DETERMINACY FOR MEASURES

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ABSTRACT. For a given finite positive measure we determine the minimal information that is needed from its Fourier transform to determine the measure completely. In particular, we show that if the support of a measure doesn't contain a sequence of Beurling-Malliavin interior density $d > 0$ then the interval of determinacy for this measure must be of length larger than πd. As a consequence, we provide a sharp lower estimate of the rate of oscillation of high pass signals in terms of their spectral gap. This gives a considerably shorter and more informative proof of the estimate previously obtained by A. Eremenko and D. Novikov.

1. INTRODUCTION

We say that a positive finite measure μ is a-determinate if there exists no other positive finite measure ν such that their Fourier transforms coincide on $[-a, a]$, i. e.,

$$\hat{\mu}(x) = \int e^{ixt} d\mu(t) = \int e^{ixt} d\nu(t) = \hat{\nu}(x)$$

for all $x \in [-a, a]$. It is easy to see that this definition does not depend on the interval $[-a, a]$, but only on its length. One of the main problems that we consider here is the following.

Problem 1.1. *Suppose that we are given a finite positive measure μ. For a given $a > 0$, how can we tell whether μ is a-determinate?*

This is the analog of the determinacy part of the classical moment problem. The later problem has received a lot of attention throughout the years. Still it appears that no explicit solution is known yet even in the classical polynomial case.

The first solution that we offer here, in essence, goes back to M. Riesz who proved the corresponding result for the moment problem. A weaker version of it appeared recently in [13].

Theorem 1.2. *A positive finite measure μ is a-determinate if and only if*

$$\int \frac{\log m^{\mu}_{a/2}(x)}{1 + x^2} dx = \infty,$$

where

$$m^{\mu}_{a/2}(w) := \sup\{|F(w)| : F \in \text{span}\{e^{ixt} : t \in [-a/2, a/2]\} \text{ and } \|F\|_{L^2(\mu)} \le 1\}.$$

This result by itself, although definitive, is not very revealing since it crucially depends on the quantity $m(w)$ which is usually not easy to compute (or estimate). However, in some special cases estimates are possible. These estimates allowed us to deduce the following strengthening of the classical Beurling's gap theorem [1].

Corollary 1.3. *If μ is a positive finite measure which is supported on a set whose complement is long then μ is a-determinate for every $a > 0$.*

As usual we say that a union of open intervals $\cup(a_n, b_n)$ is long if

$$\sum_n \left(\frac{b_n - a_n}{a_n} \right)^2 = \infty.$$

This corollary says that a Fourier transform of a non-zero signed measure cannot vanish on any interval of positive length if its positive part is supported on a set whose complement is long. Notice that we don't require anything about the support of the negative part. This implies that if μ is a-indeterminate for some $a > 0$ then μ must contain a sequence of positive interior Beurling-Malliavin density in its support. It is natural to expect that a more precise relation between the density of the support of μ and the determinacy of μ exists. Our next result makes this observation precise.

Theorem 1.4. *Let μ be a positive finite measure. If μ is a-indeterminate, then the support of μ must contain an a/π-uniform sequence.*

The precise definition of a d-uniform sequence will be given in the next section. Vaguely speaking d-uniform sequences are those sequences which are in certain sense similar to arithmetic progressions with a difference term equal to $1/d$.

We actually prove a more general result. Namely, for a general regular de Branges space $\mathcal{B}_E$ we show that a positive measure μ whose support is sufficiently sparse (in certain precise sense) is uniquely determined by its action on the space, meaning that there is no other positive measure ν such that

$$\int F(t)d\mu(t) = \int F(t)d\nu(t),$$

for all $F \in \mathcal{B}_E$.

Clearly, every a-indeterminate measure is a positive part of a signed measure σ with a spectral gap $(-a, a)$, i. e., such that $\hat{\sigma}(x) = 0$, for $x \in (-a, a)$. Conversely, every non-zero measure σ with a spectral gap $(-a, a)$ gives a rise to two a-indeterminate measures σ^+ and σ^-. Therefore, the determinacy problem is closely related to the gap problem which was recently solved by the second author in [11]. Even though these two problems are not completely equivalent they are very much related. We prove the following result that gives a precise quantitive relation between them. Recall that for a closed set $X \subset \mathbb{R}$ its gap characteristic is defined by

$G(X) = \sup\{a : \text{there exists a signed measure } \mu \text{ supported on } X \text{with spectral gap } (-a, a)\}.$

Define the determinacy characteristic of X by

$Det(X) = \inf\{a > 0 : \text{all } \mu \text{ supported on } X \text{ are } a\text{-determinate}\}.$

Theorem 1.5. *For any closed set $X \subset \mathbb{R}$ the determinacy characteristic of X is equal to its gap characteristic, i. e.,*

$$Det(X) = G(X).$$

A typical property of signed measures with a spectral gap is their oscillation. Our next result gives a precise oscillation rate of measures with a spectral gap (high-pass signals).

Theorem 1.6. *Let A and B be disjoint closed subsets of $\mathbb{R}$. Let $\mathcal{M}_a(A, B)$ be the class of all finite signed measures σ with a spectral gap $(-a, a)$ such that $\operatorname{supp} \sigma^+ \subset A$ and $\operatorname{supp} \sigma^- \subset B$ The following equality holds*

$$\sup\{a > 0 : \mathcal{M}_a(A, B) \neq \{0\}\} = \pi \sup\{d \geq 0 : \exists\{\lambda_n\} \ d\text{-uniform}, \ \{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B\}$$

As an easy consequence, we give the following improvement of the celebrated Eremenko-Novikov gap result [7, 8].

Theorem 1.7. *If σ is a nonzero signed measure with a spectral gap $(-a, a)$ then there exists an a/π-uniform sequence $\{\lambda_n\}$ such that σ has at least one sign change in every $(\lambda_n - \epsilon_n, \lambda_{n+1} + \epsilon_{n+1})$, where $\epsilon_n > 0$ are arbitrary.*

2. Preliminaries

As usual throughout the paper we will denote by $\mathcal{C}_a$ the Cartwright class of entire functions of exponential type no greater than a such that such that $\log|F(t)|/(1 + t^2) \in L^1(\mathbb{R})$. We will also use the standard notation $\mathcal{PW}_a$ for the Paley-Wiener class, and $\mathcal{B}_a$ for the Bernstein class consisting of entire function of type at most a which are bounded on $\mathbb{R}$. Finally, we will denote by $\mathcal{K}_a$ the Krein class consisting of entire functions $F(z)$ of exponential type no greater that a with only simple real zeros $\{\lambda_n\}$ such that $F(iy) = O(e^{T|y|})$ and such that $\sum_n 1/|F'(\lambda_n)| < \infty$. By a well known theorem of M. Krein each function from $\mathcal{K}_a$ belongs in $\mathcal{C}_a$ and satisfies

$$\frac{1}{F(z)} = \sum_n \frac{1}{(z - \lambda_n)F'(\lambda_n)}.$$

Besides the Paley Wiener $\mathcal{PW}_a$ space we will also consider other regular de Branges spaces. We will say that a de Branges space $\mathcal{B}_E$ is regular if

    (i) $\mathcal{B}_E \subset \mathcal{C}_a$
    (ii) If $F \in \mathcal{E}$ and $F(a) = 0$, then $(F(z) - F(a))/(z - a) \in \mathcal{E}$ (as a function of z).

Every de Branges function $E(z)$ gives a rise to an inner function $\Theta(z) := E^\#(z)/E(z)$ and a model space $\mathcal{K}_\Theta$ that this inner function generates. There exists a well known isometric isomorphism between $\mathcal{B}_E$ and $\mathcal{K}_\Theta$ given by $F \to F/E$.

Each inner function $\Theta(z)$ determines a family of positive measures μ_α on $\mathbb{R}$ indexed by $|\alpha| = 1$ in the following way

$$\Re\frac{\alpha + \Theta(z)}{\alpha - \Theta(z)} = p_\alpha \Im z + \frac{1}{\Im z}\int \frac{d\mu_\alpha(t)}{|t - z|^2}, \tag{2.1}$$

for some $p_\alpha \geq 0$. The numbers p_α can be viewed as a point mass at infinity for μ_α. Each measure μ_α is singular, supported on the set $\{\Theta = \alpha\}$, and is Poisson summable. They are usually called the Clark measures for $\Theta(z)$. Throughout the paper when we say that $\Theta(z)$ corresponds to a measure μ we will always assume that $p = 0$ and $\alpha = 1$.

If $\Theta(z)$ comes from a de Branges function, then $p = 0$. In this case, each function $f \in \mathcal{K}_\Theta$ can be represented by the Clark formula $f(z) = \frac{1}{2\pi i}K(fd\mu)(z)$, where $K(fd\mu)$ stands for the Cauchy integral

$$K(fd\mu)(z) = \int \frac{f(t)}{t - z}d\mu(t).$$

This formula gives an isometry between $\mathcal{K}_\Theta$ and $L^2(\mu)$. In this case, all the Clark measures μ_α are moreover discrete and their point masses can be computed by $\mu_\alpha(\lambda) = 2\pi/|\Theta'(\lambda)|$

for $\lambda \in \{\Theta = \alpha\}$. We will call these measures μ_α spectral measures of the corresponding de Branges space.

Each inner $\Theta(z)$ coming from a de Branges function can be written as $\Theta(t) = e^{i\theta(t)}$ for $t \in \mathbb{R}$, where $\theta(t)$ is real analytic strictly increasing function. We will call such $\theta(t)$ a continuous argument of $\Theta(z)$. The phase function of the corresponding de Branges space is defined by $\phi(t) = \theta(t)/2$.

Next we recall the definition of the densities that will be used in our work. As usual we will say that the family of disjoint intervals $\{I_n\}$ on the real line is long if

$$\sum_n \left(\frac{|I_n|}{1 + \operatorname{dist}(0, I_n)} \right)^2 = \infty.$$

Otherwise, we will say that the family is short. In the case when $\{I_n\}$ is short and $\bigcup I_n = \mathbb{R}$ we will call the family $\{I_n\}$ a short partition.

Definition 2.1. A sequence $\Lambda = \{\lambda_n\}$ with no finite accumulation point is said to be regular with density $d > 0$ if there exists a short partition $\{I_n\}$ such that

$$\frac{\int_{I_n} dn_\Lambda(t) - \int_{I_n} dn_{\mathbb{Z}/d}(t)}{|I_n|} \to 0 \text{ as } |n| \to \infty$$

Definition 2.2. The interior Beurling-Malliavin density $D_{BM}^-(X)$ of a closed set $X \subset \mathbb{R}$ is defined to be the supremum of all $d > 0$ for which there exists a regular subsequence $\Lambda \subset X$ with density d. If no such subsequence exists $D_{BM}^-(X) = 0$.

Similarly, for real sequences Λ one defines (the more famous) exterior density $D_{BM}^+(\Lambda)$.

The famous result of Beurling and Malliavin [2] says that the zero set of every function $F(z) \in \mathcal{C}_{a\pi}$ with type exactly $a\pi$ must be regular with density a. Conversely, for every regular sequence of density a and every $k > a$ there exists non-zero $F(z) \in \mathcal{C}_{k\pi}$ which vanishes at that sequence. However, $F(z)$ might have additional zeros.

In [11] it was proved that a very similar theorem holds about the zeros of functions in the Krein class. It turns out that in this case the regularity notion above needs to be refined to take into account a certain delicate separation condition. Namely, for any finite interval $I \subset \mathbb{R}$ and a Borel measure μ we define the energy of μ, $E_I(\mu)$ by

$$E_I(\mu) = \iint_{I,I} \log |x - y| d\mu(x) d\mu(y).$$

This is just the usual energy of the compactly supported $1_I(x) d\mu(x)$.

Definition 2.3. We say that a real sequence $\Lambda = \{\lambda_n\}$ with no finite accumulation point is said to be uniformly distributed if there exists a short partition $\{I_n\}$ such that

$$\sum_n \frac{E_{I_n}(dn_{\mathbb{Z}/a}) - E_{I_n}(dn_\Lambda)}{1 + \operatorname{dist}(0, I_n)^2} < \infty.$$

Definition 2.4. A real sequence $\Lambda = \{\lambda_n\}$ with no finite accumulation point is said to be d-uniform if it is regular with density d and is uniformly distributed.

Notice that every separated sequence must be uniformly distributed. Therefore, for separated sequences the notion of regular sequences with density d coincides with the notion of d-uniformness.

One of the main results in [11] implies that the zero set of every function $F(z) \in \mathcal{K}_{a\pi}$ in the Krein class with type exactly $a\pi$ must be a-uniform. Conversely, for every a-uniform sequence and every $k > a$ there exist non-zero $F(z) \in \mathcal{K}_{k\pi}$ which vanishes at that sequence. However, this function might have additional zeros.

Another property that was proved in [11] which will be useful for us is that a sequence Λ is d-uniform if and only if $n_\Lambda(x) - dx$ is equal to a Hilbert transform of some function in $L^1(dt/(1+t^2))$.

3. M. Riesz-type criterion and its consequences

For a given set of functions $\mathcal{A}$ we will say that a positive measure μ is $\mathcal{A}$-determinate if there is no other positive measure $\nu \neq \mu$ such that

$$\int f(t)d\mu(t) = \int f(t)d\nu(t),$$

for all $f \in \mathcal{A}$. Otherwise, we say that μ is $\mathcal{A}$-determinate. We will study determinacy on linear spaces of entire functions $\mathcal{E}$ satisfying the following properties:

(E1) If $F(z) \in \mathcal{E}$, then $F^{\#}(z) := \overline{F(\bar{z})} \in \mathcal{E}$,

(E2) If $F(z) \in \mathcal{E}$ and $F(a) = 0$, then $(F(z) - F(a))/(z - a) \in \mathcal{E}$ (as a function of z).

(E3) There exists $a \geq 0$ such that $\mathcal{E} \subset \mathcal{C}_a$.

There are many examples of sets $\mathcal{E}$ that satisfy (E1)-(E3). The set of all polynomials $\mathcal{P}$ is one such example. Other examples that we will use include the Bernstein class $\mathcal{B}_a$, the Paley-Wiener space $\mathcal{PW}_a$, and other regular de Branges spaces (those de Branges spaces which satisfy (E2) and (E3)).

It is not hard to see that a positive finite measure μ is a-determinate if and only if it is $\mathcal{B}_a$-determinate or equivalently $\mathcal{PW}_a$-determinate.

The next result gives a criterion for determinacy which generalizes the classical M. Riesz criterion for determinacy in the moment problem. Our proof will also rely on the M. Riesz original idea. However, we will use the de Branges spaces as a substitute for the Gauss quadrature formula. We will use the following notation

$$\mathcal{E} * \mathcal{E} = \{F : F = \sum_{i=1}^{n} G_i H_i, \text{ for some } G_i, H_i \in \mathcal{E}\}.$$

Theorem 3.1. *Let $\mathcal{E}$ be a set which satisfy (E1)-(E3) and let μ be a positive measure such that $\mathcal{E} \subset L^1(\mu) \cap L^2(\mu)$. The following are equivalent:*

(a) *μ is $\mathcal{E} * \mathcal{E}$-indeterminate*

(b) *The $L^2(\mu)$-closure of $\mathcal{E}$ is a de Branges space.*

(c) *The majorant $m(x) := \sup\{|F(x)| : F \in \mathcal{E}, \|F\|_{L^2(\mu)} \leq 1\}$ satisfies*

$$\int \frac{\log m(x)}{1 + x^2} dx < \infty,$$

Proof. First we prove that the integrability of $\log m(x)/(1+x^2)$ implies that the $L^2(\mu)$ closure of $\mathcal{E}$ is a de Branges space. We will denote this closure by $\mathcal{B}(\mu)$.

Each function F in $\mathcal{E}$ is in the Cartwright class $\mathcal{C}_a$. Therefore

$$\log |F(z)| \leq a|\Im z| + \frac{|\Im z|}{\pi} \int \frac{\log^+ |F(t)|}{|t - z|^2} dt.$$

Taking a supremum over all $F(z) \in \mathcal{E}$ with norm no greater than 1 we first obtain

$$\log|F(z)| \le a|\Im z| + \frac{|\Im z|}{\pi} \int \frac{\log^+ m(t)}{|t-z|^2} dt, \tag{3.1}$$

for all non-real z. Using the simple estimate

$$\sup_{t \in \mathbb{R}} \left| \frac{t-i}{t-z} \right| \le \frac{1+|z|}{|\Im z|},$$

which is valid for all $z \notin \mathbb{R}$, we obtain

$$\log|F(z)| \le a|\Im z| + C\frac{(1+|z|)^2}{|\Im z|}, \tag{3.2}$$

where

$$C = \frac{1}{\pi} \int \frac{\log^+ m(t)}{1+t^2} dt < \infty.$$

This immediately shows that $m(z)$ is bounded on compact sets that don't intersect the real axis. Standard application of the Levinson's log log theorem shows that $m(z)$ is also bounded on compact sets that intersect the real axis.

Let now $\{F_n\}$ be a sequence of functions in $\mathcal{E}$ which converges in $L^2(\mu)$ to some limit $f \in L^2(\mu)$. The goal is to show that f is actually a restriction of some entire function $F \in \mathcal{E}(\mu)$. To see this notice that $\{F_n\}$ being Cauchy and the inequality

$$|F_k(z) - F_l(z)| \le m(z)\|F_k - F_l\|_{L^2(\mu)},$$

together with the fact that $m(z)$ is locally bounded imply that $\{F_n\}$ converges uniformly on compact sets to some entire function F which agrees with f a.e. μ. It remains to show that this $F(z) \in \mathcal{C}_a$. But this follows immediately from the fact that F also satisfies (3.1).

Let $F \in \mathcal{B}_T(\mu)$ and let $F_n \in \mathcal{E}$ be a sequence that converges to F. Since $F_n(z)$ converges to $F(z)$ uniformly on compact sets, we have that for any $w \in \mathbb{C}$

$$\frac{|F(w)|}{\|F\|} = \lim_{n \to \infty} \frac{|F_n(w)|}{\|F_n\|} \le m(w). \tag{3.3}$$

Therefore, non-real point evaluations are bounded in $\mathcal{B}(\mu)$. It is trivial to see that $F \in \mathcal{B}(\mu)$ implies $F^\# \in \mathcal{B}(\mu)$. So, by a well known criterion of de Branges it remains to show that $F(z)(z-\bar{z}_0)/(z-z_0) \in \mathcal{B}(\mu)$ for all $F \in \mathcal{B}(\mu)$ and $z_0 \notin \mathbb{R}$ such that $F(z_0) = 0$. Since

$$F(z)\frac{z-\bar{z}_0}{z-z_0} = F(z) - (z_0 - \bar{z}_0)\frac{F(z)}{z-z_0},$$

it is enough to show that $F(z)/(z-z_0) \in \mathcal{B}(\mu)$. Let $F_n \in \mathcal{E}$ be a sequence that converges to F. We have

$$\int \left| \frac{F(t)}{t-z_0} - \frac{F_n(t)}{t-z_0} \right|^2 d\mu(t) \le \frac{1}{|\Im z_0|^2} \|F_n - F\|^2.$$

Thus, by (E2) it follows that $F(z)/(z-z_0) \in \mathcal{B}(\mu)$. So $\mathcal{B}(\mu)$ is a de Branges space.

If $\mathcal{B}(\mu)$ is a de Branges space pick any spectral measure ν of $\mathcal{B}(\mu)$ which is different from μ (in case μ is one of them). Let $F \in \mathcal{E} * \mathcal{E}$ be arbitrary. Then by definition $F = GH$ for some $G, H \in \mathcal{E}$. Since $\mathcal{B}(\mu)$ is isomorphic to $L^2(\nu)$ and is isometrically contained in $L^2(\mu)$ we obtain that

$$\int F(t)d\mu(t) = \int \sum G_i(t)\overline{H_i^\#(t)}d\mu(t) = \sum \int G_i(t)\overline{H_i^\#(t)}d\nu(t) = \int F(t)d\nu(t).$$

Thus, μ is $\mathcal{E} * \mathcal{E}$-indeterminate.

Next we show that if μ is $\mathcal{E} * \mathcal{E}$-indeterminate then

$$\int \frac{\log m(x)}{1 + x^2} dx < \infty.$$

If μ is $\mathcal{E}$-indeterminate there exists another positive finite measure $\nu \neq \mu$ such that $\int F(t) d\mu(t) = \int F(t) d\nu(t)$ for all $F \in \mathcal{E} * \mathcal{E}$. This implies

$$\int |F(t)|^2 d\mu(t) = \int |F(t)|^2 d\nu(t), \tag{3.4}$$

for all $F(z) \in \mathcal{E}$.

Consider $\sigma = \mu - \nu$. We have that

$$\int \frac{F(t) - F(z)}{t - z} d\sigma(t) = 0,$$

for every function $F(z) \in \mathcal{E}$ and every $z \notin \mathbb{R}$. This implies that

$$F(z) = \frac{1}{G(z)} \int \frac{F(t)}{t - z} d\sigma(t),$$

where $G(z) = \int \frac{d\sigma(t)}{t-z}$. In particular, for $z = x + i$ with $x \in \mathbb{R}$ we have,

$$|F(x + i)| \leq \frac{1}{|G(x + i)|} \int |F(t)| d|\sigma|(t) \leq \frac{C}{|G(x + i)|} \left(\int |F(t)|^2 d|\sigma|(t) \right)^{1/2}.$$

Now since $|\sigma| \leq \mu + \nu$ we obtain,

$$|F(x + i)| \leq \frac{\sqrt{2}C}{|G(x + i)|} \left(\int |F(t)|^2 d\mu(t) \right)^{1/2},$$

for every $F(z) \in \mathcal{E}$. Consequently,

$$m(x + i) \leq \frac{C'}{|G(x + i)|}.$$

The fact that $G(z + i)$, as a function of z, is of exponential type and bounded on $\mathbb{R}$ implies that

$$\int \frac{\log m(x + i)}{1 + x^2} dx \leq \int \frac{\log^- |G(x + i)|}{1 + x^2} dx + C'' < \infty.$$

Now, let $F(z)$ be arbitrary element of $\mathcal{E}$ with norm no greater than 1. Then $F(z + i)$ as a function of z is in the Cartwright class $\mathcal{C}_a$. Therefore, for all $x \in \mathbb{R}$ we have

$$\log |F(x)| \leq a + \frac{1}{\pi} \int \frac{\log^+ |F(t + i)|}{(t - x)^2 + 1} dt.$$

Taking a supremum over all such $F(z)$ we first obtain

$$\log |F(x)| \leq a + \frac{1}{\pi} \int \frac{\log^+ |m(t + i)|}{(t - x)^2 + 1} dt,$$

and then again by taking a supremum on the left we deduce

$$\log m(x) \leq a + \frac{1}{\pi} \int \frac{\log^+ |m(t + i)|}{(t - x)^2 + 1} dt.$$

Finally,

$$\int \frac{\log m(x)}{1+x^2}dx \leq \frac{a\pi}{2} + \frac{1}{\pi}\int\int \frac{\log^+|m(t+i)|}{(t-x)^2+1}\frac{1}{1+x^2}dtdx$$

$$= a\pi + \int \frac{2\log m(x+i)}{4+x^2}dx < \infty.$$

This completes the proof.

$\square$

Notice that the set of all polynomials $\mathcal{P}$ satisfies $\mathcal{P} * \mathcal{P} = \mathcal{P}$, so we derive the classical M. Riesz criterion as a special case. In the case $\mathcal{E} = \mathcal{B}_a$ we obtain Theorem 1.2 from the introduction.

As we mentioned in the introduction, the usefulness of the criterion above depends on the possibility to estimate the majorant $m(w)$. Below we consider some cases when this is doable.

Corollary 3.2. *Let μ be a positive finite measure. If there exists a non negative uniformly continuous function $w(t)$ on $\mathbb{R}$ satisfying $e^{w(t)} \in L^1(\mu)$ and*

$$\int \frac{w(t)}{1+t^2}dt = \infty,$$

then μ is a-determinate for every $a > 0$.

Proof. First note that we may replace the uniform continuity assumption on $w(t)$ by the stronger one that $w(t)$ is uniformly Lipschitz (see [9, pg. 96]). Next notice that every entire function $F(z)$ of exponential type at most a for which $F(t)/e^{w(t)/2} \leq 1$ on $\mathbb{R}$ must also satisfy $\int |F(t)|^2 d\mu(t) \leq 1$. Therefore, by combining the corollary in [9, pg. 236] (for $W(t) = e^{w(t)/2}$) with Theorem 1.2 we obtain

$$\int \frac{\log m_a^\mu(t)}{1+t^2}dt = \infty,$$

for every $a > 0$. The result now follows from Theorem 3.1. It should be noted that the idea illustrated here is due to de Branges. $\square$

We next prove a strengthening of Levinson's gap result [10].

Corollary 3.3. *Let μ be a positive finite measure. If there exists a non negative function $w(t)$ on $\mathbb{R}$ which is increasing on $[0, \infty)$ and satisfies $e^{w(t)} \in L^1(\mu)$ and*

$$\int_1^\infty \frac{w(t)}{1+t^2}dt = \infty,$$

then μ is a-determinate for every $a > 0$.

Proof. Let $v(t)$ be the largest minorant of $w(t)$ which is uniformly Lipschitz with Lipschitz constant 1. Then (see the lemma in [9, pg. 239])

$$\int \frac{v(t)}{1+t^2}dt = \infty,$$

and the result follows from the previous corollary. $\square$

Finally, we prove a strengthening of the Beurling's gap result.

Corollary 1.3. *If μ is a positive finite measure which is supported on a set whose complement is long then μ is a-determinate for every $a > 0$.*

Proof. Define $w(t)$ to be a function which is zero outside of the intervals (a_n, b_n) and on each of those intervals let the graph of $w(t)$ be a right isosceles triangle with hypothenuse (a_n, b_n). This function is clearly non negative, uniformly continuous and satisfies $e^{w(t)} \in L^1(\mu)$. Moreover, the assumption that $\cup(a_n, b_n)$ is long yields that

$$\int \frac{w(t)}{1 + t^2} dt = \infty.$$

The result now follows from Corollary 3.2. $\qquad\square$

4. Extreme measures in the indeterminate case

Let $\mathcal{A}$ be a set of functions on $\mathbb{R}$. For a given positive measure μ with $\mathcal{A} \subset L^1(\mu)$ denote by $\mathcal{M}_\mathcal{A}(\mu)$ the set of all positive measures ν such that $\int F(t) d\mu(t) = \int F(t) d\nu(t)$ for all $F \in \mathcal{A}$. Clearly, this set is convex and therefore it is interesting to describe its extreme points. The following result, which in essence goes back to M. Naimark, gives a description of the extreme points of $\mathcal{M}_\mathcal{A}(\mu)$.

Theorem 4.1. *A positive finite measure $\nu \in \mathcal{M}_\mathcal{A}(\mu)$ is an extreme point of $\mathcal{M}_\mathcal{A}(\mu)$ if and only if the span of the set $\mathcal{A}$ is dense in $L^1(\nu)$.*

Proof. Assume that $\nu \in \mathcal{M}_\mathcal{A}(\mu)$ is not an extreme point of $\mathcal{M}_\mathcal{A}(\mu)$. Then $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ for some $\nu_1, \nu_2 \in \mathcal{M}_\mathcal{A}(\mu)$ and $0 < \alpha < 1$. It is easy to see that

$$\phi(f) = \int f(t) d\nu(t) - \int f(t) d\nu_1(t),$$

is a non trivial bounded linear functional on $L^1(\nu)$ which vanishes on the set $\mathcal{A}$. Therefore, the span of $\mathcal{A}$ cannot be dense in $L^1(\nu)$.

Conversely, assume that the span of $\mathcal{A}$ is not dense in $L^1(\nu)$. Then there exists a nontrivial bounded linear functional ϕ on $L^1(\nu)$ which vanish identically on $\mathcal{A}$. We can assume that its norm is 1. In this case both $\phi_1(f) = \int f(t) d\nu(t) - \phi(f)$ and $\phi_2(f) = \int f(t) d\nu(t) + \phi(f)$ are nonnegative linear functionals on $C_0(\mathbb{R})$ in the sense that both $\phi_1(f) \geq 0$ and $\phi_2(f) \geq 0$ whenever $f(t) \geq 0$ on $\mathbb{R}$. Therefore, there exist positive finite measures ν_1 and ν_2 such that $\phi_1(f) = \int f(t) d\nu_1(t)$ and $\phi_2(f) = \int f(t) d\nu_2(t)$ for all $f \in L^1(\nu)$. Notice that $\phi \neq 0$ implies that $\phi_1 \neq \phi \neq \phi_2$ and hence $\nu_1 \neq \nu \neq \nu_2$. Also for all $F \in \mathcal{A}$ we have that $\int F(t) d\nu_1(t) = \int F(t) d\nu(t) = \int F(t) d\nu_2(t)$ and therefore $\nu_1, \nu_2 \in \mathcal{M}_\mathcal{A}(\mu)$. Finally, since clearly

$$\nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2$$

we obtain that ν cannot be an extreme point for $\mathcal{M}_\mathcal{A}(\mu)$. $\qquad\square$

Next, we consider the set of finite signed measure which annihilate a given de Branges space $\mathcal{B}_E$. Notice that their positive and negative variations are $\mathcal{B}_E$-indeterminate. Conversely, every $\mathcal{B}_E$ indeterminate measure is a positive part of an annihilating signed measure. It will be crucial for us to examine the properties of the extreme points of the set of all annihilating measures of a given de Branges space $\mathcal{B}_E$. Our approach is based on de Branges' extreme point method [3–6].

Let A and B be two disjoint closed subsets of $\mathbb{R}$. Denote by $\mathcal{M}_E(A, B)$ the set of all signed Radon measures σ of total variation $\|\sigma\| \leq 1$, which annihilate $\mathcal{B}_E$, and such that $\operatorname{supp} \sigma^+ \subset A$, $\operatorname{supp} \sigma^- \subset B$. Here, as usual, σ^+ and σ^- denote the positive and negative parts of σ in the canonical Jordan decomposition $\sigma = \sigma^+ - \sigma^-$.

Lemma 4.2. *The set $\mathcal{M}_E(A, B)$ is weak-* compact and convex. In particular, it has nonzero extreme points.*

Proof. Convexity part is obvious. To show that $\mathcal{M}_E(A, B)$ is weak-* compact it is enough to show that it is weak-* closed. Let $\nu_n \in \mathcal{M}_E(A, B)$ be a sequence of measures that converges to some measure ν in the weak-* sense. It is clear that ν is another Radon signed measure with total variation no greater than 1. Moreover, the fact that each ν_n annihilates $\mathcal{B}_E$ implies that $\int F(t) d\nu_n(t) = 0$ for every function F in $\mathcal{B}_E$. Since each such function F is continuous on the real line and vanishes at infinity, the weak-* convergence on ν_n implies that $\int F(t) d\nu(t) = 0$. Therefore, ν must also annihilate $\mathcal{B}_E$. So, it remains to show that $\operatorname{supp} \nu^+ \subset A$ and $\operatorname{supp} \nu^- \subset B$.

Suppose that there exists a Borel set S disjoint from A such that $\nu^+(S) > 0$. Without loss of generality we can assume that $\nu^-(S) = 0$ because otherwise we can take S to be $S \cap P$. Inner regularity of ν^+ implies that there exists a compact set $K \subset S$ with $\nu^+(K) > \nu^+(S)/2$. Similarly, outer regularity of ν^- implies existence of an open set G such that $S \subset G \subset A^c$ and $\nu^-(G) < \nu^-(S) + \nu^+(S)/4 = \nu^+(S)/4$. By Urysohn's lemma there exists a continuous function f which vanishes at infinity and such that $f = 1$ on K, $f = 0$ outside of G and $0 \leq f \leq 1$ everywhere. Now, it is easy to see that

$$\int f d\nu \geq \nu^+(K) - \nu^-(G) \geq \nu^+(S)/4 > 0.$$

On the other hand for all $n \in \mathbb{N}$ we have $\int f d\nu_n = \int f d\nu_n^- \leq 0$. We have a contradiction. The proof that $\operatorname{supp} \nu^- \subset B$ is similar.

The Krein-Milman theorem now implies that $\mathcal{M}_E(A, B)$ is a closed convex hull of its extreme points, and clearly some of them must be non-zero since $\mathcal{M}_E(A, B)$ contains a non-zero element.

$\square$

Lemma 4.3. *Let ν be an extreme point in $\mathcal{M}_E(A, B)$. Then any function $f \in L^1(|\nu|)$ which is not in the $L^1(|\nu|)$-closure of $\mathcal{B}_E$ must satisfy*

$$\int f(t) d\nu(t) \neq 0.$$

Moreover, the space of all bounded linear functionals on $L^1(|\nu|)$ which vanish on $\mathcal{B}_E$ is one-dimensional.

Proof. Let $f \in L^1(|\nu|)$ be a non-zero function which is not in the $L^1(|\nu|)$-closure of $\mathcal{B}_E$. Then there exists a bounded linear functional which is zero on $\mathcal{B}_E$ and non-zero at f. Each such functional L on $L^1(|\nu|)$ can be represented as $L(f) = \int f(t) h(t) d|\nu|(t)$ for some $h \in L^\infty(|\nu|)$. Therefore, we have $\int F(t) h(t) d|\nu|(t) = 0$, for every $F \in \mathcal{B}_E$ and $\int f(t) h(t) d|\nu|(t) \neq 0$. Let $d\nu(t) = b(t) d|\nu|(t)$ be the polar decomposition of ν. Using the fact that $F \in \mathcal{B}_E$ implies $F^\# \in \mathcal{B}_E$ we can assume with no loss in generality that h/b is real valued. Also, by adding a constant if necessary, we may assume that h/b is a nonnegative function. Choose $\lambda \in (0, 1)$ such that $\lambda h/b$ is bounded by 1 on the real line. Then $\nu_1 = \lambda h d|\nu|$ and $\nu_2 =$

$(\nu - \lambda h d |\nu|)/(1-\lambda)$ are both members of $\mathcal{M}_a(\mu)$ such that $\nu = \lambda \nu_1 + (1-\lambda)\nu_2$. Since ν is an extreme point and clearly $\nu_1 \neq 0$ we must have that $\nu_2 = 0$. Therefore, $\nu = \lambda h |\nu|$ and hence

$$\int f(t)d\nu(t) \neq 0.$$

For the second part, notice that $\nu = \lambda h |\nu|$ implies $\lambda h = b$ and therefore $L(f) = \lambda \int f(t)d\nu(t)$.
□

Lemma 4.4. *Let ν be a finite measure and let $\mathcal{B}_E$ be a regular de Branges space such that $\mathcal{B}_E \subset \mathcal{C}_a$. Then for every function $f \in L^1(|\nu|)$ which is in the $L^1(|\nu|)$-closure of $\mathcal{B}_E$ there exists an entire function F in the Cartwright class $\mathcal{C}_a$ such that $F = f$ a.e. $|\nu|$.*

Proof. The proof is very similar to the proof of Theorem 3.1. Let

$$m(w) = \sup\{|F(w)| : F \in \mathcal{PW}_a \text{ and } \|F\|_{L^1(|\nu|)} \leq 1\}.$$

Using the fact that ν annihilates $\mathcal{B}_E$ it follows that

$$\int \frac{F(t) - F(z)}{t-z} d\nu(t),$$

for all $F \in \mathcal{PW}_a$ and non-real z. Then clearly

$$m(z)\left|\int \frac{d\nu(t)}{t-z}\right| \leq \frac{1}{|\Im z|}.$$

Exactly as in the proof of Theorem 3.1 this implies that $m(t)$ is Poisson summable, that $m(w)$ is locally bounded, and that

$$\log m(w) \leq a|\Im w| + \frac{|\Im w|}{\pi}\int \frac{\log^+ m(t)}{|t-w|^2}dt,$$

for all non-real w. Let now $\{F_n(z)\}$ be a sequence of functions in $\mathcal{B}_E$ which converges in $L^1(|\nu|)$ to some limit $f \in L^1(|\nu|)$. Then $\{F_n\}$ being Cauchy and the inequality

$$|F_k(z) - F_l(z)| \leq m(z)\|F_k - F_l\|_{L^1(|\nu|)},$$

together with the fact that $m(z)$ is locally bounded imply that $\{F_n\}$ converges uniformly in compact sets to some entire function F which agrees with f a.e. μ. Finally, it is then easy to see that $F \in \mathcal{C}_a$.
□

Lemma 4.5. *The support of any extreme point $\nu \in \mathcal{M}_E(A, B)$ is a discrete set (with no finite accumulation points).*

Proof. Since $|\nu|(\mathbb{R}) = 1$ and $\nu(\mathbb{R}) = 0$, the support of ν contains more than one point. Let (a, b) be any finite interval that contains at least two points of the support of ν. There exists a function $f \in L^1(|\nu|)$ which vanishes outside of (a, b) such that $\int |f(t)|d|\nu|(t) = 1$ and $\int f(t)d\nu(t) = 0$. By Lemma 4.3 and Lemma 4.4, there exists an entire function $F \in \mathcal{C}_a$ such that $F = f$ a.e. $|\nu|$. It follows that the support of ν outside of (a, b) must be contained in the zeros of F. Since F is non zero, the support of ν outside of (a, b) cannot have a finite accumulation point. Finally, since (a, b) was arbitrary, the whole support of ν must be a discrete set.
□

To summarize, we obtained that there always exists a non-zero extreme measure $\nu \in \mathcal{M}_E(A, B)$. Moreover, this ν is discrete and has the property that the only (up to a constant) bounded linear functional on $L^1(|\nu|)$ which annihilates $\mathcal{B}_E$ is given by $L(f) = \int f(t)d\nu(t)$.

Let $\Theta = E^\#/E$ be the inner function that corresponds to the de Branges function E. If ν annihilates $\mathcal{B}_E$, then $Ed\nu$ annihilates and hence the Cauchy integral of $\bar{E}d\nu$ is divisible by Θ. In particular, if $\nu \in \mathcal{M}_E(A, B)$ is an extreme point, then, by Lemma 4.3, the only (up to a constant) bounded linear functional on $L^1(|\nu|)$ which annihilates the model space $\mathcal{K}_\Theta$ is given by $L(f) = \int f(t)E(t)d\nu(t)$. Therefore, there is no non-constant bounded function h for which the Cauchy integral $K(h\bar{E}d\nu)$ is divisible by Θ. We will use this observation in the proof of our next result.

Theorem 4.6. *Let $\mathcal{B}_E$ be a regular de Branges space and let $\phi(t)$ be the corresponding phase function. If μ is a $\mathcal{B}_E$-indeterminate measure, then the support of μ contains a sequence Λ whose counting function satisfies $\pi n_\Lambda(t) - \phi(t) = \tilde{h}(t)$ for some $h \in L^1(\frac{dt}{1+t^2})$.*

Proof. Let μ_1 be another $\mathcal{B}_E$-indeterminate measure. Then $\sigma = \mu - \mu_1$ is a finite signed measure which annihilates $\mathcal{B}_E$ and (after scaling) $\sigma \in \mathcal{M}_E(A, B)$ for $A := \operatorname{supp} \sigma^+, B = \operatorname{supp} \sigma^-$. Notice that clearly $A \subset \operatorname{supp} \mu$ and $B \subset \operatorname{supp} \mu_1$. By Lemma 4.2 there exists an extreme measure $\nu \in \mathcal{M}_E(A, B)$ which has to be discrete by Lemma 4.5.

Let $\Theta = E^\#/E$ be the inner function that corresponds to the de Branges function E. Then $\Theta(t) = e^{2i\phi(t)}$. Let Ψ be the inner function corresponding to the Clark measure $|E|\,|\nu|$. The goal is to show that there exists an outer function $h \in \ker T_{\bar{\Psi}\Theta}$ such that $\Psi\bar{\Theta}\bar{h} = h$ is also outer.

Define $g = (1 - \Psi)K(\bar{E}\nu)$. Then clearly $\bar{E}d\nu = g|E|\,|d\nu|$ and $g \in \mathcal{K}_\Psi$. Moreover, the fact that ν is real and annihilates $\mathcal{B}_E$ implies that $K(\bar{E}\nu)$ is divisible by Θ. Therefore, $g = \Theta h$ for some $h \in \mathcal{H}^2$. Assume that h is not outer, i. e., there exists an inner function Φ such that $h = \Phi k$ for some $k \in \mathcal{H}^2$. Then $\Theta(1 + \Phi)k \in \mathcal{K}_\Psi$ and hence, by Clark's formula

$$\Theta(1 + \Phi)k = (1 - \Psi)K(\Theta(1 + \Phi)k|E|\,|d\nu|).$$

So, the Cauchy integral of the measure $\Theta(1 + \Phi)k|E|\,|d\nu|$ is also divisible by Θ. However,

$$\Theta(1 + \Phi)k|E|\,|d\nu| = \frac{1 + \Psi}{\Psi}\bar{E}d\nu.$$

Thus, we obtained a non-constant bounded function $(1 + \Psi)\bar{\Psi}$ such that the Cauchy integral of $(1 + \Psi)\bar{\Psi}\bar{E}d\nu$ is also bounded by Θ. This is in contradiction with the fact that ν is an extreme point. Therefore, h is outer.

To show that $\Psi\bar{\Theta}\bar{h} = h$ is simple. Just notice that

$$\Psi\bar{\Theta}\bar{h} = \Psi\bar{g} = (1 - \Psi)K(\bar{\Theta}\bar{E}\nu) = (1 - \Psi)K(\bar{\Theta}g|E|\,|d\nu|) = (1 - \Psi)K(h|E|\,|d\nu|) = h.$$

Therefore,

$$e^{i(\psi(t)-2\phi(t))} = \bar{\Psi}(t)\Theta(t) = \frac{h\bar{(t)}}{h(t)} = \frac{e^{h(t)-i\tilde{h}(t)}}{e^{h(t)+i\tilde{h}(t)}} = e^{-2i\tilde{h}(t)},$$

where $\psi(t)$ denotes the increasing argument of the inner function Ψ. Thus, $\psi(t) - 2\phi(t) \in \tilde{L}^1(\frac{dt}{1+t^2})$.

Let $\Lambda := \operatorname{supp} \nu \subset A$. Then $\{\Psi = 1\} = \Lambda$ and hence $|2\pi n_\Lambda(t) - \psi(t)| \leq 2\pi$. So, we also have that $2\pi n_\Lambda(t) - \psi(t) \in \tilde{L}^1(\frac{dt}{1+t^2})$ and we get the desired conclusion

$$\pi n_\Lambda(t) - \phi(t) \in \tilde{L}^1\left(\frac{dt}{1+t^2}\right).$$

$\square$

In the case $\mathcal{B}_E = \mathcal{PW}_a$ we have the following more precise result.

Theorem 1.4. *Let μ be a positive finite measure. If μ is a-indeterminate, then the support of μ must contain an a/π-uniform sequence.*

Proof. If μ is a-indeterminate, then the previous theorem implies that the support of μ contains a sequence Λ whose counting function satisfies $\pi n_\Lambda(t) - at = \tilde{h}(t)$ for some $h \in L^1(\frac{dt}{1+t^2})$. As mentioned in the introduction, it was proved in [11] that this is equivalent to the fact that Λ is a/π-uniform.

$\square$

As we mentioned in the introduction there is a strong connection between the determinacy and the gap problems.

Theorem 1.5. *For any closed set $X \subset \mathbb{R}$ the determinacy characteristic of X is equal to its gap characteristic, i. e.,*

$$Det(X) = G(X).$$

Proof. Let $Det(X) = d$. By definition, for any $\epsilon > 0$ there exists a $(d - \epsilon)$-indeterminate measure μ with a support included in X. By Theorem 1.4 we have that $\operatorname{supp} \mu$ contains a $(d-\epsilon)/\pi$-uniform sequence. Therefore, by the gap theorem in [11] we have that $G(X) \geq d-\epsilon$. So, since $\epsilon > 0$ is arbitrary we obtain that $G(X) \geq Det(X)$.

To prove the other inequality let $G(X) = a$. By definition, for any $\epsilon > 0$ there exists a finite signed measure σ with a spectral gap $(-a + \epsilon, a - \epsilon)$. Its positive part σ^+ is $(a - \epsilon)$-indeterminate measure supported on X as well. Therefore, $Det(X) \geq a - \epsilon$ and hence $Det(X) \geq G(X)$.

$\square$

Lemma 4.7. *Let $\mathcal{B}_E \subset \mathcal{C}_a$. If $\nu \in \mathcal{M}_E(A, B)$ is extreme, then*

$$G = \frac{1}{K d\nu} \in \mathcal{K}_a.$$

The location of the zeroes of $G(z)$ coincides with the support of ν. Moreover, the supports of ν^+ and ν^- are interlaced, i.e., for any two points a, b in the support of ν^+ there exists a point c from the support of ν^- such that $a < c < b$ and vice versa.

Proof. The proof of the first part is the same as the proof of Theorem 66 in [6]. Therefore, we prove only the second part. Clearly, the function $G(z)$ must vanish at every point c in the support of ν, and for all such c, $G'(c) = 1/\nu(c)$. For any two consecutive zeros a and b of $G(z)$, the following inequality trivially must hold $G'(a)G'(b) < 0$. Therefore, $\nu(a)\nu(b) < 0$ for any two consecutive points in the support of ν. Thus, the supports of ν^+ and ν^- must be interlaced.

$\square$

The following result represents a refinement of de Branges Theorem 66 in [6].

Theorem 4.8. *Let A and B be disjoint closed subsets of $\mathbb{R}$. The necessary and sufficient condition for $\mathcal{M}_a(A, B)$ to contain a non-zero measure, is that there exists an entire function $G(z)$ in the Krein class $\mathcal{K}_a$ with zero set $\Lambda = \{\lambda_n\}$ satisfying $\{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B$.*

Proof. Assume that $\mathcal{M}_a(A, B)$ contains a non-zero measure. It follows from Theorem 4.7 that there exists an entire function $G(z)$ in the Krein class $\mathcal{K}_a$ with zero set $\Lambda = \{\lambda_n\}$ satisfying $\{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B$. Conversely, if such function exists define a signed discrete measure σ supported on Λ simply by $\sigma(\{\lambda_n\}) = 1/G'(\lambda_n)$. Using the properties of $G(z)$ it is easy to show that this measure (or perhaps $-\sigma$) belongs in $\mathcal{M}_a(A, B)$. $\square$

To prove our oscillation result below we will need the following technical lemma. It basically says that any subsequence of real zeros of a given function $F \in \mathcal{PW}_a$ can be replaced by double zeroes without changing the exponential type and still keeping some integrability properties on the real line.

Lemma 4.9. *Let $F \in \mathcal{PW}_a$ with infinitely many real zeros and let $\Lambda = \{\lambda_n\}$ be a subsequence of the real zeros of F indexed so that*

$$\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \ldots$$

Then there exists $c \in \mathbb{R}$ and a sequence $\Gamma := \{\gamma_n\}$ such that $\gamma_n \in (\lambda_{2n}, \lambda_{2n+1})$ for n positive and $\gamma_n \in (\lambda_{2n+1}, \lambda_{2n})$ for n negative such that the entire function $G(z)$ defined by

$$G(z) = e^{-cz} \frac{\prod_{\gamma \in \Gamma} \left(1 - \frac{z}{\gamma}\right)^2 e^{-\frac{2z}{\gamma}}}{\prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) e^{-\frac{z}{\lambda}}} F(z)$$

satisfies

$$\int \frac{|G(x)|}{1 + x^2} dx < \infty.$$

Proof. For each n positive let $\gamma_n := \sqrt{\lambda_{2n-1} \lambda_{2n}}$ and for n negative $\gamma_n := -\sqrt{\lambda_{2n+1} \lambda_{2n}}$. Clearly, this sequence $\Gamma = \{\gamma_n\}$ satisfies $\gamma_n \in (\lambda_{2n}, \lambda_{2n+1})$ for n positive and $\gamma_n \in (\lambda_{2n+1}, \lambda_{2n})$ for n negative. Define

$$c := \int \frac{2n_\Gamma(t) - n_\Lambda(t)}{t^2} dt < \infty.$$

On the real line we have

$$\log |G(x)| = -cx + x^2 \int \frac{1}{x - t} \frac{2n_\Gamma(t) - n_\Lambda(t)}{t^2} dt + \log |F(x)|.$$

Simple algebra implies

$$\log |G(x)| = -cx + x \int \frac{2n_\Gamma(t) - n_\Lambda(t)}{t^2} dt + \int \frac{2n_\Gamma(t) - n_\Lambda(t)}{t} dt + \int \frac{2n_\Gamma(t) - n_\Lambda(t)}{x - t} dt + \log |F(x)|. \tag{4.1}$$

By the choice of Γ we have $\int (2n_\Gamma(t) - n_\Lambda(t))/t \, dt = 0$. Therefore,

$$\log |G(x)| = \tilde{u}(x) + \log |F(x)|,$$

where $u(x) = 2n_\Gamma(x) - n_\Lambda(x)$. Since $\|u\|_\infty \leq 1 < \pi/2$ by the Smirnov-Kolmogorov estimate

$$\int \frac{e^{\tilde{u}(x)}}{1 + x^2} dx < \infty.$$

Therefore, using the fact that F is bounded on $\mathbb{R}$ we obtain

$$\int \frac{|G(x)|}{1+x^2}dx = \int \frac{e^{\tilde{u}(x)}\,|F(x)|}{1+x^2}dx < \infty.$$

$\square$

Now we can prove our main result stated in the introduction.

Theorem 1.6. *Let A and B be disjoint closed subsets of $\mathbb{R}$. Let $\mathcal{M}_a(A,B)$ be the class of all finite signed measures σ with a spectral gap $(-a,a)$ such that $\operatorname{supp}\sigma^+ \subset A$ and $\operatorname{supp}\sigma^- \subset B$ The following equality holds*

$$\sup\{a>0 : \mathcal{M}_a(A,B) \neq \{0\}\} = \pi\sup\{d \geq 0 : \exists\{\lambda_n\}\ d\text{-uniform},\ \{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B\}$$

Proof. Assume that $\mathcal{M}_a(A,B)$ contains a non-zero measure. By the previous theorem, it then must contain a non-zero extreme measure ν with discrete support $\{\lambda_n\}$ such that $\{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B$. It follows from Theorem 1.4 that the supports of the indeterminate measures ν^+ and ν^- must be a/π-uniform sequences. Since $\operatorname{supp}\nu^+ = \{\lambda_{2n}\}, \operatorname{supp}\nu^- = \{\lambda_{2n+1}\}$ we are done with one of the inclusions.

In the opposite direction, let $\Lambda = \{\lambda_n\}$ be d-uniform sequence with $\{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B$. Let $\epsilon > 0$ be arbitrary small number. We will construct a finite signed measure $\sigma \in \mathcal{M}_{\pi d - 2\epsilon}(A,B)$ which will finish the proof. From [11] we know first that there exists a finite signed measure σ_1 with a spectral gap $\pi d - \epsilon$. Using the extreme point procedure above if needed we can assume that this measure σ_1 oscillates between consecutive points in its support. More precisely, if we denote the support of σ_1 by $\Lambda' := \{\lambda'_n\} \subset \Lambda$, we have $\sigma_1(\lambda'_n)\sigma_1(\lambda'_{n+1}) < 0$ for all n. This measure however may not be in $\mathcal{M}_{\pi d - \epsilon}(A,B)$ since we don't know that $\Lambda' = \Lambda$.

Consider the intervals $(\lambda'_n, \lambda'_{n+1})$ which contain odd number of points from Λ. These are exactly the bad intervals since in this case λ'_n and λ'_{n+1} are both in A or both in B even though σ_1 has opposite sign at these points. To fix this form a sequence Γ by picking one point from Λ in each such interval. Since $\Lambda \setminus \Lambda'$ is $(d - \epsilon/2)$-uniform it is clear that $D^+_{BM}(\Gamma) \leq D^+_{BM}(\Lambda \setminus \Lambda') = \epsilon/2$. Let $\xi \notin \Gamma$ be an arbitrary fixed real number not in Γ. This number ξ will be useful in the end. By the Beurling-Malliavin theorem there exists an entire function $F(z) \in \mathcal{PW}_\epsilon$ which vanishes at Γ and has a double zero at ξ. We may assume that F is real on the real line (otherwise take $F + F^\#$). If the real zero set of $F(z)$ is exactly Γ, then we will be done by taking $d\sigma := F d\sigma_1$. However, $F(z)$ may have additional real zeros. Denote by Γ' these zeroes. If Γ' is finite, then we can divide them out and still remain with an entire function $F_1(z) \in \mathcal{PW}_\epsilon$ which will make do for us. In the case when Γ' is infinite we index it so that

$$\cdots < \gamma_{-2} < \gamma_{-1} < 0 < \gamma_1 < \gamma_2 < \ldots$$

If there are finitely many positive (negative) terms in Γ we divide them out and work with the negative (positive) terms only. Form the sequence $\Delta = \{\delta_n\}$ from the geometric means of consecutive pairs in Γ' or more precisely for n positive $\delta_n := \sqrt{\gamma_{2n-1}\gamma_{2n}}$ and for n negative $\delta_n := -\sqrt{\gamma_{2n+1}\gamma_{2n}}$. We can now apply the previous lemma to $F(z)$ and obtain an entire function $G(z) \in \mathcal{C}_\epsilon$ such that

(i) The zero set of G is $\Lambda' \cup \Delta \cup \xi$ and G has double zeros on $\Delta \cup \{\xi\}$.

(ii)

$$\int \frac{|G(x)|}{1+x^2}dx < \infty.$$

Consider $H(z) = G(z)/(z + \xi)^2$. This H is an entire function form $\mathcal{C}_\epsilon$ which is real on the real line. Moreover, by (ii) H is integrable and hence must be bounded on $\mathbb{R}$. Consider the measure $d\sigma = Hd\sigma_1$. Since H is bounded this is a finite signed measure. The fact that H is of exponential type no greater than ϵ implies that $d\sigma$ has a spectral gap $d - 2\epsilon$. In addition, in each of the bad intervals $(\lambda'_n, \lambda'_{n+1})$ the function H has exactly one zero and the rest of its zeroes are double zeroes. Therefore, σ has equal sign point masses at λ'_n and λ'_{n+1} and has the same good behavior as σ_1 on the rest of Λ'. Thus, $\sigma \in \mathcal{M}_{\pi d - 2\epsilon}(A, B)$ which is exactly what we needed.   □

4.1. **Sign changes of measures with spectral gap.** If σ is an absolutely continuous signed measure with continuous density f then it is clear what the sign change of σ mean. The sign change will occur exactly at the places where f has zeros (of odd order). Therefore we can count the number of sign changes by counting these zeros. For general signed measures it is not immediately clear how to define sign changes. One natural definition, that was used in [7], is as follows.

Definition 4.10. The number of sign changes of σ on an interval (a, b) is defined to be the minimal degree of a polynomial p for which $pd\sigma$ is a positive measure on $[a, b]$.

The following lemma follows immediately from this definition.

Lemma 4.11. *A signed measure σ has at least one sign change on an interval (a, b) if there exist Borel sets $P, N \subset (a, b)$ such that $\sigma(P) > 0$ and $\sigma(N) < 0$.*

As a consequence of our oscillation theorem we obtain the following improvement of the main result in [7, 8] stated in the introduction.

Theorem 1.7. *If σ is a nonzero signed measure with a spectral gap $(-a, a)$ then there exists an a/π-uniform sequence $\{\lambda_n\}$ such that σ has at least one sign change in every $(\lambda_n - \epsilon_n, \lambda_{n+1} + \epsilon_{n+1})$, where $\epsilon_n > 0$ are arbitrary.*

Proof. Let $A = \operatorname{supp} \sigma^+$, $B = \operatorname{supp} \sigma^-$. Here, as usual, σ^+ and σ^- denote the positive and negative parts of σ in the canonical Jordan decomposition $\sigma = \sigma^+ - \sigma^-$. Then $\sigma \in \mathcal{M}_a(A, B)$. It follows from Lemma 4.2 that $\mathcal{M}_a(A, B)$ contains an extreme point (measure). By Theorem 1.6 this measure is discrete and supported on a/π-uniform sequence $\{\lambda_n\}$ such that $\{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B$ or vice versa. It is clear now that on any double-sided enlargement $(\lambda_n - \epsilon_n, \lambda_{n+1} + \epsilon_{n+1})$ of $(\lambda_n, \lambda_{n+1})$ the conditions from the previous lemma hold. Therefore, σ has at least one sign change on each of these intervals.   □

Next, we show how to deduce the main result from [7, 8] as a simple corollary of Theorem 1.7.

Theorem 4.12. *If σ is a nonzero signed measure with a spectral gap $(-a/2, a/2)$ then the number of sign changes $s(r, \sigma)$ of σ on the interval $(0, r)$ satisfies*

$$\liminf_{r \to \infty} \frac{s(r, \sigma)}{r} \geq \frac{a}{2\pi}.$$

Proof. Let $\Lambda = \{\lambda_n\}$ be the $a/2\pi$-uniform sequence from Theorem 1.7. It is clear that $s(r, \sigma) \geq n_\Lambda(r)$, where $n_\Lambda(r)$ is the usual counting function that counts the number of elements in the set $\Lambda \cap (0, r)$. The fact that Λ is $a/2\pi$-uniform sequence immediately implies that

$$\lim_{r \to \infty} \frac{n_\Lambda(r)}{r} = \frac{a}{2\pi}.$$

We are done. □

Remark. Notice that we don't need to use Beurling-Malliavin result to deduce the previous theorem. The much more elementary Levinson's theorem on distribution of zeros of Cartwright class functions is also sufficient. We want to stress this to illustrate how shorter and simpler our proof is in comparison to the original proof offered in [7].

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