MATH 846 Algebraic Graph Theory

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2-Homogeneous bipartite distance-regular graphs

Brian Curtin 1997

9. The Q-polynomial property

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- 2-homogeneous DRG: $\forall x, y, z \in X$ with $\partial(x,y)=2$, $\partial(x,z)=i=\partial(y,z)$
  $\gamma_i = |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)|$ $(1 \le i \le D-1)$ only depends on i.
- Antipodal 2-cover. $\forall x$, $\exists! y \in X$, s.t. $\partial(x,y)=D$.

<u>Lemma 40</u> DRG Γ. TFAE

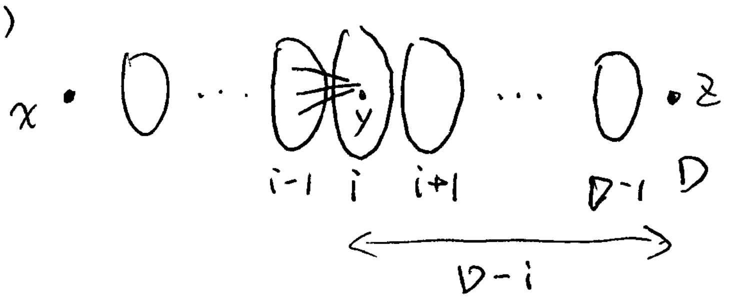
(i) $c_i = b_{D-i}$ (ii) $k_i = k_{D-i}$ (iii) $k_D = 1$.

Pf. See also remarks from Lecture 24.

(i) $\Rightarrow$ (ii) Use $k_i = \dfrac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i}$ $(0 \le i \le D)$.

(ii) $\Rightarrow$ (iii) Set $i=D$: $k_D = k_0 = 1$.

(iii) $\Rightarrow$ (i)



$c_i = b_{D-i}$

□.

<u>Lemma 37</u> DRG $\Gamma=(X,R)$, bipartite. $D \ge 3$. eigenvalue θ. Let $\theta_0^*, \ldots, \theta_D^*$ be the associated dual eigenvalues. TFAE:

(i) Γ is Q-polynomial wrt θ.

(ii) $\theta_0^*, \ldots, \theta_D^*$ are distinct, and $\exists \beta, \gamma^* \in \mathbb{F}$ s.t. $\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* = \gamma^*$. $(1 \le i \le D-1)$.

This is Thm 31 from Lec 32. p.10 (Pascasio).

<u>Lemma 38</u> DRG $\Gamma=(X,R)$. bipartite, $D \ge 3$. eigenvalue θ, associated primitive idempotent $E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$. TFAE:

(i) Γ is Q-polynomial wrt θ and $q_{1i}^i = 0$. $(0 \le i \le D-1)$

(ii) $\theta_0^* \ne \theta_i^*$ $(1 \le i \le D)$, and $\exists x, y \in X$ with $\partial(x,y)=2$ s.t.

$$\sum_{w \in \Gamma_1(x) \cap \Gamma_1(y)} E\hat{w} \in \operatorname{span}\{E\hat{x}, E\hat{y}\}.$$

Suppose (i), (ii) hold. then $q_{1D}^D = 0$.

Recall Thm 19 from section 6. With above notation, TFAE.

(iii) For all $x, y \in X$ with $\partial(x,y)=2$, $\displaystyle\sum_{w \in \Gamma_1(x) \cap \Gamma_1(y)} E\hat{w} = \mu \frac{\theta_1^*}{\theta_0^* + \theta_2^*}(E\hat{x} + E\hat{y})$ $(\mu = c_2)$

(iv) $\exists x, y \in X$ s.t. $\partial(x,y)=2$. and $\displaystyle\sum_{w \in \Gamma_1(x) \cap \Gamma_1(y)} E\hat{w} \in \operatorname{span}\{E\hat{x}, E\hat{y}\}$.

Citation [15] Thm 3 (in the language of DRGs): ②

Note that [15] is NOT assuming bipartite. TFAE:

(ii) $\sum_{w \in \Gamma_i(x) \cap \Gamma_i(y)} E\hat{w} \in \mathrm{span}\{E\hat{x}, E\hat{y}\}$ for all $x, y \in X$.

(iv) Γ is Q-polynomial wrt θ and $q^i_{ii} = 0$ $(0 \le i \le D-1)$.

(v) Possible intersection numbers, 4 cases.

Pf of 38. 38(i) ⇒ 38(ii): 38(i) = [15](iv) ⇒ [15](ii) ⇒ 38(ii);

38(ii) ⇒ 38(i): $38(ii) \xrightarrow{19} 19(iii) \Rightarrow [15](ii) \Rightarrow [15](ii) = 38(i)$

$\exists \xrightarrow[\text{bipartite}]{} \forall$

If 38(i), (ii) hold, so does [15].(v). Adding the bipartite condition leaves only one case, ~~with~~ which implies $q^D_{iD} = 0$. □.

Lemma 39. Bipartite DRG Γ. $D \ge 3$. θ an eigenvalue. Γ is Q-poly wrt θ. $\theta^*_0, \cdots, \theta^*_D$ are the associated dual eigenvalues, with $\beta, \gamma^* \in \mathbb{F}$ s.t. $\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} = \gamma^*$ $(1 \le i \le D-1)$ as in Lem 37. TFAE:

(i) $\gamma^* = 0$

(ii) All conditions of Thm 19, 22 hold for θ, E.

(iii) $q^i_{1i} = 0$ $(0 \le i \le D-1)$.

(iv) $q^i_{ii} = 0$ $(0 \le i \le D)$

(v) $\theta^*_i = -\theta^*_{D-i}$ $(0 \le i \le D)$

Pf. Routine checks with previous lemmas and theorems.

Note that (iii) ⇒ (iv) follows imediately from Lem 38.

Lemma 41 ~~Bipartite~~ (Adapted) Bipartite DRG $\Gamma = (X, R)$. $D \ge 3$. is an antipodal 2-cover. $\theta^*_0, \theta^*_1, \cdots, \theta^*_D$ the associated dual eigenvalues then either $\theta^*_i = -\theta^*_{D-i}$ $(0 \le i \le D)$ or $\theta^*_i = \theta^*_{D-i}$ $(0 \le i \le D)$.

Assume in addition that Γ is Q-poly wrt θ, then $\theta^*_i = -\theta^*_{D-i}$ $(0 \le i \le D)$

Pf. The original statement refers to [2].

Only the ~~last par~~ latter part is needed and proved here.

Recall: When trying to compute the spectrum of a DRG from the intersection numbers, we defined the polynomials u_i, v_i's and similarly u_i^*, v_i^*'s. Some facts. (3)

- $v_i(A) = A_i$ (Lem 11. Lec 8)
- $u_i = \frac{v_i}{k_i}$ $(0 \le i \le D)$ (Lem 18, Lec 9)
- $A_j = \sum_{i=0}^{D} v_j(\theta_i) E_i$ $(0 \le i \le D)$ (Lem 20. Lec 9). — $v_j(\theta_i)$ are eigenvalues of A_j.
- $v_1^*(\lambda) = \lambda$. $u_1^*(\lambda) = \frac{\lambda}{m_1} = \frac{\lambda}{\theta_0^*}$ (Lem 93, Lec 17)

Also recall Askey-Wilson duality (Thm 92. Lec 17):

$$u_i(\theta_j) = u_j^*(\theta_i^*) \qquad 0 \le i, j \le D.$$

Pf of 41. Since T is an antipodal 2-cover, $\forall x \in X$ $\exists! x' \in X$ s.t. $A_D \hat{x} = \hat{x}'$ and ~~$\hat{x}' = A_D$~~ $A_D \hat{x}' = \hat{x}$. Hence $A_D^2 \hat{x} = \hat{x} \Rightarrow A_D^2 = I$.

Thus, eigenvalues of $A_D = \pm 1$.

Take $i = D$, $j = 1$ in A-W duality:

$$\underbrace{\pm 1}_{\text{eigenvalues of } A_D} = v_D(\theta_1) \underset{k_D = 1}{=} u_D(\theta_1) = u_1^*(\theta_D^*) = \frac{\theta_D^*}{\theta_0^*}.$$

Thus $\theta_0^* + \theta_D^* = 0$. By Cor 44. Lec 12, fix $x, y \in X$ with $\partial(x,y) = D$ then $E\hat{y} = -E\hat{x}$. For any $z \in X$,

$$\begin{aligned}
0 &= \langle E\hat{z}, E\hat{x} + E\hat{y} \rangle = \langle E\hat{z}, E\hat{x} \rangle + \langle E\hat{z}, E\hat{y} \rangle \\
&= |X|^{-1} m_1 u_i(\theta_1) + |X|^{-1} m_1 u_{D-i}(\theta_1) \\
&= |X|^{-1} m_1 u_1^*(\theta_i^*) + |X|^{-1} m_1 u_1^*(\theta_{D-i}^*) \\
&= |X|^{-1} (\theta_i^* + \theta_{D-i}^*) \qquad (*)
\end{aligned}$$

where $i = \partial(x, z)$ and $D - i = \partial(y, z)$.

Since z is arbitrary, $(*)$ holds for every $0 \le i \le D$. □

Theorem 42 Bipartite DRG $\Gamma=(X,R)$, $D\geq 3$, $k\geq 3$. TFAE:

(i) Γ is 2-homogeneous

(ii) Γ is an antipodal 2-cover & Q-polynomial.

(iii) Γ has a Q-polynomial structure for which $q^i_{ii}=0$ $(0\leq i\leq D-1)$

~~Pf~~ If (i)-(iii) hold,

(C) $q^i_{ii}=0$ $(0\leq i\leq D)$ directly by Lem 39.

(A) D even, Γ is Q-polynomial wrt θ iff $\theta\in\{\theta_1, \theta_{D+1}\}$.

(B) D odd, Γ is Q-polynomial wrt θ iff $\theta=\theta_1$.

(A)(B) follow mainly from Lem 32 and ~~24~~ Thm 24.

Pf of 42 (i) $\Rightarrow$ (ii) $c_i=b_{D-i}$ $(0\leq i\leq D)$ by Thm 34 (ii) when $c_2=2$.
by Thm 35 (iii) when $c_2\neq 2$.
Lemma 41 $\Rightarrow$ 2-cover ✓

By Lemma 24 using the notion of geometric 2-homogeneity, the second largest eigenvalue θ_1 satisfies all conditions of Lem 27, which guarantees the $(\beta-\gamma^*)$-recurrence in Lemma $\Rightarrow$ Q-poly ✓.

(ii) $\Rightarrow$ (iii) 2-cover $\xRightarrow{\text{Lem 41}}$ $\theta^*_i=-\theta^*_{D-i}$ $\xRightarrow{\text{Lem 39}}$ $q^i_{ii}=0$ $(0\leq i\leq D-1)$. ✓

(iii) $\Rightarrow$ (i) $q^i_{ii}=0$ $\xRightarrow{\text{Lem 38}}$ all conditions of Thm 24 holds $\Rightarrow$ 2-homo ✓

$\square$.

Note that when (i)-(iii) hold, (C) means $a^*_i=q^i_{ii}=0$ $(0\leq i\leq D-1)$; that is, Γ is dual bipartite.

In Lec 23, we proved Γ is an antipodal 2-cover if Γ is bipartite and dual bipartite wrt a Q-poly ordering (Lem 23, 25 in Lec 23) ~~without saying it.~~

In Lec 26, we classified all such graphs and called them 2-homogeneous (Thm 43 from Lec 26, Kazumasa Nomura).

We've been implictly using Thm 42 here in class.