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THE INCIDENCE ALGEBRA OF A UNIFORM POSET

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Abstract. Let $P, \leq$ denote a finite graded poset of rank $N \geq 2$, with fibers $P_0, P_1, \ldots, P_N$. Let the matrices L_i, R_i, E_i^* $(0 \leq i \leq N)$ have rows and columns indexed by P, and entries

$$(L_i)_{xy} = 1 \text{ if } x \in P_{i-1},\ y \in P_i,\ x \leq y, \text{ and } 0 \text{ otherwise} \quad (1 \leq i \leq N),$$

$$(R_i)_{xy} = 1 \text{ if } x \in P_{i+1},\ y \in P_i,\ y \leq x, \text{ and } 0 \text{ otherwise} \quad (0 \leq i \leq N-1),$$

$$(E_i^*)_{xy} = 1 \text{ if } x, y \in P_i,\ x = y, \text{ and } 0 \text{ otherwise} \quad (0 \leq i \leq N),$$

and $L_0 = R_N = 0$. The *incidence algebra* of P is the real matrix algebra generated by L_i, R_i, E_i^* $(0 \leq i \leq N)$. P is *uniform* if there exists real numbers e_i^+, e_i^-, f_i $(1 \leq i \leq N)$ (satisfying a certain condition) such that

$$e_i^- R_{i-2} L_{i-1} L_i + L_i R_{i-1} L_i + e_i^+ L_i L_{i+1} R_i = f_i L_i \quad (1 \leq i \leq N) \quad (R_{-1} = L_{N+1} = 0).$$

We show the incidence algebra T of a uniform poset takes a very simple form, and present a method for computing the irreducible T-modules. We give 11 families of examples that show many of the classical geometries are uniform. In each case we compute the irreducible T-modules. We present some open problems, and discuss a connection with P- and Q-polynomial association schemes.

Key words. Graded poset, Partial geometry, Partial geometric lattice, Association scheme.

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1. Introduction. In [2], Bose introduced a semi-linear incidence structure of points and lines called an (R, K, T)-*partial geometry*. The structure provided a uniform way of studying examples such as the *Steiner systems* (where $2 \leq T = K \leq R$), *transversal designs* or equivalently *nets* (where $2 \leq T = K - 1 \leq R$), and *generalized quadrangles* (where $T = 1 < R, K$). To include more examples, the concept has since been generalized in two main directions. To get the higher rank analogs of the nets and generalized quadrangles, namely the d-*nets* and *polar spaces*, respectively, the point-line system has been replaced with a ranked semi-lattice satisfying various axioms. This has given rise to the *partial d-space* of Laskar [17], the *partial geometric lattice* of Bose and Miskimins [5], and the *regular semi-lattice* of Delsarte [9]. We refer the reader to the papers of Liebler and Meyerowitz [20], Laskar and Dunbar [18], Laskar and Sprague [19], Meyerowitz [21], and Meyerowitz and Miskimins [22] for more information on partial geometric lattices. Another approach is to eliminate the semi-linear condition on the point-line system, in order to get more general partially balanced incomplete block designs as examples. This approach has given rise to the *partial geometric design* of Bose, Shrikhande and

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Singhi [6], (see also Bose, Bridges, and Shrikhande [3], [4]) and the equivalent $1\frac{1}{2}$-*design* of Neumaier [24], [25]. The M_n- and S_n-*designs* of Neumaier [23] combine both approaches to an extent. They are certain rank N graded posets, where the removal of the upper fiber P_N yields a semi-lattice.

In this paper we introduce the notion of a *uniform poset*. This is a certain finite graded poset with arbitrary rank $N \geq 2$, that retains the simple algebraic properties of a partial geometry, but neither it, nor any truncation or interval is assumed to be a semi-lattice. In spite of this, uniform posets are in a sense less general than the above constructions. Very roughly, the axioms for a partial geometric lattice and regular semi-lattice endow the upper fiber of the poset with a simple algebraic structure. In a uniform poset, this structure is extended to all fibers. This gives a more complete description of the structure of the poset, and also happens to simplify many of the calculations. We note that a uniform poset that is also a semi-lattice has a very restricted structure, and can probably be classified if the rank is sufficiently large. (See Conjecture 3 in Section 4). We will consider this special case in a future paper, and focus here on the algebraic properties of arbitrary uniform posets.

For the rest of this section we define our terms and give background information. In Section 2 we define a uniform poset, and give a method for finding its algebraic structure. The main result is Theorem 2.5. In Section 3, we give 11 infinite families of uniform posets, and in each case compute the algebraic structure using the method of Section 2. We acknowledge that in many cases some or all of this structure has previously been found by authors such as Delsarte [9], [10], [11], Dunkl [13], [14], and Stanton [29], [30], [31], but we wish to stress the essential similarity of the examples. In Section 4, we give some conjectures relating uniform posets and P- and Q-polynomial association schemes.

In this paper, $P, \leq$ is always assumed to be a finite, partially ordered set, or *poset*. Usually we just refer explicitly to P. If x and y are elements of P, then we write $x < y$ if $x \leq y$ and $x \neq y$. We say y *covers* x if $x < y$, but there is no $z \in P$ with $x < z < y$. A *grading* of P is a partition of P into disjoint non-empty sets $P_0, P_1, \ldots, P_N$, called *fibers*, such that for all $x, y \in P$, $x \in P_i$ and y covers x implies $i \leq N-1$ and $y \in P_{i+1}$. The *height function* $h: P \to \{0, 1, \ldots, N\}$ of the grading satisfies $h(x) = i$ if $x \in P_i$ ($x \in P$, $0 \leq i \leq N$). The integer N is the *rank* of the grading. A *graded poset* is a poset, together with a grading. Now let P be a graded poset of some rank $N \geq 2$, with fibers $P_0, P_1, \ldots, P_N$. The *dual* of P is the poset $\overline{P} = P$ with grading $\overline{P}_i = P_{N-i}$ $(0 \leq i \leq N)$, where $x \leq y$ in $\overline{P}$ if and only if $y \leq x$ in P. Also, P is said to be *ϕ-regular* if the following conditions (1), (2), (3) hold.

(1) For all integers i, j, k $(0 \leq i \leq j \leq k \leq N)$ and all $x, y \in P$ with $x \in P_i$, $y \in P_k$, and $x \leq y$, the number of $z \in P_j$ with $x \leq z \leq y$ is a constant denoted $\phi(i,j,k)$.

(2) For all integers i, j $(0 \leq i \leq j \leq N)$ and all $x \in P_i$, the number of $z \in P_j$ with $x \leq z$ is a positive constant denoted $\phi(i,j,\infty)$.

(3) For all integers $j,\ k\ (0 \le j \le k \le N)$ and all $y \in P_k$, the number of $z \in P_j$ with $z \le y$ is a positive constant denoted $\phi(-\infty, j, k)$.

Now again let P be any graded poset of rank $N \ge 2$. Define the *lowering matrices* L_i, the *raising matrices* R_i, and the *projection matrices* E_i^* $(0 \le i \le N)$ of P to have rows and columns indexed by P, and entries

$$(L_i)_{xy} = \begin{cases} 1 & \text{if} \quad x \in P_{i-1},\ y \in P_i,\ x \le y \\ 0 & \text{otherwise} \end{cases} \qquad (1 \le i \le N)$$

$$(R_i)_{xy} = \begin{cases} 1 & \text{if} \quad x \in P_{i+1},\ y \in P_i,\ y \le x \\ 0 & \text{otherwise} \end{cases} \qquad (0 \le i \le N-1)$$

$$(E_i^*)_{xy} = \begin{cases} 1 & \text{if} \quad x, y \in P_i,\ x = y \\ 0 & \text{otherwise} \end{cases} \qquad (0 \le i \le N),$$

and $L_0 = R_N = 0$. The *incidence algebra* T of P is the real matrix algebra generated by the $L_i,\ R_i,\ E_i^*\ (0 \le i \le N)$. The irreducible T-modules are of interest in several contexts. For example, they can give those irreducible representations of the automorphism group of P that exist in $\mathbf{R}P$. See Dunkl [13] for the subset lattice, Dunkl [14] for the subspace lattice, Stanton [29] for the polar spaces, and Stanton [30], [31] for certain posets associated with the bilinear, alternating, hermitian, and quadratic forms (i.e. $A_q(N,M)$, $Alt_q(N)$, $Her_q(N)$ and $Quad_q(N)$ in Section 3 of this paper). We refer the reader to Stanton [32], [33] for related information on group representations. In another application, the integrality of the T-module multiplicities can give feasibility conditions governing the existence of certain posets. This was done (implicitly) for (R, K, T)-partial geometries by Bose [2] and extended (implicitly) to partial geometric lattices by Liebler and Meyerowitz [20], Meyerowitz [21], and Meyerowitz and Miskimins [22]. The irreducible T-modules can also be used to compute eigenvalues of graphs that may exist on a fiber of P. We refer the reader to Delsarte [9], [10], [11] and the above mentioned papers by Stanton and Meyerowitz for further details.

Now let P be a graded poset of rank N, with incidence algebra T. Let V be the vector space $\mathbf{R}^m$ $(m = |P|)$, with standard basis identified with P. We view V as a Euclidean space with the usual inner product $\langle\ ,\ \rangle$, and call V the *standard module* of P. A subspace W of V is a *module* of T if it is invariant under T, that is, if $t(w) \in W$ for all $w \in W$ and all $t \in T$. A module W of T is *irreducible* if it is non-zero, and the only non-zero module it contains is W itself. T-modules W and Y are *isomorphic* if there exists an isomorphism of vector spaces $\sigma : W \to Y$ such that $t(\sigma(w)) = \sigma(t(w))$ for all $w \in W$ and all $t \in T$. We do not distinguish between isomorphic T-modules.

2. The structure of a uniform poset. In this section we define a uniform poset, and give a method for computing the irreducible modules of its incidence algebra.

DEFINITION 2.1. Let N be an integer at least 2. A *parameter matrix of order* N is a tri-diagonal real matrix $E := (e_{ij})_{1\le i,j\le N}$ satisfying

(1) $e_{ii} = 1 \quad (1 \le i \le N)$,

(2) $e_{i,i-1} \ne 0 \quad (2 \le i \le N)$ or $e_{i,i+1} \ne 0 \quad (1 \le i \le N-1)$,

(3) The principal submatrix $E(r,p) := (e_{ij})_{r+1\le i,j\le p}$ is nonsingular for all integers $r,\ p\ (0 \le r \le p \le N)$.

We denote $e_i^- := e_{i,i-1}\ (2 \le i \le N)$, $e_i^+ := e_{i,i+1}\ (1 \le i \le N-1)$, and for convenience set $e_1^- = e_N^+ = 0$. The purpose of (2) will become clear from the proof of part (2) of Theorem 2.5, and the purpose of (3) will become clear from part (1) of Definition 2.4.

DEFINITION 2.2. Let P be a finite graded poset of rank $N \ge 2$, with lowering and raising matrices $L_i,\ R_i\ (0 \le i \le N)$. Then P is *uniform* if there exists a parameter matrix E of order N and a vector $F := (f_i)_{1\le i\le N}$ in $\mathbb{R}^N$ satisfying

$$(2.1)\quad e_i^- R_{i-2}L_{i-1}L_i + L_iR_{i-1}L_i + e_i^+ L_iL_{i+1}R_i = f_iL_i \quad (1 \le i \le N)$$
$$(R_{-1},\ L_{N+1} = 0).$$

Here e_i^+, e_i^- are entries in E as indicated in Definition 2.1. We call E, F a *set of parameters* for P. These parameters need not be unique.

Note 1. The *partial geometric design* of Bose, Shrikhande, and Singhi [6] is equivalent to a ϕ-regular uniform poset with $N = 2$ and $|P_0| = 1$.

Note 2. The transpose of R_i is L_{i+1} $(0 \le i \le N-1)$. In particular, If P is a uniform poset then the dual poset $\overline{P}$ is also uniform, with parameters $\overline{e}_i^{\pm} = e_{N-i+1}^{\mp}$, $\overline{f}_i = f_{N-i+1}$ $(1 \le i \le N)$.

Some infinite families of uniform posets with unbounded rank can be found in Section 3. Our purpose for the rest of this section is to show how the incidence algebra of a uniform poset has a very simple structure, and can be readily computed. We first define three sets of constants $c(r,p)$, $x_i(r,p)$, $m(r,p)$, whose meaning will become clear in our main Theorem 2.5.

DEFINITION 2.3. Let P be a uniform poset of rank $N \ge 2$. For each pair of integers $r,\ p\ (0 \le r \le p \le N)$, denote by $c(r,p)$ the number of sequences $(x_r, x_{r+1}, \ldots, x_p, x'_r, x'_{r+1}, \ldots, x'_p)$, where $x_i,\ x'_i \in P_i\ (r \le i \le p)$, $x_r = x'_r$, $x_p = x'_p$, and $x_i < x_{i+1},\ x'_i < x'_{i+1}\ (r \le i \le p-1)$. We note $x_i,\ x'_i$ may coincide for any and all $i\ (r \le i \le p)$.

Note. If P is ϕ-regular (see Introduction) then

$$c(r,p) = |P_r| \prod_{j=r}^{p-1} \phi(j, j+1, \infty) \prod_{j=r+1}^{p} \phi(r, j-1, j) \qquad (0 \le r \le p \le N),$$

where

$$|P_r| = |P_0| \prod_{j=0}^{r-1} \phi(j, j+1, \infty) \left(\prod_{j=1}^{r} \phi(0, j-1, j) \right)^{-1} \qquad (0 \le r \le N).$$

DEFINITION 2.4. Let P be a uniform poset of rank $N \ge 2$, with parameters E, F.

(1) For each pair of integers r, p $(0 \le r < p \le N)$, define the real numbers $x_i(r,p)$ $(r+1 \le i \le p)$ to be the solution to the linear system

$$E(r,p) \begin{pmatrix} x_{r+1}(r,p) \\ x_{r+2}(r,p) \\ \vdots \\ x_p(r,p) \end{pmatrix} = \begin{pmatrix} f_{r+1} \\ f_{r+2} \\ \vdots \\ f_p \end{pmatrix}$$

Here $E(r,p)$ is from Definition 2.1.

(2) For each pair of integers r, p $(0 \le r \le p \le N)$, let $m(r,p)$ be the unique real number satisfying

(2a) $m(r,p) = 0$ if $r < p$ and $x_{r+1}(r,p)x_{r+2}(r,p)\cdots x_p(r,p) = 0$, and

(2b) $c(r,p) = \sum_{r'=0}^{r} \sum_{p'=p}^{N} m(r',p')x_{r+1}(r',p')x_{r+2}(r',p')\cdots x_p(r',p')$ otherwise.

Here $c(r,p)$ is from Definition 2.3.

We note the $m(r,p)$ constants can be found recursively by solving (2a), (2b) in the order $(r,p) = (0,N), (0,N-1), (1,N), (0,N-2), (1,N-1), (2,N), \ldots$.

We are now ready to give the irreducible modules for the incidence algebra of any uniform poset.

THEOREM 2.5. *Let P be a uniform poset of rank $N \ge 2$, with lowering, raising, and projection matrices L_i, R_i, E_i^* $(0 \le i \le N)$. Let T be the incidence algebra of P acting on its standard module V. Then*

(1) V decomposes into an orthogonal direct sum of irreducible T-modules.

(2) Each irreducible T-module has a basis of the form $w_r, w_{r+1}, \cdots, w_p$ for some integers r, p $(0 \le r \le p \le N)$, where

(2a) $w_i \in E_i^* V$ $(r \le i \le p)$,

(2b) $L_i w_i = w_{i-1}$ $(r+1 \le i \le p)$, and $L_r w_r = 0$,

(2c) $R_i w_i = x_{i+1}(r,p) w_{i+1}$ $(r \le i \le p-1)$, and $R_p w_p = 0$.

(The $x_i(r,p)$ are from Definition 2.4). In particular, the isomorphism class of an irreducible T-module is determined by the integers r, p.

We refer to the integers r, p in (2) as the endpoints of the module.

(3) *Let r, p be any integers ($0 \le r \le p \le N$). Then the irreducible T-module with endpoints r, p occurs in V with multiplicity $m(r,p)$. ($m(r,p)$ is from Definition 2.4.) If there is no such T-module then $m(r,p) = 0$.*

Proof of (1). Let $W \subseteq V$ be any T-module. Then it suffices to show $W^{\perp} := \{v \mid v \in V,\ \langle v,\ w\rangle = 0 \text{ for all } w \in W\}$ is also a T-module. Now T is closed under transposition, since each E_i^* is symmetric ($0 \le i \le N$), and since $R_j^t = L_{j+1}$ ($0 \le j \le N-1$). Thus for all $w \in W$, $v \in W^{\perp}$, and $A \in T$, we have $A^t \in T$ and therefore $A^t w \in W$, forcing $\langle Av, w\rangle = \langle v, A^t w\rangle = 0$ and $Av \in W^{\perp}$. Thus $W^{\perp}$ is a T-module, as desired.

Proof of (2). From (2) in Definition 2.1, we can assume either (a) $e_i^- \neq 0$ ($2 \le i \le N$), or (b) $e_i^+ \neq 0$ ($1 \le i \le N-1$). First assume Case (a).

Let $W \subseteq V$ be an irreducible T-module, and let p be the maximal integer where $E_p^* W \neq 0$. Then $E_p^* W$ is a module for the symmetric matrix $R_{p-1}L_p = L_p^t L_p$, so there exists a nonzero vector $w_p \in E_p^* W$, and a real number λ such that $R_{p-1}L_p w_p = \lambda w_p$. Note $R_p w_p = 0$ by the maximality of p. Now define $w_i := L_{i+1}L_{i+2}\cdots L_p w_p$ ($-1 \le i \le p$), and let r denote the minimal integer where $w_r \neq 0$. Then $w_i \neq 0$ ($r \le i \le p$). Now (2a) and (2b) hold. If $r = p$ there is nothing further to prove, so assume $r < p$. From our above remarks, and upon applying (2.1) to w_i, we obtain

$$R_{p-1}w_{p-1} = \lambda w_p \tag{2.2}$$

$$e_i^- R_{i-2}w_{i-2} + L_i R_{i-1} w_{i-1} + e_i^+ L_i L_{i+1} R_i w_i = f_i w_{i-1} \quad (r+1 \le i \le p). \tag{2.3}$$

Now by induction on $i = p, p-1, \ldots$ in (2.3), we find $R_i w_i \in \mathrm{Span}\{w_{i+1}\}$ ($r \le i \le p-1$). Now let y_i ($r+1 \le i \le p$) denote the real number satisfying $R_i w_i = y_{i+1} w_{i+1}$ ($r \le i \le p-1$), and set $y_r = y_{p+1} = 0$. Substituting this in (2.3), we find

$$e_i^- y_{i-1} + y_i + e_i^+ y_{i+1} = f_i \qquad (r+1 \le i \le p).$$

But by (1) of Definition 2.4, $y_i = x_i(r,p)$ ($r+1 \le i \le p$) is the unique solution to this system. This proves (2c), and we are done with Case (a).

Now assume Case (b), and again let W be an irreducible T-module. Applying Case (a) to the dual of P, we conclude W has a basis $w'_r, w'_{r+1}, \ldots, w'_p$ where, (in the original poset P), $w'_i \in E_i^* V$ ($r \le i \le p$), $L_i w'_i = x_i(r,p) w'_{i-1}$ ($r+1 \le i \le p$), $L_r w'_r = 0$, $R_i w'_i = w'_{i+1}$ ($r \le i \le p-1$), and $R_p w'_p = 0$. Note $x_i(r,p) \neq 0$ ($r+1 \le i \le p$), otherwise $\mathrm{Span}\{w'_i, \ldots, w'_p\}$ is a T-module, contradicting the irreducibility of W. Now set $w_r = w'_r$ and

$$w_i = w'_i \left(\prod_{j=r+1}^{i} x_j(r,p)^{-1} \right) \qquad (r+1 \le i \le p).$$

Then w_i $(r \le i \le p)$ is the desired basis for W.

Proof of (3). Let $m(r,p)$ be the multiplicity of the irreducible T-module with endpoints r, p, if this module exists, and set $m(r,p)=0$ if this module does not exist. We must show $m(r,p)$ satisfies (2a), (2b) of Definition 2.4. To prove (2a), suppose $x_i(r,p)=0$ for some integer i $(r+1 \le i \le p)$. If there exists an irreducible T-module with endpoints r, p, then, using the notation of (2) in the present theorem, $\text{Span}\{w_r, w_{r+1}, \ldots w_{i-1}\}$ would be a T-module, contradicting the irreducibility of W. Hence W does not exist, and $m(r,p)=0$, as desired. To prove (2b), we consider the trace of the element $A = L_{r+1}L_{r+2}\cdots L_pR_{p-1}R_{p-2}\cdots R_{r+1}R_r$ of T. Elementary counting arguments give $\text{trace}(A) = c(r,p)$. Now fix a decomposition of V into a direct sum of irreducible T-modules. Then $\text{trace}(A)$ is equal to the sum of the traces of the restrictions of A to these modules, and this sum is just the right side of (2b). □

As a consequence of Theorem 2.5, the incidence algebra of a uniform poset has the following simple basis.

COROLLARY 2.6. *Let P be a uniform poset of rank $N \ge 2$, with lowering, raising, and projection matrices L_i, R_i, E_i^* $(0 \le i \le N)$, and incidence algebra T. For each 4-tuple of integers s, r, p, t $(0 \le s \le r \le p \le t \le N)$, define*

$$\theta^+(s,r,p,t) = L_{p+1}L_{p+2}\cdots L_{t-1}L_tR_{t-1}R_{t-2}\cdots R_{s+1}R_sL_{s+1}L_{s+2}\cdots L_{r-1}L_r$$
$$\theta^-(s,r,p,t) = R_{r-1}R_{r-2}\cdots R_{s+1}R_sL_{s+1}L_{s+2}\cdots L_{t-1}L_tR_{t-1}R_{t-2}\cdots R_{p+1}R_p$$
$$= \text{transpose of } \theta^+(s,r,p,t).$$

We interpret $\theta^\pm(r,r,r,r) = E_r^*$ $(0 \le r \le N)$.

Then T has a basis B of the form

$$B = \{\theta^+(s,r,p,t) \mid 0 \le s \le r \le p \le t \le N, \quad m(s,t) \ne 0\} \tag{2.4}$$

$$\cup\, \{\theta^-(s,r,p,t) \mid 0 \le s \le r < p \le t \le N, \quad m(s,t) \ne 0\}. \tag{2.5}$$

Proof. The dimension of T as a real vector space is certainly at most $\Sigma(t-s+1)^2$, where the sum is over all pairs $\{s,t \mid 0 \le s \le t \le N,\ m(s,t) \ne 0\}$. Since this is the number of matrices in B, it suffices to show those matrices are independent. If not, let C be a minimal subset of B containing dependent matrices. Then C is a subset of (2.4) or (2.5), and the indices r,p of all matrices in C are identical. Without loss, we can assume C is contained in the set (2.4). Suppose

$$\Sigma a_{st}\theta^+(s,r,p,t) \tag{2.6}$$

is a dependency among the matrices in C. Pick an element $\theta^+(s',r,p,t') \in C$ where $t'-s'$ is maximal, and consider the restriction of (2.6) to an irreducible T-module with endpoints s', t'. Then every element in C is 0 on this module except $\theta^+(s',r,p,t')$ itself. But this forces $a_{s't'}=0$, contradicting the minimality of C. □

COROLLARY 2.7. *Let P be a uniform poset of rank $N \geq 2$, with incidence algebra T. Then for each integer i $(0 \leq i \leq N)$, the subalgebra $E_i^* T E_i^* \subseteq T$ is commutative with dimension at most $(N-i+1)(i+1)$.*

Proof. Pick any integer i $(0 \leq i \leq N)$, and pick any matrices a, $b \in E_i^* T E_i^*$. Then a and b commute, since by Theorem 2.5, this is true of their restrictions to any irreducible T-module. The dimension bound is obtained by counting matrices in (2.4), (2.5). □

3. Examples of Uniform Posets. In this section we give 11 infinite families of uniform posets. For each example we give the irreducible modules of the incidence algebra using Theorem 2.5. We suppress the details of our calculations.

EXAMPLE 3.1. *In each of the 11 examples that follow, $P, \leq$ is a ϕ-regular graded poset with $|P_0| = 1$ and height function h. Basic combinatorial information on the examples can be found in the stated references.*

1. The truncated subset semi-lattice $S(N, M)$ $(2 \leq N \leq M)$ **[9], [13].**

P = all subsets of $\{1, 2, \ldots, M\}$ with size at most N,
$u \leq v$ if u is a subset of v $(u, v \in P)$,
$h(u)$ = size of u $(u \in P)$,
$\phi(i, j-1, j) = j - i, \quad \phi(i, i+1, \infty) = M - i \quad (0 \leq i < j \leq N)$.

2. The Hamming semilattice $H(N, M)$ $(N \geq 2,\ M \geq 3)$ **[9].**

P = all N-tuples of elements from the set $\{0, 1, 2, \ldots, M-1\}$,
$u \leq v$ if u, v agree on all non-zero coordinates of u $(u, v \in P)$,
$h(u)$ = number of non-zero coordinates of u $(u \in P)$,
$\phi(i, j-1, j) = j - i, \quad \phi(i, i+1, \infty) = (M-1)(N-i) \quad (0 \leq i < j \leq N)$.

This the poset of type II (with $q = 1$) in [9].

3. The folded Hamming poset $H^*(N, 3)$ $(N \geq 2)$ **[9]**

Let u, v be any elements of the poset $H(N, 3)$ in the above example, with $u = (u_1, \ldots, u_N)$ and $v = (v_1, \ldots, v_N)$ $(u_i, v_i \in \{0, 1, 2\},\ 1 \leq i \leq N)$. Call u, v ***antipodal opposites*** if $(u_i, v_i) = (0, 0)$, $(1, 2)$, or $(2, 1)$ for all integers i $(1 \leq i \leq N)$.

P = all unordered pairs (u, v) where u, v are antipodal opposites in $H(N, 3)$
$(u, v) \leq (u', v')$ if at least one of $u \leq u', \quad u \leq v', \quad v \leq u', \quad v \leq v'$
holds in $H(N, 3)$ $((u, v), (u', v') \in P)$,
$h(u, v)$ = number of non-zero coordinates in u or v $((u, v) \in P)$,
$\phi(i, j-1, j) = j - i \quad (0 \leq i < j \leq N)$,
$\phi(0, 1, \infty) = N$,
$\phi(i, i+1, \infty) = 2(N-i) \quad (1 \leq i < N)$.

4. The bipartition poset $Part(2N)$ $(N \geq 3)$ **[9].** Let $X = \{1, 2, \dots, 2N\}$.

$$P = \text{all unordered pairs } (u,v) \text{ where } u, v \subseteq X, \quad u \cap v = \phi, \quad u \cup v = X.$$
$$(u,v) \leq (u',v') \text{ if at least one of } u \subseteq u', \; u \subseteq v', \; v \subseteq u', \; v \subseteq v', \; ((u,v),(u',v') \in P),$$
$$h(u,v) = \min\{|u|, |v|\} \quad ((u,v) \in P),$$
$$\phi(i, j-1, j) = j - i \quad (0 \leq i < j \leq N) \quad (i,j) \neq (0,N),$$
$$\phi(0, N-1, N) = 2N,$$
$$\phi(i, i+1, \infty) = 2N - i \quad (0 \leq i < N).$$

5. The truncated subspace semi-lattice $S_q(N,M)$ $(2 \leq N \leq M)$ **[7, Section 9.3], [9], [14].**

$P =$ all subspaces of dimension at most N in an M-dimensional vector space over the finite field $GF(q)$,
$u \leq v$ if u is a subspace of v $(u, v \in P)$,
$h(u) =$ dimension of u $(u \in P)$,
$$\phi(i, j-1, j) = \frac{q^{j-i} - 1}{q - 1}, \quad \phi(i, i+1, \infty) = \frac{q^{M-i} - 1}{q - 1} \qquad (0 \leq i < j \leq N).$$

6. The polar spaces of rank N $(N \geq 2)$ **[7, Section 9.4], [29].**

Let H be a vector space over $GF(q)$ that possesses one of the following nondegenerate forms:

name	dim(H)	form	e
$B_N(q)$	$2N+1$	quadratic	0
$C_N(q)$	$2N$	symplectic	0
$D_N(q)$	$2N$	quadratic (Witt index N)	-1
${}^2D_{N+1}(q)$	$2N+2$	quadratic (Witt index N)	1
${}^2A_{2N}(r)$	$2N+1$	Hermitian $(q=r^2)$	$\frac{1}{2}$
${}^2A_{2N-1}(r)$	$2N$	Hermitian $(q=r^2)$	$-\frac{1}{2}$

The parameter e will appear in later calculations. A subspace of H is called *isotropic* whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is N.

$P =$ all isotropic subspaces of H,
$u \leq v$ if u is a subspace of v $(u, v \in P)$,
$h(u) =$ dimension of u $(u \in P)$,
$$\phi(i, j-1, j) = \frac{q^{j-i}-1}{q-1}, \quad \phi(i, i+1, \infty) = \frac{(q^{N-i+e}+1)(q^{N-i}-1)}{q-1} \quad (0 \leq i < j \leq N).$$

7. The attenuated space $A_q(N,M)$ $(2 \le N \le M)$ [7, Section 9.5], [9], [28], [31].

Let H be a vector space of dimension $M+N$ over $GF(q)$, and fix a subspace $w \subseteq H$ of dimension M.

P = all subspaces u of H where $u \cap w = 0$,
$u \le v$ if u is a subspace of v $(u, v \in P)$,
$h(u)$ = dimension of u $(u \in P)$,

$$\phi(i,j-1,j) = \frac{q^{j-i}-1}{q-1}, \quad \phi(i,i+1,\infty) = \frac{q^{M+N-i}-q^M}{q-1} \qquad (0 \le i < j \le N).$$

This poset is equivalent to the poset of type II (with $q \ne 1$) in [9]. If $N \ge 3$, it is also equivalent to an N-net [28].

8. The poset $Alt_q(N)$ of alternating forms $(N \ge 2)$ [7, Section 9.5], [30], [31].

Let H be a vector space of dimension N over $GF(q)$.

P = all pairs (u, f), where u is a subspace of H and f is an alternating bilinear form on u,
$(u,f) \le (u',f')$ if $u \subseteq u'$ and $f = f'|_u$ $((u,f),(u',f') \in P)$,
$h(u,f)$ = dimension of u $((u,f) \in P)$,

$$\phi(i,j-1,j) = \frac{q^{j-i}-1}{q-1}, \quad \phi(i,i+1,\infty) = \frac{q^N-q^i}{q-1} \qquad (0 \le i < j \le N).$$

9. The poset $Her_q(N)$ of Hermitian forms $(N \ge 2)$ [7, Section 9.5], [30], [31].

Same as 8, except H is over $GF(q^2)$ and f is a Hermitian form.

$$\phi(i,j-1,j) = \frac{q^{2j-2i}-1}{q^2-1}, \quad \phi(i,i+1,\infty) = \frac{q(q^{2N}-q^{2i})}{q^2-1} \qquad (0 \le i < j \le N).$$

10. The poset $Quad_q(N)$ of quadratic forms $(N \ge 2)$ [7, Section 9.5], [30], [31].

Same as 8, except f is a quadratic form. We allow q even or odd. See Bannai and Ito [1, p. 309] for the définition of a quadratic form if q is even.

$$\phi(i,j-1,j) = \frac{q^{j-i}-1}{q-1}, \quad \phi(i,i+1,\infty) = \frac{q(q^N-q^i)}{q-1} \qquad (0 \le i < j \le N).$$

11. Hemmeters' poset $Hem_q(N)$ $(N \ge 2)$ (q odd) [15].

Let X^+, X^- denote two copies of the graph of the dual polar space $X = C_{N-1}(q)$, (q odd) (Bannai and Ito [1, p. 303]). Let Y be the bipartite graph with vertex set $X^+ \cup X^-$, where vertices $x^+ \in X^+$, $y^- \in X^-$ are adjacent in Y if and only if $x = y$ or x, y are adjacent in X. Then Y is the new distance-regular graph discovered by Hemmeter [15]. (Y has the same intersection numbers as the graph $D_N(q)$). Now let ∂ be the usual distance function in Y, and fix any vertex $u_0 \in Y$ (the choice of u_0 is irrelevant since Y is vertex transitive). Now define

$$\begin{aligned}
&P = Y, \\
&u \leq v \quad \text{if} \quad \partial(u_0, u) + \partial(u, v) = \partial(u_0, v) \quad (u, v \in P), \\
&h(u) = \partial(u_0, u) \quad (u \in P), \\
&\phi(i, j-1, j) = \frac{q^{j-i} - 1}{q - 1}, \quad \phi(i, i+1, \infty) = \frac{q^N - q^i}{q - 1} \quad (0 \leq i < j \leq N).
\end{aligned}$$

THEOREM 3.2. *Let P denote any of the posets in Example 3.1. Then P is uniform. Indeed*

$$e_i^- R_{i-2} L_{i-1} L_i \; + \; L_i R_{i-1} L_i \; + \; e_i^+ L_i L_{i+1} R_i \; = \; f_i L_i \qquad (1 \leq i \leq N), \tag{3.1}$$

where $e_i^{\mp}, f_i$ are given below

Example	e_i^- $(2\le i\le N)$	e_i^+ $(1\le i\le N-1)$	f_i $(1\le i\le N)$	Case
1. $S(N,M)$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$1\le i\le N-1$
	-1		$M-2N+2$	$i=N$
2. $H(N,M)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$M-1$	
3. $H^*(N,3)$		$-\frac{1}{4}$	1	$i=1$
	-1	$-\frac{1}{2}$	2	$i=2$
	$-\frac{1}{2}$	$-\frac{1}{2}$	2	$3\le i$
4. $Part(2N)$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$i\le N-2$
	-1	0	4	$i=N-1$
	-2		4	$i=N$
5. $S_q(N,M)$	$-q(q+1)^{-1}$	$-(q+1)^{-1}$	q^{M-i}	$1\le i\le N-1$
	-1		$\frac{q^{M-N+1}-q^{N-1}}{q-1}$,	$i=N$
6. *Polar spaces of rank N*	$-(q+1)^{-1}$	$-q(q+1)^{-1}$	$q^{e+2N+1-2i}+q^{i-1}$	
7. $A_q(N,M)$	$-q(q+1)^{-1}$	$-(q+1)^{-1}$	q^{N+M-i}	
8. $Alt_q(N)$	$-q^2(q+1)^{-1}$	$-q^{-1}(q+1)^{-1}$	q^{N-1}	
9. $Her_q(N)$	$-q^4(q^2+1)^{-1}$	$-q^{-2}(q^2+1)^{-1}$	q^{2N-1}	
10. $Quad_q(N)$	$-q^2(q+1)^{-1}$	$-q^{-1}(q+1)^{-1}$	q^N	
11. $Hem_q(N)$	$-q^2(q+1)^{-1}$	$-q^{-1}(q+1)^{-1}$	q^{N-1}	

Proof. Pick any integer i $(1\le i\le N)$ and pick any $x\in P_{i-1}$, $y\in P_i$. Then $(L_iR_{i-1}L_i)_{xy}$ counts the number of ordered pairs w, $z\in P$, where $z\in P_{i-1}$, $w\in P_i$, $x\le w$, $z\le w$, and $z\le y$. The x, y entries of $R_{i-2}L_{i-1}L_i$, $L_iL_{i+1}R_i$, and L_i have similar interpretations. The following table displays these counts in all cases where at least one of them is nonzero.

Example	$(R_{i-2}L_{i-1}L_i)_{xy}$	$(L_iR_{i-1}L_i)_{xy}$	$(L_iL_{i+1}R_i)_{xy}$	$(L_i)_{xy}$	*Comments*
$S(N,M)$	$2(i-1)$	M	$2(M-i)$	1	$i<N$
	$2(N-1)$	M	0	1	$i=N$
	2	2	2	0	$2\le i<N$
	2	2	0	0	$i=N$
$H(N,M)$	$2(i-1)$	$N(M-1)-(i-1)(M-2)$	$2(N-i)(M-1)$	1	
	2	2	2	0	$2\le i<N$
	2	1	0	0	$2\le i$
$H^*(N,3)$	0	N	$4(N-1)$	1	$i=1$
	2	$2N$	$4(N-2)$	1	$i=2$
	$2(i-1)$	$2N-i+1$	$4(N-i)$	1	$3\le i$
	4	2	0	0	$i=3$
	2	1	0	0	$3<i$
	2	4	4	0	$i=2$
	2	2	2	0	$3\le i<N$
$Part(2N)$	$2(i-1)$	$2N$	$4N-2i$	1	$i\le N-1$
	$2(N-1)$	$4N$	0	1	$i=N$
	2	2	2	0	$2\le i\le N-1$
	2	4	0	0	$i=N$
	0	0	6	0	$i=N-1,\ N\ge 4$
$S_q(N,M)$	$\frac{(q^{i-1}-1)(q+1)}{q-1}$	$\frac{q^i+q^{M-i+1}-q-1}{q-1}$	$\frac{(q^{M-i}-1)(q+1)}{q-1}$	1	$i<N$
	$\frac{(q^{N-1}-1)(q+1)}{q-1}$	$\frac{q^N+q^{M-N+1}-q-1}{q-1}$	0	1	$i=N$
	$q+1$	$q+1$	$q+1$	0	$2\le i<N$
	$q+1$	$q+1$	0	0	$i=N$
$PolarSpaces$	$\frac{(q^{i-1}-1)(q+1)}{q-1}$	$\frac{(q^{N-i+1}-1)(q^{N-i+1+e}+1)+q^i-q}{q-1}$	$\frac{(q^{N-i+e}+1)(q^{N-i}-1)(q+1)}{q-1}$	1	
	$q+1$	$q+1$	$q+1$	0	$2\le i<N$
	$q+1$	1	0	0	$2\le i$
$A_q(N,M)$	$\frac{(q^{i-1}-1)(q+1)}{q-1}$	$\frac{q^{N+M-i+1}-q^M+q^i-q}{q-1}$	$\frac{(q^{M+N-i}-q^M)(q+1)}{q-1}$	1	
	$q+1$	$q+1$	$q+1$	0	$2\le i<N$
	$q+1$	q	0	0	$2\le i$
$Alt_q(N)$	$\frac{(q^{i-1}-1)(q+1)}{q-1}$	$\frac{q^N-q^2+q^{i+1}-q^{i-1}}{q-1}$	$\frac{(q^N-q^i)(q+1)}{q-1}$	1	
	$q+1$	$q(q+1)$	$q^2(q+1)$	0	$2\le i<N$
	$(q+1)^2$	$q^2(q+1)$	0	0	$3\le i$
$Her_q(N)$	$\frac{(q^{2i-2}-1)(q^2+1)}{q^2-1}$	$\frac{q^{2N+1}-q^4+q^{2i+2}-q^{2i-1}}{q^2-1}$	$\frac{q(q^{2N}-q^{2i})(q^2+1)}{q^2-1}$	1	
	q^2+1	$q^2(q^2+1)$	$q^4(q^2+1)$	0	$2\le i<N$
	q^2+1	q^4	0	0	$2\le i$
	$(q+1)(q^2+1)$	$q^4(q+1)$	0	0	$3\le i$
$Quad_q(N)$	$\frac{(q^{i-1}-1)(q+1)}{q-1}$	$\frac{q^{N+1}-q^2+q^{i+1}-q^i}{q-1}$	$\frac{q(q^N-q^i)(q+1)}{q-1}$	1	
	$q+1$	$q(q+1)$	$q^2(q+1)$	0	$2\le i<N$
	$q+1$	q^2	0	0	$2\le i$
	$2(q+1)$	$2q^2$	0	0	$3\le i$
$Hem_q(N)$	$\frac{(q^{i-1}-1)(q+1)}{q-1}$	$\frac{q^N-q^2+q^{i+1}-q^{i-1}}{q-1}$	$\frac{(q^N-q^i)(q+1)}{q-1}$	1	
	$q+1$	$q(q+1)$	$q^2(q+1)$	0	$2\le i<N$
	$2(q+1)$	$(q+1)(2q-1)$	$q(q^2-1)$	0	$3\le i<N$
	0	$q+1$	$q(q+1)^2$	0	$3\le i<N$
	$(q+1)^2$	$q^2(q+1)$	0	0	$3\le i$

This establishes (3.1). Conditions (2) and (3) of Definition 2.1 must now be verified. But condition (2) is immediate and condition (3) holds since $\det E(r,p)$ in the following table is never 0.

Example	$\det E(r,\,p)\ (0\le r<p\le N)$	*Case*
$S(N,M)$	$(p-r+1)2^{r-p}$	$p<N$
	2^{r-p+1}	$p=N$
$H(N,M)$	$(p-r+1)2^{r-p}$	
$H^*(N,3)$	$(p-r+1)2^{r-p}$	
$Part(2N)$	$(p-r+1)2^{r-p}$	$p<N-1$
	2^{r-p+1}	$p=N-1$
	2^{r-p+2}	$r+1<p=N$
	1	$r+1=p=N$
$S_q(N,M)$	$(\frac{q^{p-r+1}-1}{q-1})(q+1)^{r-p}$	$p<N$
	$q^{p-r-1}(q+1)^{r-p+1}$	$p=N$
Polar spaces	$(\frac{q^{p-r+1}-1}{q-1})(q+1)^{r-p}$	
$A_q(N,M)$	same	
$Alt_q(N)$	same	
$Quad_q(N)$	same	
$Hem_q(N)$	same	
$Her_q(N)$	$(\frac{q^{2p-2r+2}-1}{q^2-1})(q^2+1)^{r-p}$	

□

Recall that for any integer $r\ge 0$,

$$(a)_r=1 \quad\text{if}\quad r=0,\quad\text{and}\quad a(a-1)\cdots(a-r+1)\quad\text{if}\quad r>0.$$

Also,

$$(a;q)_r=1\quad\text{if}\quad r=0,\quad\text{and}\quad (1-a)(1-aq)\cdots(1-aq^{r-1})\quad\text{if}\quad r>0.$$

THEOREM 3.3. *Let P denote any poset in Example 3.1. Then the parameters determining the irreducible modules of the incidence algebra (by Theorem 2.5) are as follows. All $m(r,p)$ not listed are 0.*

1. $S(N,M)$.

$$x_i(r,p) = \begin{cases} (i-r)(p-i+1) & \text{if } p < N \\ (i-r)(M-r-i+1) & \text{if } p = N \end{cases} \qquad (1 \le r+1 \le i \le p \le N)$$

$$m(r,p) = \frac{(M)_r(M-2r+1)}{r!(M-r+1)} \quad \text{if } p = \min\{M-r, N\} \qquad (0 \le r \le p \le N).$$

2. $H(N,M)$.

$$x_i(r,p) = (M-1)(i-r)(p-i+1) \qquad (1 \le r+1 \le i \le p \le N)$$

$$m(r,p) = \frac{(N+1)_r(M-2)^{r+p-N}(p-r+1)}{(N-p)!(r+p-N)!(N+1)} \qquad (0 \le r \le p \le N \le r+p).$$

3. $H^*(N,3)$.

$$x_i(r,p) = \begin{cases} p & \text{if } r=0 \text{ and } i=1 \\ 2(i-r)(p-i+1) & \text{otherwise} \end{cases} \qquad (1 \le r+1 \le i \le p \le N)$$

$$m(r,p) = \frac{(N+1)_r(p-r+1)}{(N-p)!(r+p-N)!(N+1)} \quad (0 \le r \le p \le N \le r+p,\ r+p-N \text{ even})$$

4. $Part(2N)$.

$$x_i(r,p) = \begin{cases} 2(N-r)(N-r+1) & \text{if } i=p=N \\ (i-r)(2N-r-i+1) & \text{if } 1 \le i \le N-1 \text{ and } p \ge N-1 \\ (i-r)(p-i+1) & \text{if } p < N-1 \end{cases}$$

$$(1 \le r+1 \le i \le p \le N)$$

$$m(r,p) = \frac{(2N)_r(2N-2r+1)}{r!(2N-r+1)} \quad \text{where } p=N \ (r \text{ even}) \quad \text{or } p=N-1 \ (r \text{ odd})$$

$$(0 \le r \le p \le N \le r+p).$$

5. $S_q(N,M)$.

$$x_i(r,p) = \begin{cases} \dfrac{(q^i - q^r)(q^p - q^{i-1})}{(q-1)^2 q^{i-1}} & \text{if } p < N \\[2ex] \dfrac{(q^i - q^r)(q^{M-r} - q^{i-1})}{(q-1)^2 q^{i-1}} & \text{if } p = N \end{cases} \qquad (1 \le r+1 \le i \le p \le N)$$

$$m(r,p) = \frac{(q^N;q^{-1})_r (q^N - q^{2r-1})}{(q;q)_r (q^N - q^{r-1})} \quad \text{if} \quad p = \min\{N, M-r\} \qquad (0 \le r \le p \le N).$$

6. Polar spaces of rank N.

$$x_i(r,p) = \frac{(q^{2N+e-r-p-i+1}+1)(q^i - q^r)(q^p - q^{i-1})}{q^{i-1}(q-1)^2} \qquad (1 \le r+1 \le i \le p \le N)$$

$m(r,p) =$

$$\frac{(q^N;q^{-1})_r(-q^{e+N};q^{-1})_{2r}(-q^{e-r};q)_{N-p}(q^{-r};q)_{N-p}(q^{r-N};q)_{N-p}(-q^{e+N-2r+2};q)_{N-p}}{(q;q)_r(-q^{e+N-2r+2};q)_r(-q^{e+1-r};q)_r(q;q)_{N-p}(-q^{e+1};q)_{N-p}(-q^{e+N-2r+1};q)_{N-p}}$$

$$\times \frac{(1+q^{e-r+2N-2p})}{(q^{r-N-1};q)_{N-p}(1+q^{e-r})} q^{(er+2r-\binom{r+1}{2}+(N-p)(2r-N+p-1)+ep-eN)}(-1)^{p-N}$$

$$(0 \le r \le p \le N \le r+p).$$

7. $A_q(N,M)$.

$$x_i(r,p) = \frac{q^{M+N-p-r-i+1}(q^i - q^r)(q^p - q^{i-1})}{(q-1)^2} \qquad (1 \le r+1 \le i \le p \le N)$$

$$m(r,p) = \frac{(q^M;q^{-1})_{r+p-N}(q^N;q^{-1})_{r+N-p}\, q^{\binom{r+p-N}{2}} q^{N-p}(-1)^{r+p-N}}{(q^{N-r+1};q^{-1})_{N-p}(q;q)_{N-p}(q;q)_{r+p-N}}$$

$$(0 \le r \le p \le N \le r+p \le M+N).$$

8. $Alt_q(N)$.

$$x_i(r,p) = \frac{q^{N-r-p}(q^i - q^r)(q^p - q^{i-1})}{(q-1)^2} \qquad (1 \le r+1 \le i \le p \le N)$$

$m(r,p) =$

$$\frac{(q^N;q^{-1})_{r+p-N}(q^{2N-r-p};q^{-1})_{N-p}(q^{N-r+1} - q^{N-p})\, q^{(\frac{r+p-N}{2})(\frac{r+p-N-2}{2})}(-1)^{\frac{r+p-N}{2}}}{(q^2;q^2)_{\frac{r+p-N}{2}}(q;q)_{N-p}(q^{N-r+1} - 1)}$$

$$(0 \le r \le p \le N \le r+p, \quad r+p-N \text{ even}).$$

9. $Her_q(N)$.

$$x_i(r,p) = \frac{q^{2N-2p-2r+1}(q^{2i} - q^{2r})(q^{2p} - q^{2i-2})}{(q^2-1)^2} \qquad (1 \leq r+1 \leq i \leq p \leq N)$$

$$m(r,p) =$$

$$\frac{(q^{2N};q^{-2})_{r+p-N}(q^{4N-2r-2p};q^{-2})_{N-p}(q^{2N-2r+2} - q^{2N-2p})(-q)^{\binom{r+p-N}{2}}(-1)^{r+p-N}}{(-q;-q)_{r+p-N}(q^2;q^2)_{N-p}(q^{2N-2r+2}-1)}$$

$$(0 \leq r \leq p \leq N \leq r+p).$$

10. $Quad_q(N)$.

$$x_i(r,p) = \frac{q^{N-p-r+1}(q^i - q^r)(q^p - q^{i-1})}{(q-1)^2} \qquad (1 \leq r+1 \leq i \leq p \leq N)$$

$$m(r,p) = \frac{(q^N;q^{-1})_{r+p-N}(q^{2N-r-p};q^{-1})_{N-p}(q^{N-r+1} - q^{N-p})(-1)^{r+p-N}}{(q^{-2};q^{-2})_{[\frac{r+p-N}{2}]}(q;q)_{N-p}(q^{N-r+1}-1)}$$

$$(0 \leq r \leq p \leq N \leq r+p).$$

11. $Hem_q(N)$. Same as $Alt_q(N)$.

Proof. The $x_i(r,p)$ constants are found by solving the linear system in (1) of Definition 2.4. Since $E(r,p)$ is tri-diagonal, this amounts to solving a 3-term linear recurrence equation. To get the $m(r,p)$ constants, we first find the $c(r,p)$ constants using the Note after Definition 2.3, and then apply (2) of Definition 2.4 and induction. We note that much of this data has been found by other authors using different methods. See Dunkl [14] for $S_q(N,M)$ and $A_q(N,M)$, and Stanton [29], [30], [31] for the polar spaces, $Alt_q(N)$, $Her_q(N)$, and $Quad_q(N)$. □

4. *Remarks.* In this section we give some directions for future research. Let X denote a d-class symmetric association scheme with Bose Mesner algebra M (defined in Bannai and Ito [1, p. 56]), and let P denote a uniform poset with rank N and incidence algebra T. Call X and P *compatible* if X can be identified with the upper fiber P_N of P so that M is a subalgebra of $E^*_N T E^*_N$. Note this implies $d \leq N$ by Corollary 2.7. Each uniform poset in Example 3.1 is compatible with at least one P- and Q-polynomial association scheme with $d \geq 2$. Denoting by A the first associate matrix of the scheme, we have, using the notation of Bannai and Ito [1, p. 301]:

Poset	*Compatible d – class scheme*	*Comments*
1. $S(N,M)$	Johnson scheme $J(N,M)$ $d=\min\{N,M-N\}$	$A = R_{N-1}L_N - NE_N^*$
2. $H(N,M)$	Hamming scheme $H(N,M-1)$ $d=N$	$A = R_{N-1}L_N - NE_N^*$
2a. $H(N,3)$	$\frac{1}{2}H(N+1,2)$ $d=[\frac{N+1}{2}]$	$A = \frac{1}{4}R_{N-1}R_{N-2}L_{N-1}L_N -$ $(N-2)R_{N-1}L_N + \frac{N(N-3)}{2}E_N^*$
3. $H^*(N,3)$	Antip. quot. of $H(N,2)$ $d=[N/2]$	$A = R_{N-1}L_N - NE_N^*$ $(N \geq 3)$
4. $Part(2N)$	Antip. quot. of $J(N,2N)$ $d=[N/2]$	$A = \frac{1}{2}R_{N-1}L_N - NE_N^*$
5. $S_q(N,M)$	q-Johnson scheme $J_q(N,M)$ $d=\min\{N,M-N\}$	$A = R_{N-1}L_N - (\frac{q^N-1}{q-1})E_N^*$
6. *Rank N polarspaces*	schemes of dual polar spaces $d=N$	$A = R_{N-1}L_N - (\frac{q^N-1}{q-1})E_N^*$
6a. $C_N(q)$ (q odd)	Ustimenko's scheme [16] $d=[\frac{N+1}{2}]$	$A = (q+1)^{-2}R_{N-1}R_{N-2}L_{N-1}L_N -$ $\frac{q(q^{N-2}-1)}{q-1}R_{N-1}L_N +$ $(\frac{q^N-1}{q-1})(\frac{q^N-q}{q^2-1} - 1)E_N^*$
6b. $B_N(q)$	$\frac{1}{2}D_{N+1}(q)$ $d=[\frac{N+1}{2}]$	same as 6a.
7. $A_q(N,M)$	bilin. forms scheme $H_q(N,M)$ $d=\min\{N,M\}$	$A = R_{N-1}L_N - (\frac{q^N-1}{q-1})E_N^*$
8. $Alt_q(N)$	alt. forms scheme Alt $d=[N/2]$	$A = (q+1)^{-1}$ $\times(R_{N-1}L_N - (\frac{q^N-1}{q-1})E_N^*)$
9. $Her_q(N)$	Herm. forms scheme Her $d=N$	$(q+1)A^2 + 2A - q(q-1)R_{N-1}L_N$ $= (\frac{q^{2N}-1}{q+1})E_N^*$
10. $Quad_q(N)$	quad. forms scheme $Quad$ $d=[\frac{N+1}{2}]$	no simple relationship
11. $Hem_q(N)$	$Quad$ $d=[N/2]$	same as 8.

We note $A \in E_N^* T E_N^*$ for Examples 9, 10. For Example 9, this follows since T is the full commuting algebra of $\mathrm{Aut}(P)$, which contains A (see [30, Theorem 4.10]). For Example 10, this follows from the averaging technique of Stanton [31, p.304]. Call a symmetric association scheme *geometric* if it is compatible with some uniform poset. Then the above information shows that most of the known P- and Q-polynomial schemes with $d \geq 7$ are geometric. Indeed, the catalog in [7, Section 8.5] shows that apart from the ordinary cycles, the known P- and Q-polynomial schemes with $d \geq 7$ that are possibly not geometric are (i) the Doob schemes [12] (with the same parameters as the Hamming schemes $H(d,4)$), and (ii) the Hemmeter schemes [15] (with the same parameters as $D_d(q)$, q odd), and (iii) the antipodal quotient of the half cube $\frac{1}{2}H(n,2)$, (with n even).

Problem 1. Determine if the examples (i), (ii), (iii) above are geometric. If not, can the definition of "geometric" be slightly generalized so that they are?

Conjecture 2. All geometric association schemes are P- and Q-polynomial.

Conjecture 3. For sufficiently large N, the only uniform ϕ-regular semi-lattices of rank N are those in 1, 2, 5, 6, 7 of Example 3.1.

Problem 4. Find a simple axiom system for the *points* P_N and *lines* P_{N-1} (and if necessary, *planes* P_{N-2}) that characterizes the 11 examples in Section 3 (or all uniform posets). See Cameron [8] for Example 6 and Sprague [26], [27], [28] for Examples 1, 2, 5, 7.

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