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Leonard systems of q-Racah type

Summary

We saw

$$\begin{tikzcd} \text{Krawtchouk polys} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] & & \text{LS of Krawtchouk type} \arrow[dl, Leftrightarrow] \\ & \text{f.d. irred } sl_2\text{-modules} & \end{tikzcd}$$

$$\begin{tikzcd} \text{polys } \{u_i\}_{i=0}^{N} \text{ from Term branch Askey-Scheme} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] & & \text{general LS} \arrow[dl, Leftrightarrow] \\ & \text{f.d. irred modules for what algebra ?} & \end{tikzcd}$$

No known canonical answer for ?

The following algebras play a role

- The Tridiagonal algebra T

  Sometimes called the q-Onsager algebra

  T is defined by 2 gens A, A^* subject to the tridiagonal relations TD1, TD2 from Th 225

- Askey-Wilson algebra $AW(3)$

  Sometimes called Zhedanov algebra

  $AW(3)$ is defined by two generators A, A^* subject to the Askey-Wilson relations AW1, AW2 from Th 231

- $U_q sl_2$

- q-tetrahedron algebra $\boxtimes_q$

- DAHA rank 1

  "Double affine Hecke algebra"

As we discuss these algebras we restrict our attention to LS of q-Racah type.

$\square$

Lec 16 Mon

Leonard systems of q-Racah type

Below thm 24.5 we gave a handout listing

all the param arrays

the "most general" family of PA was the q-Racah

(Ex 35.2 in handout)

this family involves parameters

$$q, \theta_0, \theta_0^*, h, h^*, s, s^*, r_1, r_2$$

subject to $\quad ss^* = r_1 r_2 q^{-N-1}$

(and some inequalities)

We account for 4 free parameters as follows.

Until further notice: $\quad N \geq 1$,

$$\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$$

is LS on V of q-Racah type, and PA

$$(\{\theta_i\}_{i=0}^N, \{\theta_i^*\}_{i=0}^N, \{\varphi_i\}_{i=1}^N, \{\phi_i\}_{i=1}^N)$$

Pick $r, r^*, t, t^* \in \mathbb{F}$ with $r \neq 0, r^* \neq 0$. Then

$$\left(rA + tI, \ \{E_i\}_{i=0}^N, \ r^*A^* + t^*I, \ \{E_i^*\}_{i=0}^N \right)$$

is LS on V with q-Racah type and PA

$$\left(\{r\theta_i + t\}_{i=0}^N, \ \{r^*\theta_i^* + t^*\}_{i=0}^N \ \{rr^*\varphi_i\}_{i=1}^N, \ \{rr^*\phi_i\}_{i=1}^N \right)$$

Adjusting Φ in this way

$$\theta_0, \ \theta_0^*, \ h, \ h^*$$

become whatever we like, provided $h \neq 0, \ h^* \neq 0$

To minimize the role of $\theta_0, \theta_0^*, h, h^*$ we normalize the PA of Φ as follows.

Until further notice

$\mathbb{F}$ is alg closed.

For aesthetic reasons replace q by q^2

Write

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2 \ldots$$

Assume the θ_i, θ^*_i have form

$$\theta_i = a q^{2i-N} + a^{-1} q^{N-2i}$$

$$0 \le i \le N$$

$$\theta^*_i = b q^{2i-N} + b^{-1} q^{N-2i}$$

$$0 \ne a \in \mathbb{F}, \qquad 0 \ne b \in \mathbb{F}.$$

In this case φ_i, ϕ_i take form

$$\varphi_i = a^{-1} b^{-1} q^{N+1} (q^{-i} - q^{i})(q^{N-i+1} - q^{i-N-1})(q^{-i} - abc\, q^{i-N-1})(q^{-i} - \frac{ab}{c} q^{i-N-1}),$$

$$\phi_i = a b^{-1} q^{N+1} (q^{-i} - q^{i})(q^{N-i+1} - q^{i-N-1})(q^{-i} - \frac{bc}{a} q^{i-N-1})(q^{-i} - \frac{b}{ac} q^{i-N-1}).$$

$$1 \le i \le N$$

for some $0 \ne c \in \mathbb{F}$

Note c is determined up to inverse — we can replace c by $\frac{1}{c}$ and leave the PA invariant.

Note

$\{\theta_i\}_{i=0}^N$ is (β, γ, δ)-rcc with

$$\beta = q^2 + q^{-2}$$

$$\gamma = 0$$

$$\delta = -(q^2 - q^{-2})^2$$

Sim for $\{\theta^*_i\}_{i=0}^N$. Also $\displaystyle\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} = \frac{q^i - q^{-i}}{q - q^{-1}} \, \frac{q^{N-i+1} - q^{i-N-1}}{q^N - q^{-N}}$.

$(0 \le i \le N+1)$

Sometimes we abbrev $B = A^*$

LEM 274 With above notation the TD rels become

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = -(q^2-q^{-2})^2(AB-BA) \qquad \text{TD1} \quad ✓$$

$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = -(q^2-q^{-2})^2(BA-AB) \qquad \text{TD2} \quad ✓$$

Note Write

$$[u,v]_q = quv - q^{-1}vu$$

Then the above eqs become

$$\left[A,\left[A,\left[A,B\right]\right]_q\right]_{q^{-1}} = -(q^2-q^{-2})^2[A,B], \qquad \text{TD1}$$

$$\left[B,\left[B,\left[B,A\right]\right]_q\right]_{q^{-1}} = -(q^2-q^{-2})^2[B,A] \qquad \text{TD2}$$

"q-Dolan grady" relations.

DEF 275 Let $\mathcal{O}_q$ denote the (assoc) $\mathbb{F}$-algebra defined by gens A, B subject to the q-Dolan Grady rels

$\mathcal{O}_q$ called the *q-Onsager algebra*.

Note Compare $\mathcal{O}_q$ with orig Onsager algebra, which is the Lie algebra defined by gens A, B subject to Dolan-Grady rels

$$[A,[A,[A,B]]] = 4[A,B],$$
$$[B,[B,[B,A]]] = 4[B,A].$$

(Onsager, 1949)

Note Referring to our LS Φ on V

A. sat the q-Dolan Grady relations

and therefore induce a $\mathcal{O}_q$-module structure on V.

This $\mathcal{O}_q$-module is irreducible by constr.

However, not every f.d. irred $\mathcal{O}_q$-module gives a LS. Precise classification still open

Note $\mathcal{O}_q$ is currently being used in stat. mech

to study integrable systems

XXZ-open spin chain

...

See recent papers on arXiv by Pascal Baseilhac

Next goal: AW relations for q-Racah case.

In this case it is convenient to introduce a 3d generator C

thm 276 (Hau-wen Huang 2010)

For our LS Φ on V, $\exists$ $C \in \mathrm{End}\, V$ s.t

(num $= \alpha C$)

$$\frac{qAB - q^{-1}BA}{q^2 - q^{-2}} + C = \frac{(a+a^{-1})(b+b^{-1}) + (c+c^{-1})(q^{N+1}+q^{-N-1})}{q+q^{-1}} I,$$

(num $= \alpha A$)

$$\frac{qBC - q^{-1}CB}{q^2 - q^{-2}} + A = \frac{(b+b^{-1})(c+c^{-1}) + (a+a^{-1})(q^{N+1}+q^{-N-1})}{q+q^{-1}} I,$$

(num $= \alpha B$)

$$\frac{qCA - q^{-1}AC}{q^2 - q^{-2}} + B = \frac{(c+c^{-1})(a+a^{-1}) + (b+b^{-1})(q^{N+1}+q^{-N-1})}{q+q^{-1}} I,$$

" $\mathbb{Z}_3$- symmetric Askey - Wilson relations "

pf 1 (sketch)

In the last 2 equations, if we eliminate C using the 1st equation, we get the AW relations AW1, AW2 for this q-Racah case

pf2 (sketch) Represent A, B by matrices via Φ. WLOG

$$A = \begin{pmatrix} \theta_0 & & & \mathbf{0} \\ 1 & \theta_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \theta_N \end{pmatrix} \qquad B = \begin{pmatrix} \theta_0^* & \varphi_1 & & \mathbf{0} \\ & \theta_1^* & \ddots & \\ & & \ddots & \varphi_N \\ 0 & & & \theta_N^* \end{pmatrix}$$

Def C using 1st eq. check Last 2 eqs by brute force $\square$

Aside

Given $A, B, C \in \operatorname{End} V$ call A, B, C Leonard triple whenever for each of $X \in \{A, B, C\}$ $\exists$ basis for V that makes X diagonal and the other two irred tridiag.

We cite a result.

th 277 (Hau-Wen Huang 2010)

Referring to th 276

(i) For C the roots of the char poly are

$$c q^{2i-N} + c^{-1} q^{N-2i} \qquad 0 \le i \le N \qquad \ast$$

(ii) C is mult free iff $\ast$ mut dist iff

c^2 not among $q^{2N-2}, q^{2N-4}, \ldots, q^{2-2N}$

iff A, B, C is Leonard triple

$\square$

Lec 37 Wed Dec 8

Ref to Th 276 recall antiaut $\dagger$ for A, B from L188

So $A^\dagger = A$, $B^\dagger = B$

Def $C' = C^\dagger$

LEM 278 (Hau-Wen Huang 2010) With above not

(i) A, B, C' sat the equations of Th 276 with $(q; a, b, c)$ replaced by $(q^{-1}, a^{-1}, b^{-1}, c^{-1})$

(ii)
$$C' - C = \frac{AB - BA}{q - q^{-1}}$$

pf (i) Apply $\dagger$ to the eqs of Th 276 and use the fact that $\dagger$ is antiaut

(ii) Obs

$$\frac{qAB - q^{-1}BA}{q^2 - q^{-2}} + C = \frac{\alpha_c}{q + q^{-1}} I \qquad *$$

$$\frac{qBA - q^{-1}AB}{q^2 - q^{-2}} + C' = \frac{\alpha_c}{q + q^{-1}} I \qquad **$$

Subtract ** from * and simplify

$\square$

th279 (H. Huang 2010)

With ref to th276

$$(q+q^{-1})^2-(q^{N+1}+q^{-N-1})^2-(a+a^{-1})^2-(b+b^{-1})^2$$

$$-(c+c^{-1})^2-(a+a^{-1})(b+b^{-1})(c+c^{-1})(q^{N+1}+q^{-N-1})$$

$$= qABC+q^2A^2+q^{-2}B^2+q^2C^2-q\alpha_A A-q^{-1}\alpha_B B-q\alpha_C C$$

(and cyclic perms)

$$= q^{-1}CBA+q^{-2}A^2+q^2B^2+q^{-2}C^2-q^{-1}\alpha_A A-q\alpha_B B-q^{-1}\alpha_C C$$

(and cyclic perms)

pf to get the 1st equation represent A, B, C as matrices via L7 as in pf of th276. Multiply it out

To get the 2nd equation from the 1st, apply the antiaut †

and simplify using L278 (ii) ☐

Next goal $U_q\mathfrak{sl}_2$

Until further notice $\operatorname{char} \mathbb{F} \neq 2$, q not root of 1.

Def 280 $U_q(\mathfrak{sl}_2)$ is the (assoc) $\mathbb{F}$-alg with 1 defined by gens e, f, k, k^{-1} subject to

$$kk^{-1} = k^{-1}k = 1$$

$$ke = q^2 ek$$

$$kf = q^{-2} fk$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

"Chevalley presentation"

We recall the equitable pres for $U_q\mathfrak{sl}_2$

thm 281 $U_q(\mathfrak{sl}_2)$ is iso to the $\mathbb{F}$-alg with 1 defined by gens x, x^{-1}, y, z and rels ✓

$$xx^{-1} = x^{-1}x = 1$$

$$\frac{qxy - q^{-1}yx}{q-q^{-1}} = 1,$$

"equitable presentation"

$$\frac{qyz - q^{-1}zy}{q-q^{-1}} = 1$$

$$\frac{qzx - q^{-1}xz}{q-q^{-1}} = 1$$

An iso with the presentation in Def 280 is

$$x \longrightarrow k$$

$$y \longrightarrow k^{-1} + f(q-q^{-1})$$

$$z \longrightarrow k^{-1} - k^{-1}eq(q-q^{-1})$$

the inverse of this iso is

$$k \longrightarrow x$$

$$e \longrightarrow (1-xz)q^{-1}(q-q^{-1})^{-1}$$

$$f \longrightarrow (y-x^{-1})(q-q^{-1})^{-1}$$

each map is hom of $\mathbb{F}$-algebras, and that they

LEM 282

(i) $$q(1-yz) = q^{-1}(1-zy) = \frac{yz-zy}{q-q^{-1}} \qquad (=n_x)$$

(ii) $$q(1-zx) = q^{-1}(1-xz) = \frac{zx-xz}{q-q^{-1}} \qquad (=n_y)$$

(iii) $$q(1-xy) = q^{-1}(1-yx) = \frac{xy-yx}{q-q^{-1}} \qquad (=n_z)$$

pf (i) Rearrange

$$\frac{qyz-q^{-1}zy}{q-q^{-1}} = 1$$

To get

$$q(1-yz) = q^{-1}(1-zy)$$

Call this common value m.

Obs

$$m = \frac{qm - q^{-1}m}{q-q^{-1}}$$

$$= \frac{q\left(q^{-1}(1-zy)\right) - q^{-1}\left(q(1-yz)\right)}{q-q^{-1}}$$

$$= \frac{yz-zy}{q-q^{-1}}$$

□

(ii), (iii) Sim

By Lem 2.8.2 we get the following multiplication table in $U_q sl_2$:

	x	y	z
x		$1-q^{-1}\wedge z$	$1-q\wedge y$
y	$1-q\wedge z$		$1-q^{-1}\wedge x$
z	$1-q^{-1}\wedge y$	$1-q\wedge x$	

LEM 282A We have

(i) $$\frac{qYX - q^{-1}XY}{q-q^{-1}} = 1 - (q+q^{-1})n_Z$$

(ii) $$\frac{qZY - q^{-1}YZ}{q-q^{-1}} = 1 - (q+q^{-1})n_X$$

(iii) $$\frac{qXZ - q^{-1}ZX}{q-q^{-1}} = 1 - (q+q^{-1})n_Y$$

pf (i) In LHS elim XY, YX using the above mult table.

(ii), (iii) Sim □

LEM 283

(i) $x n_y = q^2 n_y x$ $\qquad$ $x n_z = q^{-2} n_z x$

(ii) $y n_z = q^2 n_z y$ $\qquad$ $y n_x = q^{-2} n_x y$

(iii) $z n_x = q^2 n_x z$ $\qquad$ $z n_y = q^{-2} n_y z$

pf (i) We verify

$$\begin{array}{ccc} x n_y & \overset{?}{=} & q^2 n_y x \\ \| & & \| \\ x \; q(1-zx) & & q^2 \; q^{-1}(1-xz) \; x \\ \| & & \| \\ q(x-xzx) & & q(x-xzx) \end{array}$$

✓

The other equations are similar.

☐

We now consider a certain central element Λ in $U_q sl_2$ called the Casimir element.

LEM 284 The following elements in $U_q sl_2$ coincide:

$$qX + q^{-1}Y + qZ - qXYZ \qquad (= \lambda_y)$$

$$qY + q^{-1}Z + qX - qYZX \qquad (= \lambda_z)$$

$$qZ + q^{-1}X + qY - qZXY \qquad (= \lambda_x)$$

$$q^{-1}X + qY + q^{-1}Z - q^{-1}ZYX \qquad (= \Lambda_y)$$

$$q^{-1}Y + qZ + q^{-1}X - q^{-1}XZY \qquad (= \Lambda_z)$$

$$q^{-1}Z + qX + q^{-1}Y - q^{-1}YXZ \qquad (= \Lambda_x)$$

(Call it Λ)

pf $\lambda_y = \Lambda_z$ since

$$\frac{\lambda_y - \Lambda_z}{q-q^{-1}} = X\Big(\underbrace{1 - \frac{qYZ - q^{-1}ZY}{q-q^{-1}}}_{\overset{\shortparallel}{0}} \Big)$$

$$= 0$$

Also $\lambda_y = \Lambda_x$ since

$$\frac{\lambda_y - \Lambda_x}{q-q^{-1}} = \Big(\underbrace{1 - \frac{qxy-q^{-1}yx}{q-q^{-1}}}_{\overset{\shortparallel}{0}}\Big) z$$

$$= 0$$

By these comments and symmetry

$$\begin{tikzcd}
\Lambda_y \arrow[r, phantom, "="] \arrow[d, phantom, "\shortparallel"] & \lambda_x \arrow[r, phantom, "="] & \Lambda_z \arrow[d, phantom, "\shortparallel"] \\
\lambda_z \arrow[dr, phantom, "="] & & \lambda_y \arrow[dl, phantom, "="] \\
& \Lambda_x &
\end{tikzcd}$$

□

LEM 285 Δ is central in $U_q sl_2$.

pf show Δ commutes with each of X, Y, Z.

show $x\Delta = \Delta x$:

$$x\Delta - \Delta x = x\lambda_z - \lambda_y x$$

$$= x\left(qY + q^{-1}Z + qX - qYZX\right)$$

$$- \left(qX + q^{-1}Y + qZ - qXYZ\right)x$$

$$= \underbrace{qXY - q^{-1}YX}_{\substack{\shortparallel \\ q-q^{-1}}} \qquad + \underbrace{q^{-1}XZ - qZX}_{\substack{\shortparallel \\ q^{-1}-q}}$$

$$= 0$$

Rest is similar. □

DEF 285A Call Δ the <u>Casimir</u> <u>element</u> of $U_q sl_2$.

Some useful formulae

LEM 286 We have

$$n_x x = \Lambda - qy - q^{-1}z \qquad x n_x = \Lambda - q^{-1}y - qz$$

$$n_y y = \Lambda - qz - q^{-1}x \qquad y n_y = \Lambda - q^{-1}z - qx$$

$$n_z z = \Lambda - qx - q^{-1}y \qquad z n_z = \Lambda - q^{-1}x - qy$$

pf To get the first equation, in the LHS elim n_x using $n_x = q(1-yz)$ and in RHS elim Λ using 2nd expression in L284.

other eqs similar. □

LEM 287

(i) $$\frac{x n_x - n_x x}{q - q^{-1}} = y - z$$

(ii) $$\frac{y n_y - n_y y}{q - q^{-1}} = z - x$$

(iii) $$\frac{z n_z - n_z z}{q - q^{-1}} = x - y$$

pf Use L 286 □

LEM 288 Ω is equal to each of

$$\frac{q X n_x - q^{-1} n_x X}{q - q^{-1}} + (q + q^{-1}) z$$

(+ cyclic perms)

$$\frac{q n_x X - q^{-1} X n_x}{q - q^{-1}} + (q + q^{-1}) y$$

(+ cyclic perms)

pf Use L 286 $\square$

LEM 289 We have

$$n_x n_y = 1 - q^{-1}\Delta z + q^{-2}z^2$$

$$n_y n_x = 1 - q\Delta z + q^2 z^2$$

$$n_y n_z = 1 - q^{-1}\Delta x + q^{-2}x^2$$

$$n_z n_y = 1 - q\Delta x + q^2 x^2$$

$$n_z n_x = 1 - q^{-1}\Delta y + q^{-2}y^2$$

$$n_x n_z = 1 - q\Delta y + q^2 y^2$$

pf To get the 1st equation note

$$\begin{aligned} n_x n_y &= q^{-1} n_x (1 - xz) \\ &= q^{-1} n_x - q^{-1} n_x x z \\ &= 1 - yz - q^{-1}\left(\Delta - qy - q^{-1}z\right) z \\ &= 1 - q^{-1}\Delta z + q^{-2}z^2. \end{aligned}$$

Other eqs similar. □

LEM 290 In $U_q sl_2$

(i) $$\frac{q n_x n_y - q^{-1} n_y n_x}{q - q^{-1}} = 1 - z^2$$

(ii) $$\frac{q n_y n_z - q^{-1} n_z n_y}{q - q^{-1}} = 1 - x^2$$

(iii) $$\frac{q n_z n_x - q^{-1} n_x n_z}{q - q^{-1}} = 1 - y^2$$

pf Use L 289 $\square$

Now consider f.d. $U_q sl_2$ modules.

LEM 291 Given $N = 0, 1, 2..$ and $\varepsilon \in \{1, -1\}$

$\exists$ $U_q sl_2$-module $L(N, \varepsilon)$ with the following properties:

recall

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$L(N, \varepsilon)$ has a basis $\{v_i\}_{i=0}^N$ s.t

$$K v_i = \varepsilon q^{N-2i} v_i \qquad 0 \le i \le N$$

$$e v_i = \varepsilon [N-i+1] v_{i-1} \qquad 1 \le i \le N, \qquad e v_0 = 0$$

$$f v_i = [i+1] v_{i+1} \qquad 0 \le i \le N-1, \qquad f v_N = 0$$

the $U_q sl_2$-module $L(N, \varepsilon)$ is irred provided

$$q^{2i} \ne 1 \qquad 1 \le i \le N$$

pf See forexample Jantzen.

We now describe $L(N,\varepsilon)$ from pt of view of the equitable presentation.

LEM 292 $L(N,\varepsilon)$ has a basis $\{u_i\}_{i=0}^{N}$ with resp to which

$$X: \quad \varepsilon \begin{pmatrix} q^N & & & & 0 \\ & q^{N-2} & & & \\ & & q^{N-4} & & \\ & 0 & & \ddots & \\ & & & & q^{-N} \end{pmatrix}$$

$$Y: \quad \varepsilon \begin{pmatrix} q^{-N} & & & & 0 \\ q^{-N}-q^{2-N} & q^{2-N} & & & \\ & q^{-N}-q^{4-N} & q^{4-N} & & \\ & 0 & \ddots & \ddots & \\ & & & q^{-N}-q^{N} & q^N \end{pmatrix}$$

"constant row sum q^{-N}"

$$Z: \quad \varepsilon \begin{pmatrix} q^{-N} & q^N-q^{-N} & & & 0 \\ & q^{2-N} & q^N-q^{2-N} & & \\ & & q^{4-N} & \ddots & \\ & 0 & & \ddots & \\ & & & & q^N \end{pmatrix}$$

"constant row-sum" q^N

pf Ref to L291 take $u_i = \gamma_i v_i$ where $\gamma_0 = 1$

$\gamma_i = -\varepsilon\, q^{N-i}\, \gamma_{i-1} \qquad 1 \le i \le N$

$\Box$

LEM 293 For the basis $\{u_i\}_{i=0}^{N}$ of $L(N,\varepsilon)$ from L 292

$$n_y u_i = -q^{N-i}\left(q^{N-i+1} - q^{i-N-1}\right) u_{i-1} \qquad (1 \le i \le N),$$

$$n_y u_0 = 0$$

$$n_z u_i = q^{-i}\left(q^{i+1} - q^{-i-1}\right) u_{i+1} \qquad (0 \le i \le N-1),$$

$$n_z u_N = 0$$

pf Use L 292 and $n_y = q(1-zx)$,

$n_z = q(1-xy)$ □

Note By L29.3, rel the basis $\{u_i\}_{i=0}^{N}$ for $L(N,\varepsilon)$

the matrices rep n_y, n_z are:

$$n_y: \begin{pmatrix} 0 & q^{-1}-q^{2N-1} & & & & \\ & 0 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & 0 & q^{-1}-q^{3} & \\ & & & & 0 & q^{-1}-q \\ & & & & & 0 \end{pmatrix}$$

$(i-1,i)$-entry $q^{-1}-q^{2N-2i+1}$

$$n_z: \begin{pmatrix} 0 & & & & & \\ q-q^{-1} & 0 & & & & \\ & q-q^{-3} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 0 & \\ & & & & q-q^{1-2N} & 0 \end{pmatrix}$$

$(i,i-1)$-entry $q-q^{1-2i}$

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LEM 294 On $L(N, \varepsilon)$

$$\Lambda = \varepsilon \left(q^{N+1} + q^{-N-1} \right) I$$

pf By L 286

$$\Lambda = q^{-1}X + qZ + \eta_y Y.$$

In this equation represent the RHS by a matrix wrt $\{u_i\}_{i=0}^{N}$. Using L 292, L 293 we find this matrix is $\varepsilon (q^{N+1} + q^{-N-1}) I$ □

□

thm 295 Pick nonzero scalars a, b, c in $\mathbb{F}$ and let

A, B, C denote the following elements in $U_q sl_2$:

$$A = aX + a^{-1}Y + {}^{b}/_{c} \frac{XY - YX}{q - q^{-1}}$$

$$B = bY + b^{-1}Z + {}^{c}/_{a} \frac{YZ - ZY}{q - q^{-1}}$$

$$C = cZ + c^{-1}X + {}^{a}/_{b} \frac{ZX - XZ}{q - q^{-1}}$$

Then

$(num = \tilde{Z}_C)$

(i) $$\frac{qAB - q^{-1}BA}{q^2 - q^{-2}} + C = \frac{(a+a^{-1})(b+b^{-1}) + (c+c^{-1})\Delta}{q + q^{-1}}$$

$(num = \tilde{Z}_A)$

(ii) $$\frac{qBC - q^{-1}CB}{q^2 - q^{-2}} + A = \frac{(b+b^{-1})(c+c^{-1}) + (a+a^{-1})\Delta}{q + q^{-1}}$$

$(num = \tilde{Z}_B)$

(iii) $$\frac{qCA - q^{-1}AC}{q^2 - q^{-2}} + B = \frac{(c+c^{-1})(a+a^{-1}) + (b+b^{-1})\Delta}{q + q^{-1}}$$

Pf(i) Expand AB, BA

AB:

	b Y	b^{-1} Z	c/a n_x
a X	ab	a/b	c
a^{-1} Y	b/a	$\frac{1}{ab}$	$\frac{c}{a^2}$
b/c n_z	$\frac{b^2}{c}$	c^{-1}	b/a

BA:		X	Y	n_z
		a	a^{-1}	b/c
Y	b	ab	b/a	b^2/c
Z	b^{-1}	a/b	$\frac{1}{ab}$	$\frac{1}{c}$
n_x	c/a	c	$\frac{c}{a^2}$	b/a

terms	coeff	reason this coeff is 0
ab	$\frac{qXY - q^{-1}YX}{q^2-q^{-2}} - \frac{1}{q+q^{-1}}$	def
$\frac{a}{b}$	$\frac{qXZ - q^{-1}ZX}{q^2-q^{-2}} + n_y - \frac{1}{q+q^{-1}}$	L282A(iii)
$\frac{b}{a}$	$\frac{q(Y^2 + n_z n_x) - q^{-1}(Y^2 + n_x n_z)}{q^2-q^{-2}} - \frac{1}{q+q^{-1}}$	L290(iii)
$\frac{1}{ab}$	$\frac{qYZ - q^{-1}ZY}{q^2-q^{-2}} - \frac{1}{q+q^{-1}}$	def
c	$\frac{qXn_x - q^{-1}n_x X}{q^2-q^{-2}} + Z - \frac{\Delta}{q+q^{-1}}$	L288
$\frac{1}{c}$	$\frac{qn_z Z - q^{-1}Zn_z}{q^2-q^{-2}} + X - \frac{\Delta}{q+q^{-1}}$	L288
$\frac{c}{a^2}$	$\frac{qYn_x - q^{-1}n_x Y}{q^2-q^{-2}}$	L283(ii)
$\frac{b^2}{c}$	$\frac{qn_z Y - q^{-1}Yn_z}{q^2-q^{-2}}$	L283(ii)

Back to our LS Φ on V from Th 276

Th 296 Ref to Th 276

$\exists$ $U_q(sl_2)$-module str on V s.t. on V

$$A = aX + a^{-1}Y + \frac{b}{c}\frac{XY-YX}{q-q^{-1}}$$

$$B = bY + b^{-1}Z + c/a\,\frac{YZ-ZY}{q-q^{-1}}$$

$$C = cZ + c^{-1}X + a/b\,\frac{ZX-XZ}{q-q^{-1}}$$

this $U_q sl_2$-module is iso $L(N, 1)$.

pf Compare Th 276 with Th 295 using L 294

Here is abstract ver of Th 279

Thm 297 Ref to Th 295

$$(q+q^{-1})^2 - \Lambda^2 - (a+a^{-1})^2 - (b+b^{-1})^2$$

$$-(c+c^{-1})^2 - (a+a^{-1})(b+b^{-1})(c+c^{-1})\Delta$$

$$= qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - q\tilde{\alpha}_A A - q^{-1}\tilde{\alpha}_B B - q\tilde{\alpha}_C C$$

(+ cyclic perms)

$$= q^{-1}CBA + q^{-2}A^2 + q^2B^2 + q^{-2}C^2 - q^{-1}\tilde{\alpha}_A A - q\tilde{\alpha}_B B - q^{-1}\tilde{\alpha}_C C$$

(+ cyclic perms)

pf Use L282-290 (more details below) □

ABC

Z

	Y	Z	n_x
X	abc	ac/b	c^2
Y	bc/a	$\frac{c}{ab}$	c^2/a^2
n_z	b^2	1	cb/a

X

	Y	Z	n_x
X	$\frac{ab}{c}$	$a/(bc)$	1
Y	$b/(ac)$	$\frac{1}{abc}$	$\frac{1}{a^2}$
n_z	b^2/c^2	$\frac{1}{c^2}$	$b/(ac)$

n_y

	Y	Z	n_x
X	a^2	a^2/b^2	$\frac{ac}{b}$
Y	1	$\frac{1}{b^2}$	$\frac{c}{ab}$
n_z	$\frac{ab}{c}$	$\frac{a}{bc}$	1

A^2

	X	Y	nz
	a	a^{-1}	b/c
aX	a^2	1	$\frac{ab}{c}$
$a^{-1}Y$	1	$\frac{1}{a^2}$	$\frac{b}{ac}$
$\frac{b}{c}nz$	$\frac{ab}{c}$	$\frac{b}{ac}$	$\frac{b^2}{c^2}$

B^-

	bY	$b^{-1}Z$	$\frac{c}{a} n_x$
bY	b^2	1	$\frac{bc}{a}$
$b^{-1}Z$	1	$\frac{1}{b^2}$	$\frac{c}{ab}$
$\frac{c}{a} n_x$	$\frac{bc}{a}$	$\frac{c}{ab}$	$\frac{c^2}{a^2}$

C^{ru}

	cZ	$c^{-1}X$	$\frac{a}{b}\, ny$
cZ	c^2	1	$\frac{ac}{b}$
$c^{-1}X$	1	$\frac{1}{c^2}$	$\frac{a}{bc}$
$\frac{a}{b}\, ny$	$\frac{ac}{b}$	$\frac{a}{bc}$	$\frac{a^2}{b^2}$

$\widetilde{\chi}_A A$

	aX	$a^{-1}Y$	$\frac{b}{c}\Lambda_Z$
$bc\ I$	abc	$\frac{bc}{a}$	b^2
$\frac{b}{c}\ I$	$\frac{ab}{c}$	$\frac{b}{ac}$	$\frac{b^2}{c^2}$
$\frac{c}{b}\ I$	$\frac{ac}{b}$	$\frac{c}{ab}$	1
$\frac{1}{bc}\ I$	$\frac{a}{bc}$	$\frac{1}{abc}$	$\frac{1}{c^2}$
$a\ II$	a^2	1	$\frac{ab}{c}$
$a^{-1}\Delta$	1	$\frac{1}{a^2}$	$\frac{b}{ac}$

$\tilde{\alpha}_B$ B

	bY	$b^{-1}Z$	$\frac{c}{a}\, n_x$
$ac\ I$	abc	$\frac{ac}{b}$	c^2
$\frac{a}{c}\ I$	$\frac{ab}{c}$	$\frac{a}{bc}$	1
$\frac{c}{a}\ I$	$\frac{bc}{a}$	$\frac{c}{ab}$	$\frac{c^2}{a^2}$
$\frac{1}{ac}\ I$	$\frac{b}{ac}$	$\frac{1}{abc}$	$\frac{1}{a^2}$
$b\,\Delta$	b^2	1	$\frac{bc}{a}$
$b^{-1}\Delta$	1	$\frac{1}{b^2}$	$\frac{c}{ab}$

$\mathcal{Z}_c C$

	cZ	$c^{-1}X$	$\frac{a}{b}\, ny$
$ab\, I$	abc	$\frac{ab}{c}$	a^2
$\frac{a}{b}\, I$	$\frac{ac}{b}$	$\frac{a}{bc}$	$\frac{a^2}{b^2}$
$\frac{b}{a}\, I$	$\frac{bc}{a}$	$\frac{b}{ac}$	1
$\frac{1}{ab}\, I$	$\frac{c}{ab}$	$\frac{1}{abc}$	$\frac{1}{b^2}$
$c\Lambda$	c^2	1	$\frac{ac}{b}$
$c^{-1}\Lambda$	1	$\frac{1}{c^2}$	$\frac{a}{bc}$

a^2

b^2

c^2

$\frac{a^2}{b^2}$

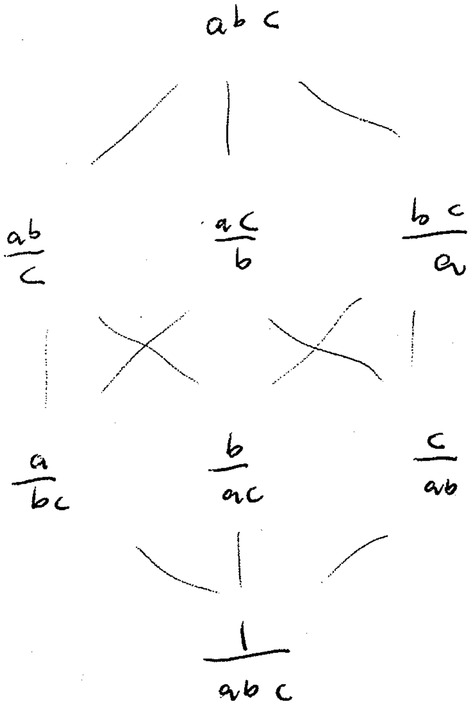
$\frac{b^2}{c^2}$

$\frac{c^2}{a^2}$

$\frac{1}{a^2}$

$\frac{1}{b^2}$

$\frac{1}{c^2}$



1

abc term

$-\Lambda =$

q	XYZ
q^2	0
q^{-2}	0
q^2	0
$-q$	X
$-q^{-1}$	Y
$-q$	Z

$$\Lambda = qX + q^{-1}Y + qZ - qXYZ$$

$\frac{1}{a^2}$ term

$-1 =$

q	$\underbrace{Y n_x X} \quad = \quad q^{-2} n_x Y X$
q^2	Y^2
q^{-2}	0
q^2	0
$-q$	ΔY
$-q^{-1}$	n_x
$-q$	0

$$-1 = q^{-1} n_x (YX - 1) + q^2 Y^2 - q \Delta Y$$

$$- n_x n_z$$

$$n_x n_z = 1 - q \Delta Y + q^2 Y^2$$

ok

$\frac{b}{c^2}$ term

$0 =$

q	$n_z YX$
q^2	n_z^2
q^{-2}	0
q^2	0
$-q$	n_z
$-q^{-1}$	0
$-q$	0

$$0 = q n_z \left(\underbrace{yX + q n_z}_{-q n_z} - 1 \right) \quad \checkmark$$

ok

$\frac{a^2}{b^2}$ term

$0 =$

q	$xz\, n_y$
q^2	0
q^{-2}	0
q^2	n_y^2
$-q$	0
$-q^{1}$	0
$-q$	n_y

$$(q\,xz + q^2 n_y - q\, r n_y = 0$$

$$q\left(\underbrace{xz - i}_{-q n_y} + n_y q\right) n_y = 0 \qquad \text{ok}$$

c^2 term

$-1 =$

q	$x\, n_x z \quad (= x z\, n_x\, q^{-2})$
q^2	0
q^{-2}	0
q^2	Z^2
$-q$	0
$-q^{-1}$	n_x
$-q$	Λz

$$-1 = \underbrace{q^{-1}(xz-1)}_{-n_y}\, n_x \quad - q\Lambda z \quad + q^2 Z^2$$

$$n_y n_x = 1 - q\Lambda z + q^2 Z^2$$ ✓

a^2 term :

$-1 =$

q	$x\cancel{y} \cdot ny$
q^2	X^2
q^{-2}	0
q^2	0
$-q$	$x\Delta$
$-q^{1}$	0
$-q$	ny

$$-1 = \underbrace{q(xy-1)}_{-nz} \; ny \quad + \quad q^2X^2 - qx\Delta$$

$$nzny = 1 - qX\Delta + q^2X^2$$ *

b^2 term

$-1 =$

q	$n_z Y Z$
q^2	0
q^{-2}	Y^2
q^2	0
$-q$	n_z
$-q^4$	ΔY
$-q$	0

$$-1 = \underbrace{q\, n_z (YZ - I)}_{-n_z n_x} + q^{-2} Y^2 - q^4 \Delta Y$$

$$n_z n_x = 1 - q^4 \Delta Y + q^{-2} Y^2$$

$$\rightarrow n_y n_z = 1 - q^4 \Delta X + q^{-2} X^2 \qquad *$$

$\frac{1}{abc}$ term

$-\Delta =$

q	YZX
q^2	0
q^{-2}	0
q^2	0
$-q$	Y
$-q^{1}$	Z
$-q$	X

ok.

$\frac{ab}{c}$ term

$\Lambda \left(q n_z - 1 \right)$ "v"

$-\Delta y x$

$-\Lambda =$

q	$XYX + n_z Y n_y$
q^2	$X n_z + n_z X$
q^{-2}	0
q^2	0
$-q$	$X + + \Delta n_z$
$-q^{-1}$	Y
$-q$	X

$\Lambda = qZ + q^{-1}X + n_y Y$ ~~X~~

$-n_z Z \; n_z X$

$-qX \quad -q^{-1}Y \quad -qZ \quad +qXYZ$

$qX + q^{-1}Y + qX \quad -\Lambda \quad -qXYX$

$-q(1-XY)Z$

$q(1-XY)X$

$n_z X$

$= q\, n_z Y n_y + q^2 n_z X + q^2 X n_z$

$q^{-1} y\, n_z n_y \qquad -q \Delta n_z$

$q^2 n_z Z$

$q n_z \left| \left(q - q^{-1} \right) \left(Z - X \right) + n_y y \right)$

$n_z X$

$q^{-1} y \left(1 - q \Delta X + q^2 X^2 \right) \qquad q n_z Y n_y$

$\left(q^2 - 1 \right) n_z Z$

$\frac{ac}{b}$ term

$-\Lambda =$

q	xzz	$+ x n_x n_y$
q^2	0	
q^{-2}	0	
q^2	$z n_y$	$+ n_y z$
$-q$	X	
$-q^{-1}$	Z	
$-q$	Z	$+ \Delta n_y$

$q^2 n_y z$

$q(1-xz)z$

$qxzz$

$qX + q^{-1}Z \quad + qZ \quad -\Lambda = qXZ^3 + qX n_x n_y$

$+ q^2 z n_y + q^2 n_y z$

$-qZ - q^{-1}X - qY$

$-q\Lambda n_y$

$-q^{-1}Z \; -qX \; -q^{-1}Y$

$+q^{-1}YXZ$

$-y \overset{?}{=} qxn_x + q^2 z - q\Lambda$

$\Lambda = q^{-1}y + qz + xn_x$

$-q^{-1}y(1-xz))$

$q^{-1}Y(XZ-$

$(q+q^{-1})^2 - \Lambda^2 - 2 - 2 - 2$

$q^2 + 2 + q^{-2}$

---

q	$n_z z^2$	$x n_x x$	$y^2 n_y$	$n_z n_x n_y$
q^2	$xy + yx$			
q^{-2}	$yz + zy$	$-q^{-1}(-q^{-1} zy + q yz)$		
q^2	$zx + xz$	$q(qzx - q^{-1}xz)$		
$-q$	n_z	$+\Lambda y$	$+\Lambda x$	
$-q^{-1}$	n_x	$+\Lambda z$	$+\Lambda y$	
$-q$	n_y	$+\Lambda x$	$+\Lambda z$	

$-q^{-1}(q-q^{-1})$ $\quad q^{-2}-1$

$q(\Lambda - qx - q^{-1}y)z$ $\quad q\Lambda z - q^2 xz - yz$

$qx\Lambda - q^2 xy - xz - xz$

$-1 \; q n_y$

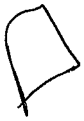
$qx(\Lambda - qY - q^{-1}z)$ $\quad qY\Lambda - Yz - q^2 Yx$

$-1 \; q^{-1} n_x$

$qY(\Lambda - q^{-1}z - qx)$

$q\Lambda z - \Lambda n_z z$

$q n_z(1 - q^{-1}\Lambda z + q^{-2} z^2)$ $\quad + q^{-1} n_z z^2$



Lec 39 Monday Dec 13

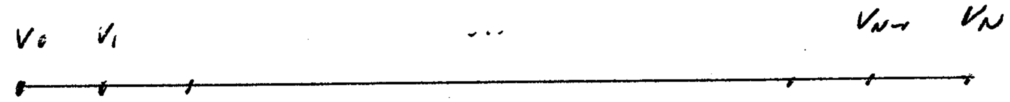
Recall the $U_q(sl_2)$-module $V = L(N, 1)$

On V each of X, Y, Z is mult free, with eigvals

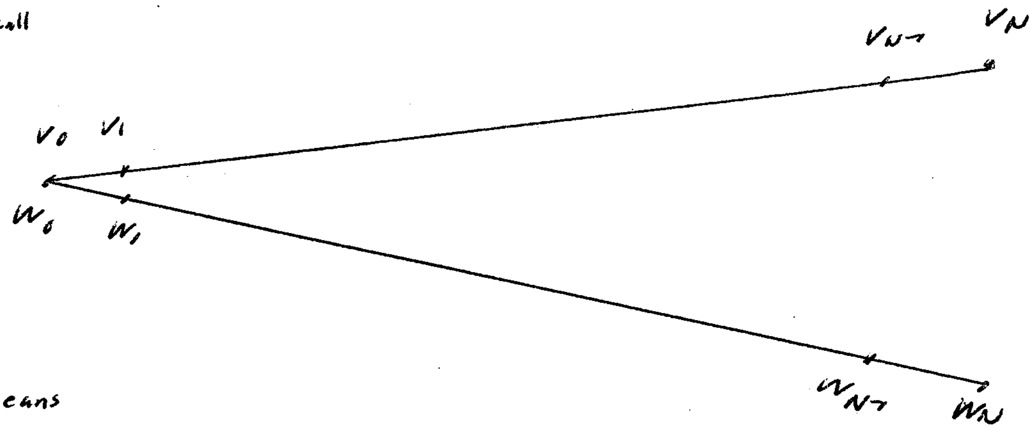
$\{ q^{N-2i} \}_{i=0}^{N}$

Next goal: how are eigenspace decomps of X, Y, Z related?

Recall notation for a decomp $\{V_i\}_{i=0}^{N}$ of V:



Recall

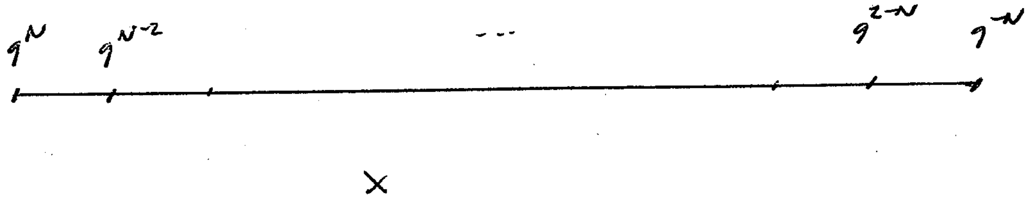


means

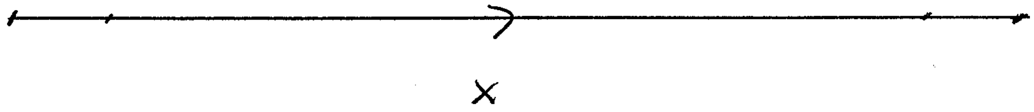
$$V_0+V_1+\cdots+V_i = W_0+W_1+\cdots+W_i \qquad 0 \le i \le N$$

To display the eigenspace decomps of X, Y, Z we label the line segment with the eigenvalues instead of eigenspaces.

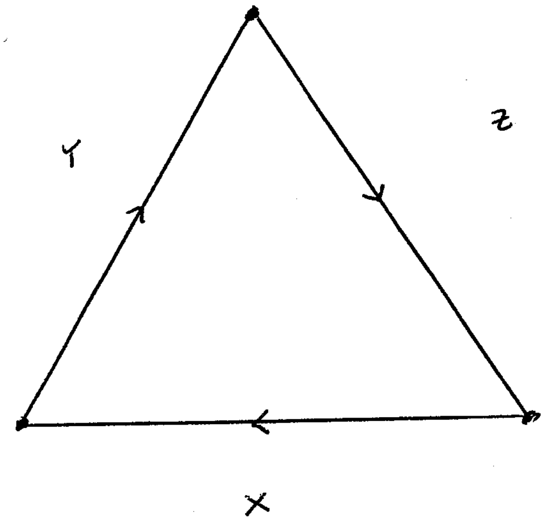
For instance the eigenspace decomp of X is denoted



or just



LEM 298 On $V = L(N,1)$ the eigenspace decomps of X, Y, Z are related as follows:



pf. Focus on $X-Y$ corner (other corners sim)

For $0 \le i \le N$ let

$U_i =$ eigenspace for X with eigval q^{N-2i}

$V_i =$ -- Y --

Show

$$V_0 + V_1 + \cdots + V_i = U_N + U_{N-1} + \cdots + U_{N-i} \qquad *$$

Recall the basis $\{u_i\}_{i=0}^{N}$ for V from L 292 rel $\{u_i\}_{i=0}^{N}$

$X: \quad \mathrm{diag}(q^N, q^{N-2}, \ldots, q^{-N})$

$$Y: \quad \begin{pmatrix} q^{-N} & & & 0 \\ & q^{2-N} & & \\ 0 & & \ddots & \\ & & & q^N \end{pmatrix} \qquad \text{Lower BD}$$

$$Z: \quad \begin{pmatrix} q^{-N} & & & 0 \\ & q^{2-N} & & \\ 0 & & \ddots & \\ & & & q^N \end{pmatrix} \qquad \text{upper BD}$$

Obs

u_i is basis for $\mathcal{U}_i$ $\quad 0 \le i \le N$

rel $\{u_i\}_{i=0}^{N}$ the matrix rep Y is lower triangular with (i,i)-entry q^{2i-N} for $0 \le i \le N$

therefore, for $0 \le i \le N$

$$U_N + U_{N-1} + \cdots + U_{N-i} \qquad **$$

is Y-inv, and restriction of Y to $**$ has eigvals

$$q^N, q^{N-2}, \ldots, q^{N-2i}$$

Line $*$ follows. □

Back to our LS Φ on V from th 276

In th 296 we obtained a $U_q(sl_2)$-module structure on V s.t. on V,

$$A = aX + a^{-1}Y + {}^{b}/_{c}\ \frac{XY-YX}{q-q^{-1}}$$

$$B = bY + b^{-1}Z + {}^{c}/_{a}\ \frac{YZ-ZY}{q-q^{-1}}$$

$$C = cZ + c^{-1}X + {}^{a}/_{b}\ \frac{ZX-XZ}{q-q^{-1}}$$

this $U_q(sl_2)$-module is iso $L(N,1)$.

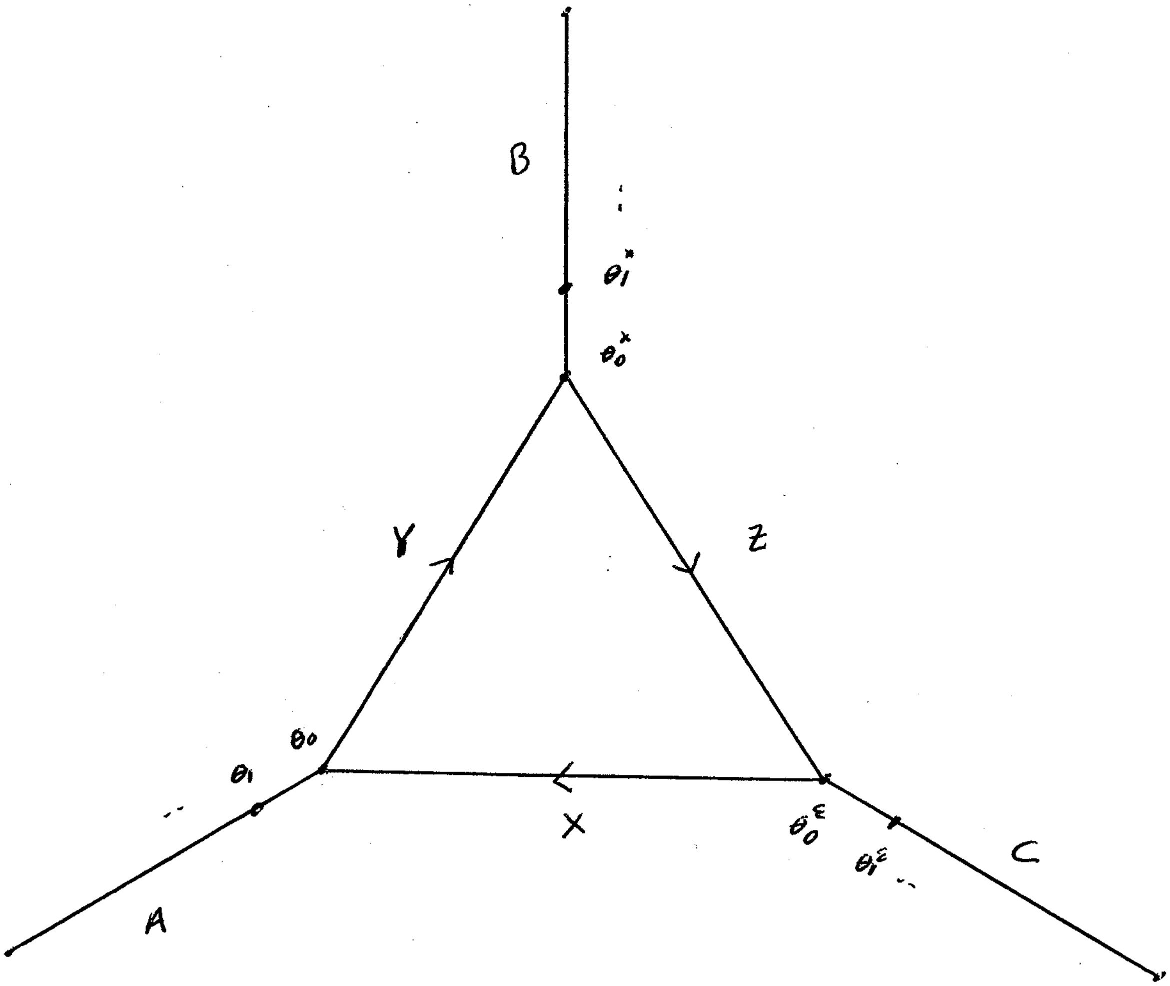
Next goal: how are eigenspace decomps of X, Y, Z related to those of A, B, C

[Lets assume C is MF so that we may speak of its eigenspace decomp]
see th 277

Recall eigvals

element	eigenvalues	
A	$\theta_i = aq^{2i-N} + a^{-1}q^{N-2i}$	$0 \le i \le N$
B	$\theta_i^* = bq^{2i-N} + b^{-1}q^{N-2i}$	..
C	$\theta_i^\varepsilon = cq^{2i-N} + c^{-1}q^{N-2i}$	..

thm 29 with the above notation / assumptions



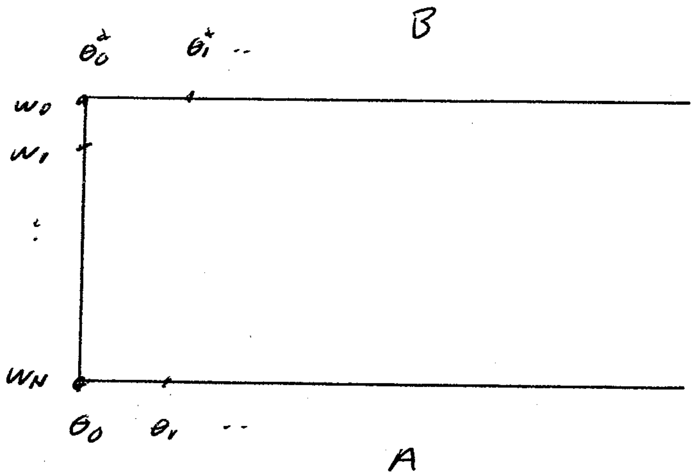
pf Focus on Y; case of Z, X are similar.

$\exists$ a basis $\{w_i\}_{i=0}^N$ for V

wrt which

$$A: \begin{pmatrix} \theta_N & & & 0 \\ 1 & \theta_{N-1} & & \\ & 1 & \ddots & \\ 0 & & \ddots & \\ & & 1 & \theta_0 \end{pmatrix} \qquad B: \begin{pmatrix} \theta_0^* & \varphi_1 & & 0 \\ & \theta_1^* & \ddots & \\ & & \ddots & \varphi_N \\ 0 & & & \theta_N^* \end{pmatrix}$$

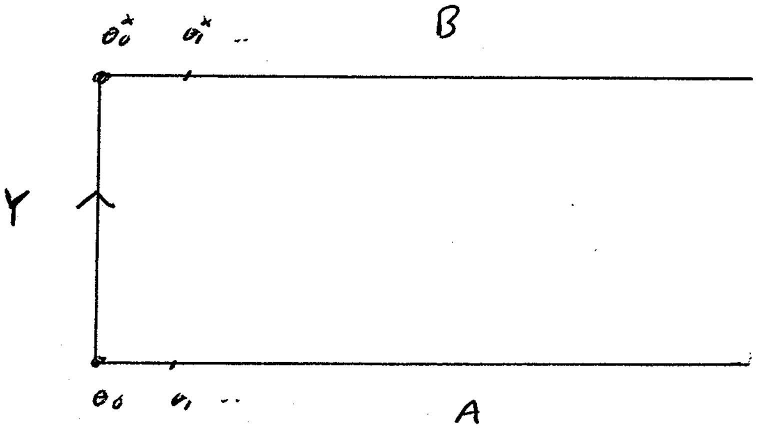
this basis is the following split basis for LP A, B:



[illegible] we saw

$$(Y - q^{2i-N} I)\, w_i = 0 \qquad 0 \le i \le N$$

Therefore



$\square$

Recall the element C' from L278

Next goal:
We expand picture of Th 299 to include C'

LEM 300 Given $0 \neq t \in \mathbb{F}$ $\exists$ automorphism $\sigma = \sigma(t)$ of $U_q sl_2$ that sends

$$x \longrightarrow x^{-1}$$

$$y \longrightarrow x + t \frac{zx - xz}{q - q^{-1}} \quad (= n_y)$$

$$z \longrightarrow x + t^{-1} \frac{xy - yx}{q - q^{-1}} \quad (= n_z)$$

Moreover σ has order 2

pf Check σ respects the defining rels for $U_q sl_2$:

$$\frac{q xy - q^{-1} yx}{q - q^{-1}} = 1$$

$$\frac{q x^{-1}(x + t n_y) - q^{-1}(x + t n_y) x^{-1}}{q - q^{-1}} \overset{?}{=} 1$$

Require $q x^{-1} n_y \overset{?}{=} q^{-1} n_y x^{-1}$

OK

$$\frac{q\, y z - q^{-1} z y}{q - q^{-1}} = 1$$

$$\frac{q\left(x + t n_y \right)\left(x + t^{-1} n_z \right) - q^{-1}\left(x + t^{-1} n_z \right)\left(x + t n_y \right)}{q - q^{-1}} \overset{?}{=} 1$$

LHS − RHS =

$$x^2 - 1 + \frac{q n_y n_z - q^{-1} n_z n_y}{q - q^{-1}}$$

$$+ \quad t \, \frac{q n_y x - q^{-1} x n_y}{q - q^{-1}}$$

$$+ \quad t^{-1} \, \frac{q x n_z - q^{-1} n_z x}{q - q^{-1}}$$

t-coef is 0 ✓

t^{-1}-coef .. ✓

constant term ..✓

$$\frac{q z x - q^{-1} x z}{q - q^{-1}} = 1$$

$$\frac{q\left(x + t^{-1} n_z \right) x^{-1} - q^{-1} x^{-1} \left(x + t^{-1} n_z \right)}{q - q^{-1}} \overset{?}{=} 1$$

Require

$$q\, n_z\, x^{-1} \overset{?}{=} q^{-1} x^{-1} n_z \qquad \text{ok}$$

To show σ is bij show σ^{-1} exists. Show $\sigma^{-1} = \sigma$ ie $\sigma^2 \overset{?}{=} 1$

$$x \xrightarrow{\sigma} x^{-1} \xrightarrow{\sigma} x \quad ✓$$

$$y \xrightarrow{\sigma} x + t\,\frac{zx - xz}{q - q^{-1}} \xrightarrow{\sigma}$$

$$x^{-1} + t\,\frac{\left(x + t^{-1} n_z \right) x^{-1} - x^{-1}\left(x + t^{-1} n_z \right)}{q - q^{-1}} \overset{?}{=} y$$

Require

$$\frac{n_z x^{-1} - x^{-1} n_z}{q - q^{-1}} \overset{?}{=} y - x^{-1}$$

mult by x

$$\frac{n_z - \overbrace{x^{-1} n_z x}^{q^2 n_z}}{q - q^{-1}} \overset{?}{=} yx - 1$$

$$LHS = -q n_z$$

By def $n_z = q^{-1}(1 - yx)$ ✓

Sim

$$z \xrightarrow{\sigma} x + t^{-1}\,\frac{xy - yx}{q - q^{-1}} \xrightarrow{\sigma} z$$

$\square$

(Aside)

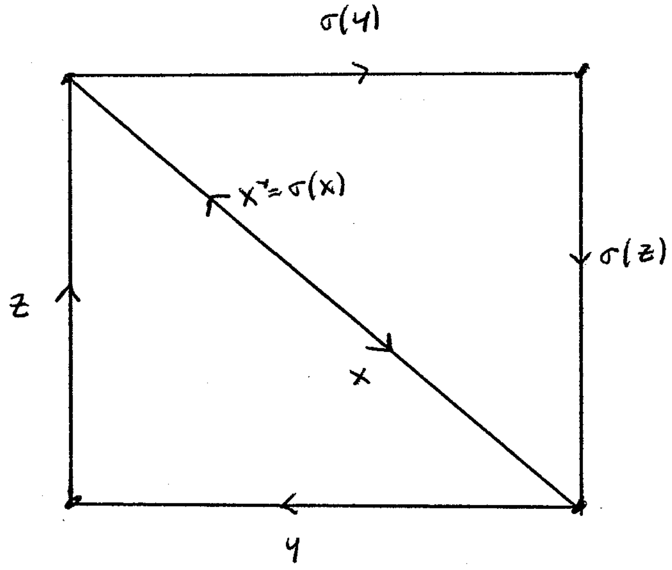
Problem Find an operator Δ that acts on f.d. $U_q sl_2$-modules, such that on each of these modules

$$\Delta^{-1} u \Delta = \sigma(u) \qquad \forall u \in U_q sl_2$$

Note Instead of working with σ directly, it might be easier to consider the composition of σ and $\rho \, n \, \rho^{-1}$, where

$$\rho : x \to y \to z \to x .$$

LEM 301 We have



where

means

$$\frac{q\,rs - q^{-1}sr}{q-q^{-1}} = I$$

pf Require

(i) $$\frac{q\,z\,\sigma(y) - q^{-1}\sigma(y)z}{q-q^{-1}} \overset{?}{=} 1$$

(ii) $$\frac{q\,\sigma(z)y - q^{-1}y\,\sigma(z)}{q-q^{-1}} \overset{?}{=} 1$$

(i)

$$\frac{qz(x+tn_y) - q^{-1}(x+tn_y)z}{q-q^{-1}} \overset{?}{=} 1$$

LHS−RHS =

$$\frac{qzx - q^{-1}xz}{q-q^{-1}} - 1 \qquad (=0)$$

$$+\ t\,\frac{qzn_y - q^{-1}n_yz}{q-q^{-1}} \qquad (=0)$$

$$= 0$$

(ii) Sim

$$\frac{q(x+t^{-1}n_z)y - q^{-1}y(x+t^{-1}n_z)}{q-q^{-1}} \overset{?}{=} 1$$

LHS−RHS =

$$\frac{qxy - q^{-1}yx}{q-q^{-1}} - 1$$

$$+\ t^{-1}\,\frac{qn_zy - q^{-1}yn_z}{q-q^{-1}}$$

$$= 0+0$$

$$= 0$$

□

Recall the q-Serre rels

$$u^3 v - [3]_q u^2 v u + [3]_q u v u^2 - v u^3 = 0$$
$$v^3 u - [3]_q v^2 u v + [3]_q v u v^2 - u v^3 = 0$$

$$[3]_q = \frac{q^3 - q^{-3}}{q - q^{-1}} = 1 + q^2 + q^{-2}$$

LEM 302 In $U_q(\mathfrak{sl}_2)$

(i) the pair $y, \sigma(y)$ sat q-Serre rels

(ii) -- $z, \sigma(z)$ ...

pf (i) show

$$y^3 \sigma(y) - [3]_q y^2 \sigma(y) y + [3]_q y \sigma(y) y^2 - \sigma(y) y^3 = 0$$

$\sigma(y) = x + t n_y$

show

$$y^3 x - [3]_q y^2 x y + [3]_q y x y^2 - x y^3 = 0 \qquad *$$
$$y^3 n_y - [3]_q y^2 n_y y + [3]_q y n_y y^2 - n_y y^3 = 0 \qquad **$$

*: LHS = $\left[y, \underbrace{\left[y, \underbrace{[y, x]_{q^{-1}}}_{(q^{-1}-q)1}\right]}_{0}\right]_q$

$= 0$

**: LHS = $\left[y, \left[y, \underbrace{[y, n_y]_q}_{(q-q^{-1})\Lambda - (q^2 - q^{-2})x}\right]\right]_{q^{-1}}$

$$\Big[y, \underbrace{[y, \Delta]}_{0} \Big]_{q^{-1}} = 0$$

$$\Big[y, [y, x] \Big]_{q^{-1}} = \Big[y, \underbrace{[y, x]_{q^{-1}}}_{\substack{\shortparallel \\ (q^{-1}-q) I}} \Big]$$

$$= 0$$

We have shown $y, \sigma(y)$ sat the 1st q-Serre relation. To verify the 2nd q-Serre rel apply σ to $y, \sigma(y)$ and recall $\sigma^2 = 1$

□

(ii) Sim.

LEM. 303 Ref to L295 assume t not among

$$q^{N-1}, q^{N-3}, \ldots, q^{3-N}, q^{1-N}$$

then the pair $y, \sigma(y)$ is a LP on $V = L(N,1)$

An eigval sequence is $\{q^{2i-N}\}_{i=0}^{N}$

A dual eigval sequence is $\{q^{N-2i}\}_{i=0}^{N}$

the corresp 1st split sequence is

$$t(q^{i}-q^{-i})(q^{N-i+1}-q^{i-N-1}) \qquad 1 \leq i \leq N$$

the corresp 2nd split sequence is

$$(t-q^{N-2i+1})(q^{i}-q^{-i})(q^{N-i+1}-q^{i-N-1}) \qquad 1 \leq i \leq N$$

pf Note above data gives a PA over $\mathbb{F}$, So it corresponds to some LS on V which we denote by Φ'

For $0 \leq i \leq N$ let U_i = eigenspace of X on V for eigval q^{N-2i}

By constr

$$(y - q^{2i-N} I) U_i \subseteq U_{i+1} \qquad 0 \leq i \leq N, \quad U_{N+1} = 0$$

$$(\sigma(y) - q^{N-2i} I) U_i \subseteq U_{i-1} \qquad 0 \leq i \leq N, \quad U_{-1} = 0$$

Also for $1 \leq i \leq N$ U_i is inv under

$$(y - q^{2i-2-N} I)(\sigma(y) - q^{N-2i} I)$$

corresp eigval is

$$t(q^{i}-q^{-i})(q^{N-i+1}-q^{i-N-1})$$

Therefore we may identify $A', A^{*\prime}$ with $y, \sigma(y)$ resp.

the result follows. $\square$

result holds for $Z, \sigma(Z)$

[Last class day]

Lec 40 Wed Dec 15

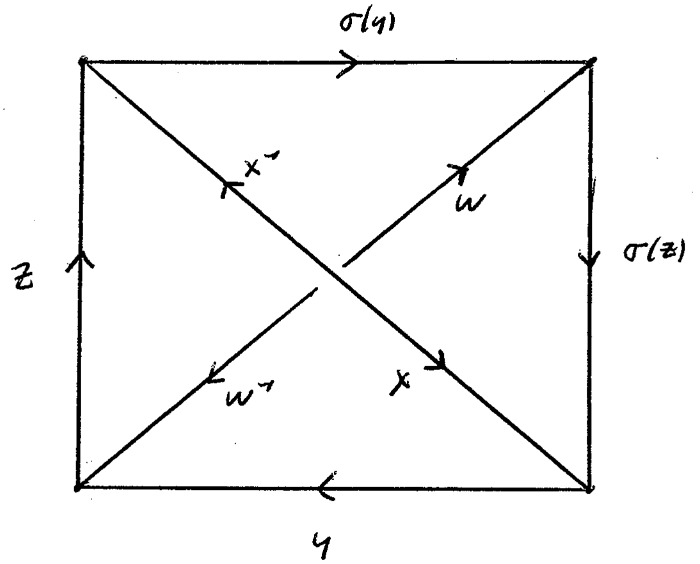
LEM 304 Ref to L300: assume t not among

$$q^{N-1}, q^{N-3}, \dots, q^{1-N}$$

and consider $V = L(N,1)$. Then $\exists$ $W \in \mathrm{End} V$ s.t.

(i) W is mult free with eigvals $\{q^{N-2i}\}_{i=0}^{N}$

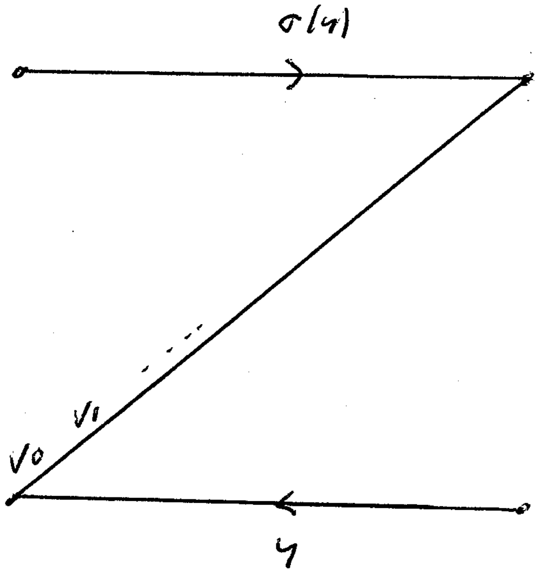
(ii) We have



$*$

where the diagram is interpreted either as above L298 or as in L301.

pf For the LP $y, \sigma(y)$ all four split decomps exist. Consider the split decomp $\{V_i\}_{i=0}^{N}$ as shown below



Define $W \in \mathrm{End} V$ s.t. for $0 \leq i \leq N$ V_i is eigspace for W
with eigval q^{N-2i}. Then the diagram $*$
holds as interpreted above L 298. We now show
Need to show that on V,
$*$ holds as interpreted in L 301

$$\frac{qyW - q^{-1}Wy}{q - q^{-1}} = 1 \qquad **$$

$$\frac{qW^{-1}z - q^{-1}zW^{-1}}{q - q^{-1}} = 1$$

$$\frac{qW\sigma(z) - q^{-1}\sigma(z)W}{q - q^{-1}} = 1$$

$$\frac{q\sigma(y)W^{-1} - q^{-1}W^{-1}\sigma(y)}{q - q^{-1}} = 1$$

We check $\ast\ast$; the other 3 are similar.

Since W comes from a split decomp for $y, \sigma(y)$

For $0 \leq i \leq N$

$$(y - q^{N-2i} I) V_i \subseteq V_{i-1}$$

Also by constr

$$(W - q^{N-2i} I) V_i = 0$$

From these facts we routinely obtain $\ast\ast$.

$\square$

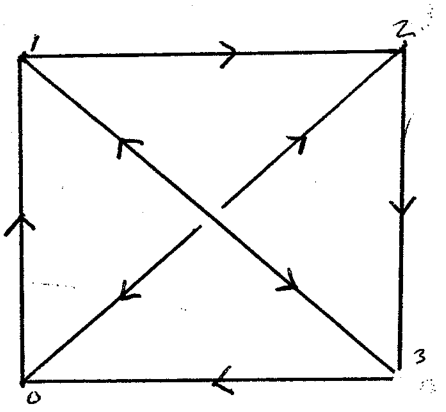
12/15/10

Write $Z_4 = \mathbb{Z}/4\mathbb{Z}$ for cyclic gp order 4.

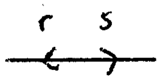
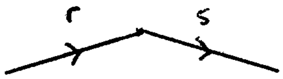
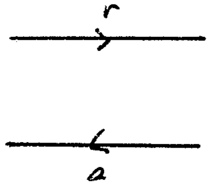
Def 3.05 Let $\boxtimes_q$ denote the (assoc) $\mathbb{F}$-algebra with 1 defined by gens

$$\{ x_{ij} \mid i,j \in \mathbb{Z}_4,\ j-i=1 \text{ or } j-i=2 \}$$

and rels



$i \longrightarrow j$ represents x_{ij}

Notation	Meaning
	$rs = sr = 1$
	$\frac{qrs - q^{-1}sr}{q-q^{-1}} = 1$
	r, s satisfy the q-Serre rels

LEM 306 Ref to Lem 300. assume t not among

$q^{N-1}, q^{N-3}, \ldots, q^{1-N}$

Consider $V = L(N, 1)$. Then $\exists$ $\boxtimes_q$-mod structure on V s.t.

gen	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}	x_{20}	x_{31}
action on V	Z	$\sigma(Y)$	$\sigma(Z)$	Y	W	X	W^{-1}	X^{-1}

This $\boxtimes_q$-mod str is uned.

pf By L304 and Def 305 $\square$

12/15/10

Back to our LS Φ on V from p. 276

thm 3.07 $\exists$ $\boxtimes_q$-module str on V s.t. on V

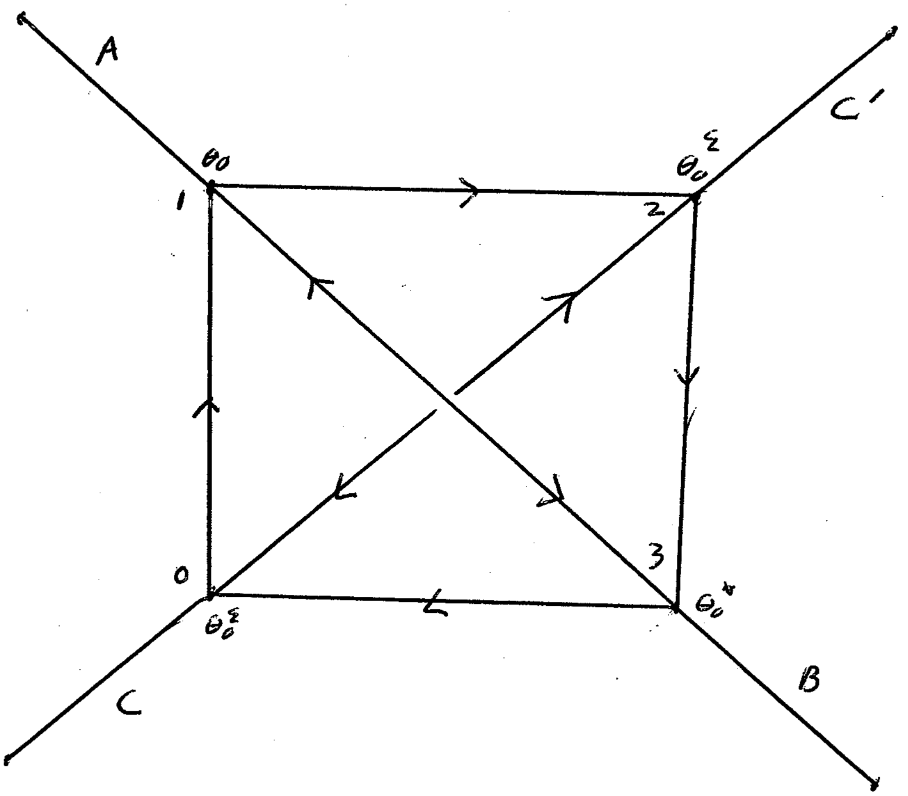
$$A = a X_{01} + a^{-1} X_{12}$$

$$B = b X_{23} + b^{-1} X_{30}$$

$$C = c X_{30} + c^{-1} X_{01} + a/b \frac{X_{30} X_{01} - X_{01} X_{30}}{q - q^{-1}}$$

$$C' = c X_{12} + c^{-1} X_{23} + b/a \frac{X_{12} X_{23} - X_{23} X_{12}}{q - q^{-1}}$$

This $\boxtimes_q$-module is irred. Moreover



pf (sketch) We saw ∃ $U_q sl_2$ module str on V s.t on V

$$A = aX + a^{-1}Y + b/c\, n_Z$$

$$B = bY + b^{-1}Z + c/a\, n_X$$

$$C = cZ + c^{-1}X + a/b\, n_Y$$

For not. convenience we adjust $X \to Z \to Y \to X$

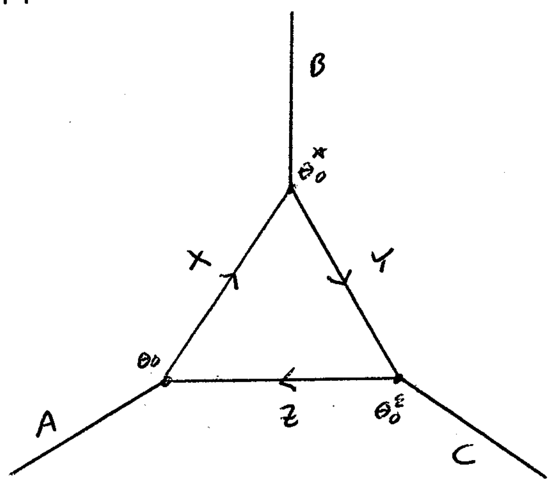
So ∃ $U_q sl_2$ module str on V s.t on V

$$A = aZ + a^{-1}X + b/c\, n_Y$$

$$B = bX + b^{-1}Y + c/a\, n_Z$$

$$C = cY + c^{-1}Z + a/b\, n_X$$

By th 299



claim $\quad \frac{ab}{c}$ Not among

$$q^{N-1}, q^{N-3}, \dots, q^{1-N}$$

pf cl $\quad$ Consider 1st split sequence $\{\varphi_i\}_{i=1}^{N}$ of Φ

We saw earlier φ_i has factor

$$q^{-i} - \frac{ab}{c} q^{i-N-1} \qquad *$$

so $*$ is nonzero for $1 \le i \le N$. $\quad$ claim follows.

Recall aut $\sigma = \sigma(t)$ of $U_q sl_2$ from:

take $t = \frac{ab}{c}$. $\quad$ So

$$\sigma: \quad \begin{aligned} x &\longrightarrow x^{-1} \\ y &\longrightarrow x + \frac{ab}{c} n_y \\ z &\longrightarrow x + \frac{c}{ab} n_z \end{aligned}$$

Consider $\boxtimes_q$-module str on V from L 306. We show this str satisfies the requirements of the theorem.

12/15/10

9

show

$$A \overset{?}{=} aX_{01} + a^{-1}X_{12} \quad (\text{on } V)$$

$$aX_{01} = aZ \qquad a^{-1}X_{12} = a^{-1}\sigma(Y) = a^{-1}\left(X + \frac{ab}{c} n_y\right)$$

$$A = aZ + a^{-1}X + \frac{b}{c} n_y$$

ok

show

$$B \overset{?}{=} b X_{23} + b^{-1} X_{30} \quad (\text{on } V)$$

$$bX_{23} = b\sigma(Z) = b\left(X + \frac{c}{ab} n_Z\right) \qquad b^{-1}X_{30} = b^{-1}Y$$

$$B = bX + b^{-1}Y + c/a \, n_Z$$

ok

show

$$C = cX_{30} + c^{-1}X_{01} + a/b \, \frac{X_{30}X_{01} - X_{01}X_{30}}{q - q^{-1}}$$

$$cX_{30} = cY \qquad c^{-1}X_{01} = c^{-1}Z \qquad a/b \, \frac{X_{30}X_{01} - X_{01}X_{30}}{q - q^{-1}} = \frac{a}{b}\,\frac{YZ - ZY}{q - q^{-1}} = \frac{a}{b} n_x$$

ok

12/18/10

Show

(on V)

$$\begin{array}{ccccccc} C' & = & c\,X_{12} & + & c^{-1}\,X_{23} & + & b/a\,\dfrac{X_{12}X_{23}-X_{23}X_{12}}{q-q^{-1}} \\ & & \parallel & & \parallel & & \parallel \\ & & c\,\sigma(y) & & c^{-1}\sigma(z) & & b/a\,\dfrac{\sigma(y)\sigma(z)-\sigma(z)\sigma(y)}{q-q^{-1}} \end{array}$$

To verify write each side in terms of X, Y, Z

Recall

$$C'-C = \frac{AB-BA}{q-q^{-1}}$$

A, B, C given earlier in terms of X, Y, Z

Also $\sigma(y)$, $\sigma(z)$ given in terms of X, Y, Z using def of σ.

One checks the two reductions give same ans.

Cor 308 Given a Leonard pair A, B on V of q-Racah type, with an eigval seq

$$a q^{2i-N} + a^{-1} q^{N-2i} \qquad 0 \le i \le N$$

and a dual eigval seq

$$b q^{2i-N} + b^{-1} q^{N-2i} \qquad 0 \le i \le N$$

then $\exists$ $\boxtimes_q$-module str on V s.t.

$$A = a x_{01} + a^{-1} x_{12} \qquad (\text{on } V)$$

$$B = b x_{23} + b^{-1} x_{30}$$

this $\boxtimes_q$-mod str is irred.

□

Pf By thm 307

THE END ♡