

Math 846: Algebraic Combinatorics

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I Some algebraic graph theory (Lectures 1–10)

Lecture 1: Graph theory concepts such as walk, path, connectivity, diameter D , automorphism group, transitivity and distance-transitivity, adjacency matrix A and its eigenvalues, the adjacency algebra M , the standard module V . The definition of a hypercube.

Lecture 2: Basic facts about the hypercube, such as eigenvalues and their eigenspaces, bipartiteness, the automorphism group, distance-transitivity.

Lecture 3: Regularity and distance-regularity. Basic concepts for a distance-regular graph, such as the intersection numbers, the valencies $\{k_i\}_{i=0}^D$, the distance matrices $\{A_i\}_{i=0}^D$, the polynomials $\{v_i\}_{i=0}^D$. The eigenvalues of the hypercube are computed.

Lecture 4: Topics about distance-regular graphs, such as the polynomials $\{u_i\}_{i=0}^D$, the bases $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$ for M , the transition matrices between $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$, the entrywise product for matrices, the Krein parameters, the dual primitive idempotents $\{E_i^*\}_{i=0}^D$, the dual adjacency algebra M^* .

Lecture 5: Topics about distance-regular graphs, such as the bijection $\rho : M \rightarrow M^*$, the dual distance matrices $\{A_i^*\}_{i=0}^D$, the bases $\{E_i^*\}_{i=0}^D$ and $\{A_i^*\}_{i=0}^D$ for M^* , the transition matrices between $\{E_i^*\}_{i=0}^D$ and $\{A_i^*\}_{i=0}^D$, the subconstituent algebra T , a bilinear form for matrices.

Lecture 6: Topics about distance-regular graphs, such as the triple product relations, the Q -polynomial property, the dual adjacency matrix A^* and its eigenvalues, the polynomials $\{v_i^*\}_{i=0}^D$ and $\{u_i^*\}_{i=0}^D$.

Lecture 7: Topics about Q -polynomial distance-regular graphs, such as Askey-Wilson duality and the actions of A, A^* on each others eigenspaces. Topics about the hypercube, such as the raising matrix R , the lowering matrix L , the relations between R, L, A^* and the relations between A, A^* .

Lecture 8: Topics for the hypercube, such as the dual raising matrix R^* , the dual lowering matrix L^* , relations between R^*, L^*, A . Topics about distance-regular graphs, such as T -modules, irreducible T -modules, isomorphism of T -modules, nonisomorphic irreducible T -modules are orthogonal, the multiplicity of an irreducible T -module.

Lecture 9: The irreducible T -modules for the hypercube.

Lecture 10: Lie algebras and their universal enveloping algebras, the Lie algebra $\mathfrak{sl}_2$

and the hypercube, the Onsager Lie algebra $\mathcal{O}$.

Lecture 10 Appendix: The Onsager Lie algebra and the $\mathfrak{sl}_2$ loop algebra.

II Tridiagonal pairs and Leonard pairs (Lectures 11–18)

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Lecture 12: Tridiagonal pairs from NonNil finite-dimensional irreducible U_q^+ -modules; how Leonard pairs and tridiagonal pairs are related.

Lecture 13: The primitive idempotents and their standard orderings; tridiagonal systems; the relatives of a tridiagonal system and the D_4 -action; the diameter of a tridiagonal system.

Lecture 14: The shape vector of a tridiagonal system; the shape vector is symmetric and unimodal; the raising and lowering maps.

Lecture 15: The split decompositions of a tridiagonal system; the tetrahedron diagram.

Lecture 16: Recurrent sequences; the tridiagonal relations.

Lecture 17: The eigenvalues/dual eigenvalues of a tridiagonal system satisfy a three-term recurrence; special cases of the tridiagonal relations; the q -Onsager algebra $\mathcal{O}_q$ and its finite-dimensional irreducible modules.

Lecture 18: The tridiagonal algebra and its finite-dimensional irreducible modules.

III The classification of Leonard systems (Lectures 19–26)

Lecture 19: Pre Leonard systems; the eigenvalue sequence $\{\theta_i\}_{i=0}^d$; the dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$; the diagonal sequence $\{a_i\}_{i=0}^d$; the dual diagonal sequence $\{a_i^*\}_{i=0}^d$; the antiautomorphism $\dagger$.

Lecture 20: The 3/4 theorem; the polynomials $\{\tau_i\}_{i=0}^d$ and $\{\eta_i\}_{i=0}^d$; the normalizing property for E_0^* .

Lecture 21: Describing a pre Leonard system using upper/lower bidiagonal matrices; the scalars $\{\varphi_i\}_{i=1}^d$.

Lecture 22: How $\{a_i\}_{i=0}^d$ and $\{a_i^*\}_{i=0}^d$ are related to $\{\varphi_i\}_{i=1}^d$; the wrap around theorem.

Lecture 23: The first and second split sequence of a Leonard system; the parameter array of a Leonard system; statement of the classification theorem for Leonard systems.

Lecture 24: A sum involving beta-recurrent sequences.

Lecture 25: What the tridiagonal relations imply about $\{\varphi_i\}_{i=1}^d$; the proof of the classification theorem (first part).

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IV Some orthogonal polynomials attached to a Leonard system Φ (Lectures 27–32)

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Lecture 28: Some polynomials $\{p_i\}_{i=0}^{d+1}$ attached to Φ .

Lecture 29: The parameters $\{m_i\}_{i=0}^d$ and $\{k_i\}_{i=0}^d$ and ν ; the normalizations u_i and v_i of p_i .

Lecture 30: A bilinear form attached to Φ ; Askey-Wilson duality revisited.

Lecture 31: Some orthogonality relations for $\{u_i\}_{i=0}^d$, $\{v_i\}_{i=0}^d$, $\{p_i\}_{i=0}^d$; the P -matrix; the polynomials $\{u_i\}_{i=0}^d$ in closed form.

Lecture 32: The polynomials $\{p_i\}_{i=0}^d$ in closed form; everything in terms of the parameter array; expressing $u_i(\theta_j)$ ($0 \leq i, j \leq d$) using hypergeometric series and basic hypergeometric series; a detailed example.

V The tetrahedron Lie algebra $\boxtimes$ (Six Lectures including Lecture 33)

Lecture 33: Motivating $\boxtimes$ using the Leonard systems from the hypercube; defining $\boxtimes$ by generators and relations.

Five student presentations: covering the paper by Brian Hartwig about $\boxtimes$.

VI The q -tetrahedron algebra $\boxtimes_q$ and the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$ (Lectures 34–37)

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Lecture 35: Defining $\boxtimes_q$ by generators and relations; using a tridiagonal pair of q -geometric type to get a $\boxtimes_q$ -module; the Chevalley presentation of $U_q(\mathfrak{sl}_2)$.

Lecture 36: The equitable presentation of $U_q(\mathfrak{sl}_2)$; the Chevalley and equitable presentations of $U_q(\widehat{\mathfrak{sl}}_2)$; an algebra homomorphism $U_q(\widehat{\mathfrak{sl}}_2) \rightarrow \boxtimes_q$; using a tridiagonal pair of q -geometric type to get a $U_q(\widehat{\mathfrak{sl}}_2)$ -module.

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VII The universal Askey-Wilson algebra Δ_q (Lectures 38–41)

Lecture 38: Motivating Δ_q via Leonard systems of q -Racah type; a $\mathrm{PSL}_q(\mathbb{Z})$ -action on Δ_q ; the Casimir element of Δ_q ; an algebra homomorphism $\mathcal{O}_q \rightarrow \Delta_q$.

Lecture 39: The Casimir element of $U_q(\mathfrak{sl}_2)$; how Δ_q is related to $U_q(\mathfrak{sl}_2)$; how Δ_q is related to the DAHA of type $(C_1^\vee, C_1)$.

Lecture 40: The Lusztig automorphisms of $\mathcal{O}_q$.

Lecture 41: Sharp tridiagonal systems; statement of their classification.

2 Topic Summary

Let $\mathbb{F}$ denote a field. By a scalar, we mean an element of $\mathbb{F}$. Unless stated otherwise, a given vector space, algebra, or tensor product is understood to be over $\mathbb{F}$. Unless stated otherwise, a given algebra without the Lie prefix is understood to be associative and have a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. Unless stated otherwise, a scalar denoted q is understood to be nonzero and $q^4 \neq 1$.

The following topics will be discussed in the course.

Definition 2.1. The Lie algebra $\mathfrak{sl}_2$ has a basis E, F, H and Lie bracket

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

This Lie algebra has another basis X, Y, Z such that

$$[X, Y] = 2X + 2Y, \quad [Y, Z] = 2Y + 2Z, \quad [Z, X] = 2Z + 2X.$$

The basis X, Y, Z is called equitable. Primary [1, 2, 60].

Definition 2.2. The $\mathfrak{sl}_2$ loop algebra is the Lie algebra consisting of the vector space $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}]$ (t an indeterminate) and the following Lie bracket: for $u, v \in \mathfrak{sl}_2$ and $f, g \in \mathbb{F}[t, t^{-1}]$,

$$[u \otimes f, v \otimes g] = [u, v] \otimes fg.$$

Definition 2.3. The Onsager Lie algebra $\mathcal{O}$ is defined by generators W_0, W_1 and relations

$$[W_0, [W_0, [W_0, W_1]]] = 4[W_0, W_1], \quad [W_1, [W_1, [W_1, W_0]]] = 4[W_1, W_0].$$

There is an injective Lie algebra homomorphism from $\mathcal{O}$ into the $\mathfrak{sl}_2$ loop algebra. Secondary [3].

Definition 2.4. The tetrahedron Lie algebra $\boxtimes$ is defined by generators

$$\{x_{i,j} | i, j \in \mathbb{I}, i \neq j\} \quad \mathbb{I} = \{0, 1, 2, 3\}$$

and the following relations.

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{i,j} + x_{j,i} = 0.$$

(ii) For mutually distinct $h, i, j \in \mathbb{I}$,

$$[x_{h,i}, x_{i,j}] = 2x_{h,i} + 2x_{i,j}.$$

(iii) For mutually distinct $h, i, j, k \in \mathbb{I}$,

$$[x_{h,i}, [x_{h,i}, [x_{h,i}, x_{j,k}]]] = 4[x_{h,i}, x_{j,k}].$$

There exists a Lie algebra isomorphism from $\boxtimes$ to the 3-point $\mathfrak{sl}_2$ loop algebra $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$. Primary [4–6].

Definition 2.5. The q -deformed enveloping algebra $U_q(\mathfrak{sl}_2)$ is defined by generators E, F, K, K^{-1} and relations

$$\begin{aligned} KK^{-1} &= 1, & K^{-1}K &= 1, \\ KE &= q^2EK, & KF &= q^{-2}FK, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

This algebra has another presentation with generators X, Y, Y^{-1}, Z and relations

$$\begin{aligned} YY^{-1} &= 1, & Y^{-1}Y &= 1, \\ \frac{qXY - q^{-1}YX}{q - q^{-1}} &= 1, & \frac{qYZ - q^{-1}ZY}{q - q^{-1}} &= 1, & \frac{qZX - q^{-1}XZ}{q - q^{-1}} &= 1. \end{aligned}$$

This presentation is called equitable. Primary [7, 8], Secondary [9–11].

Definition 2.6. The algebra U_q^+ is defined by generators W_0, W_1 and relations

$$[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = 0, \quad [W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] = 0,$$

where $[R, S]_q = qRS - q^{-1}SR$. The algebra U_q^+ is called the positive part of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. Primary [12–14], Secondary [15–19].

Definition 2.7. The q -Onsager algebra $\mathcal{O}_q$ is defined by generators W_0, W_1 and relations

$$\begin{aligned} [W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] &= (q^2 - q^{-2})^2 [W_1, W_0], \\ [W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] &= (q^2 - q^{-2})^2 [W_0, W_1]. \end{aligned}$$

The algebra $\mathcal{O}_q$ is a coideal subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$. The Lusztig automorphism of $\mathcal{O}_q$ sends

$$W_0 \mapsto W_0, \quad W_1 \mapsto W_1 + \frac{[W_0, [W_0, W_1]_q]}{(q - q^{-1})(q^2 - q^{-2})}.$$

Primary [20], Secondary [21–26].

Definition 2.8. Consider the cyclic group $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$. The q -tetrahedron algebra $\boxtimes_q$ is defined by generators

$$\{x_{i,j} | i, j \in \mathbb{Z}_4, j - i = 1 \text{ or } j - i = 2\}$$

and the following relations.

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{i,j}x_{j,i} = 1.$$

(i) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i - h, j - i)$ is one of $(1, 1), (1, 2), (2, 1)$,

$$\frac{qx_{h,i}x_{i,j} - q^{-1}x_{i,j}x_{h,i}}{q - q^{-1}} = 1.$$

(ii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,

$$[x_{h,i}, [x_{h,i}, [x_{h,i}, x_{j,k}]_q]_{q^{-1}}] = 0.$$

For nonzero $a, b \in \mathbb{F}$ there exists an injective algebra homomorphism $\mathcal{O}_q \rightarrow \boxtimes_q$ that sends $W_0 \mapsto ax_{0,1} + a^{-1}x_{1,2}$ and $W_1 \mapsto bx_{2,3} + b^{-1}x_{3,0}$. Primary [27–30], Secondary [31].

Definition 2.9. For a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ the corresponding tridiagonal algebra $T = T(\beta, \gamma, \gamma^*, \varrho, \varrho^*)$ is defined by generators A, A^* and relations

$$\begin{aligned} 0 &= [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \\ 0 &= [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A]. \end{aligned}$$

Special cases of T include the enveloping algebra $U(\mathcal{O})$ along with U_q^+ and $\mathcal{O}_q$. Primary [32].

Definition 2.10. For a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ the corresponding Askey-Wilson algebra $AW = AW(\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*)$ is defined by generators A, A^* and relations

$$\begin{aligned} A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(AA^* + A^* A) - \varrho A^* &= \gamma^* A^2 + \omega A + \eta I, \\ A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^*(A^* A + A A^*) - \varrho^* A &= \gamma A^{*2} + \omega A^* + \eta^* I. \end{aligned}$$

There exists an algebra homomorphism $T \rightarrow AW$ that sends $A \mapsto A$ and $A^* \mapsto A^*$. Primary [33].

Definition 2.11. The universal Askey-Wilson algebra UAW has three generators A, B, C . The relations assert that each of the following is central:

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}.$$

There is an algebra homomorphism $\mathcal{O}_q \rightarrow UAW$ that sends $W_0 \mapsto A$ and $W_1 \mapsto B$. The algebra UAW is generated by its center and the $\mathcal{O}_q$ image. The modular group $\text{PSL}_2(\mathbb{Z})$ acts faithfully on UAW as a group of automorphisms. There exists an injective algebra homomorphism $UAW \rightarrow U_q(\mathfrak{sl}_2) \otimes [a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ (a, b, c indeterminates) that sends

$$\begin{aligned} A &\mapsto X \otimes a + Y \otimes a^{-1} + \frac{XY - YX}{q - q^{-1}} \otimes bc^{-1}, \\ B &\mapsto Y \otimes b + Z \otimes b^{-1} + \frac{YZ - ZY}{q - q^{-1}} \otimes ca^{-1}, \\ C &\mapsto Z \otimes c + X \otimes c^{-1} + \frac{ZX - XZ}{q - q^{-1}} \otimes ab^{-1}. \end{aligned}$$

Primary [34, 35], Secondary [36].

Definition 2.12. The universal double affine Hecke algebra (UDAHA) of type $(C_1^\vee, C_1)$ is defined by generators

$$\{t_i, t_i^{-1} | i \in \mathbb{I}\} \quad \mathbb{I} = \{0, 1, 2, 3\}$$

and the following relations:

$$\begin{aligned} t_i t_i^{-1} &= t_i^{-1} t_i = 1 & i \in \mathbb{I}, \\ t_i + t_i^{-1} &\text{ is central } & i \in \mathbb{I}, \\ t_0 t_1 t_2 t_3 &= q^{-1}. \end{aligned}$$

There exists an injective algebra homomorphism $UAW \rightarrow UDAHA$ that sends

$$A \mapsto t_1 t_0 + (t_1 t_0)^{-1}, \quad B \mapsto t_3 t_0 + (t_3 t_0)^{-1}, \quad C \mapsto t_2 t_0 + (t_2 t_0)^{-1}.$$

Primary [37, 38], Secondary [39, 40].

Definition 2.13. Leonard’s theorem (1985) classifies the 3-term recurrent polynomial sequences $\{u_i\}_{i=0}^d$ whose duals are also 3-term recurrent polynomial sequences. This phenomenon is sometimes called Askey-Wilson duality or bispectrality. The famous list of solutions consists of the following polynomials: q -Racah, q -Hahn, dual q -Hahn, q -Krawtchouk, dual q -Krawtchouk, quantum q -Krawtchouk, affine q -Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito. The listed polynomials make up the terminating branch of the Askey scheme of orthogonal polynomials. These polynomials are traditionally described using hypergeometric series or basic hypergeometric series. The closely related concept of a Leonard pair is discussed below.

Definition 2.14. A tridiagonal pair is an ordered pair A, A^* of diagonalizable linear maps on a nonzero finite-dimensional vector space, such that each of A, A^* acts on the eigenspaces of the other one in a block-tridiagonal fashion. Every tridiagonal pair supports a finite-dimensional irreducible module of a tridiagonal algebra. Primary [41, 45, 46], Secondary [42–44, 47–59].

Definition 2.15. A Leonard pair is a tridiagonal pair A, A^* such that for each of A, A^* the eigenspaces all have dimension one. The classification of the Leonard pairs amounts to a linear-algebraic version of Leonard’s theorem. Every Leonard pair supports a finite-dimensional irreducible module of an Askey-Wilson algebra. Primary [60–64], Secondary [65–70].

Definition 2.16. Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and adjacency relation $\mathcal{R}$. The $(0/1)$ -adjacency matrix $A \in \text{Mat}_X(\mathbb{R})$ is symmetric and hence diagonalizable. Fix a vertex $x \in X$, and note that A acts in a block-tridiagonal fashion on the subconstituents of Γ with respect to x . By a dual adjacency matrix of Γ with respect to x , we mean a diagonal matrix $A^* \in \text{Mat}_X(\mathbb{R})$ whose eigenspaces are the subconstituents of Γ with respect to x , and A^* acts on the eigenspaces of A in a block-tridiagonal fashion. We say that Γ is Q -polynomial with respect to x whenever A^* exists. An elementary example is the hypercube graph. There are many known examples of Q -polynomial graphs that have a combinatorial property called distance-regularity. Primary [1, 71]. Secondary [77–80].

Definition 2.17. Referring to (vii), assume that Γ is Q -polynomial with respect to x . The subconstituent algebra $T = T(x)$ is the subalgebra of $\text{Mat}_X(\mathbb{R})$ generated by A, A^* . The algebra T is semisimple, so the standard module $\mathbb{R}^X$ is a direct sum of irreducible T -modules. The matrices A, A^* act on each irreducible T -module as a tridiagonal pair. Moreover, T is a homomorphic image of a tridiagonal algebra. Primary [1, 71].

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