_2^* = \{\lambda_{2n+1}\}_{n \in \mathbf{Z}, \, n \neq 0}$. Let $\Theta$ be an MIF with $\sigma(\Theta) = \Lambda_1$ and $\Theta(\Lambda_2^*) = -1$, then there is an $x \in (\lambda_0, \lambda_2)$ such that the solution of $\Theta(y) = i$ in the interval $(\lambda_0, \lambda_2)$ must lie in $(\lambda_0, x]$.*

For example, if $\Lambda_1 = \{2n\}_{n \in \mathbb{Z}}$ and $\Lambda_2^* = \{2n + 1\}_{n \in \mathbb{Z} \setminus \{0\}}$, then in the interval $(0, 2)$ the solution to $\Theta(y) = i$ lies in the interval $(0, 1.5]$. Thus, here $x = 1.5$. In fact, one could remove any finite number of points from the second spectrum and still have that an $x$ exists such that the solution to $\Theta(y) = i$ must lie in the interval $(0, x)$. A formula for $x$, in terms of the set $\Lambda_2^*$ can be obtained, thus specifying the uncertainty created by the reduction of the second spectrum.

The following illustrates the application to spectral problems for Schrödinger operators.

**Proposition 3.** *Consider any Schrödinger operator $L$ on $(a, b)$, with a fixed boundary condition $\beta$ at $b$. If the spectra of $L$ satisfy*

$$\sigma(L, D, \beta) = \{2n\}_n, \qquad \sigma(L, N, \beta) \subset \{2n + 1\}_{n \neq 0},$$

*then there is no eigenvalue for $(L, e^{i\pi/2}D, \beta)$, in the interval $(1.5, 2)$. Here $D$ and $N$ refer to the Dirichlet and Neumann conditions at $a$.*

Further study of existence of solutions of various interpolation problems by MIFs and implications of such results for spectral problems presents an interesting direction for new research.

# References

[1] ALEKSANDROV, A.B., *Multiplicity of boundary values of inner functions* Izv. Akad. Nauk Armyan. SSR Ser. Mat. 22 (1987), no. 5, pp. 490–503.

[2] ALEKSANDROV, A.B., *Inner functions and related spaces of pseudocontinuable functions,* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 170 (1989), no. Issled. Linein. Oper. Teorii Funktsii. 17, pp. 7–33. English transl., J. Soviet Math. 63 (1993), no. 2, pp. 115–129.

[3] ALEKSANDROV, A.B., *On the existence of angular boundary values of pseudocontinuable functions,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222 (1995), no. Issled. po Linein. Oper. i Teor. Funktsii. 23, pp. 5–17.

[4] ALEKSANDROV, A.B., *On the maximum principle for pseudocontinuable functions,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 217 (1994), no. Issled. po Linein. Oper. i Teor. Funktsii. 22, pp. 16–25.

[5] ALEKSANDROV, A.B., *Isometric embeddings of co-invariant subspaces of the shift operator,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 232 (1996), no. Issled. po Linein. Oper. i Teor. Funktsii. 24, pp. 5–15.

[6] AMBARZUMIAN, V., *Über eine Frage der Eigenwerttheorie,* Zeitschrift für Physik 53 (2009).

[7] BARANOV, A., *de Branges mistake,* Private Communications, 2011.

[8] BEURLING, A., MALLIAVIN, P., *On Fourier transforms of measures with compact support,* Acta Math. 107 (1962), pp. 291-302.

[9] BEURLING, A., MALLIAVIN, P., *On the closure of characters and the zeros of entire functions,* Acta Math. 118 (1967), pp. 79-93.

[10] Borg, G., *Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$,* Proc. 11-th Scandinavian Congress of Mathematicians, Johan Grundt Tanums Forlag, Oslo, 1952, pp. 276-287.

[11] Cima, J., Matheson, A., Ross, W. *The Cauchy Transform,* Mathematical Surveys and Monographs, American Mathematical Society (2006).

[12] Clark, D.N., *One dimensional perturbations of restricted shifts,* Journal d'Analyse Mathématique, December 1972 , Volume 25, Issue 1, pp. 169-191.

[13] De Branges, L., *Hilbert spaces of entire functions,* Prentice - Hall (1968), pp. 281-282.

[14] Del Rio, R., Gesztesy, F., Simon, B., *Inverse Spectral Analysis with partial information on the potential, III. Updating boundary conditions,* International Math. Research Notices, 15 (1997), pp. 751-758.

[15] Del Rio, R., Gesztesy, F., Simon, B., *Corrections and addendums to 'Inverse Spectral Analysis with partial information on the potential, III. Updating boundary conditions',* International Math. Research Notices, 11 (1999), pp. 623-625.

[16] Gesztesy, F., Simon, B., *On the determination of potential from three spectra,* Transactions of the American Mathematical Society, Vol 189(2) (1999), pp. 85-92.

[17] Gesztesy, F., Simon, B., *Inverse Spectral Analysis with partial information on the Potential, II. the case of discrete spectrum,* Transactions of the American Mathematical Society, Vol. 352 (2000), pp. 2765-2787.

[18] Hochstadt, H., Lieberman, B., *An inverse Sturm-Liouville problem with mixed given data,* SIAM J. Appl. Math. 34 (1978), pp. 676-680.

[19] Horváth, M., *Inverse Spectral theory and closed exponential systems,* Annals of Mathematics 162 (2005), pp. 885-918.

[20] Levinson, N., *The Inverse Sturm Liouville problem,* Mat. Tidsskr. B (1949), pp. 25-30.

[21] Levitan, B.M., Sargsjan, I.S., *Sturm-Liouville and Dirac operators,* Kluwer, Dordrecht, (1991).

[22] Makarov, N., Poltoratski, A., *Beurling - Malliavin theory for Toeplitz kernels,* Inventiones Mathematicae, Volume 180, Issue 3 (June 2010), pp. 443-480.

[23] MAKAROV, N., POLTORATSKI, A., *Meromorphic inner functions, Toeplitz kernels and the uncertainty principle,* Perspectives in Analysis, Mathematical Physics Studies Volume 27 (2005), pp. 185-252.

[24] MARCHENKO, V., *Some questions in the theory of one-dimensional linear differential operators of the second order, I,* Trudy Mosk. Mat. Obsch. 1 (1952), pp. 327-420.

[25] MARCHENKO, V., *Sturm-Liouville operators and applications,* Birkhauser, Basel, 1986.

[26] MITKOVSKI, M., POLTORATSKI, A., *Pólya sequences, Toeplitz kernels and gap theorems,* Advances in Mathematics, 224 (2010), pp. 1057-1070.

[27] MITKOVSKI, M., POLTORATSKI, A., *Determinacy of sets and measures,* Inventiones Math., to appear.

[28] MITKOVSKI, M., RUPAM, R., *defining sets for MIFs and uniqueness of potentials,* in preparation.

[29] POLTORATSKI, A., *A Problem Of Completeness of Exponentials,* Annals of Mathematics, Volume 178 (2013), Issue 3, pp. 983-1016.

[30] POLTORATSKI, A., *Spectral Gaps for Sets and Measures,* Acta Math., Volume 208, Issue 1 (2012), pp. 151-200.

[31] POLTORATSKI, A. *Toeplitz Approach to Problems of the Uncertainty Principle (CBMS Regional Conference Series in Mathematics),* AMS (2015).

[32] POLTORATSKI, A., SARASON, D., *Aleksandrov Clark measures,* Recent Advances in Operator related Function Theory, Contemp. Math., Vol. 393 (2006), pp. 1-14.

[33] RUPAM, R., *Uniform Boundedness of derivatives of Meromorphic Inner functions on the Real line,* Journal d'Analyse Mathématique, to appear.

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