Lec 32 Wednesday Nov 24

Terminating Branch of Askey Scheme: treatment using Leonard systems

We classified the LS in thm 222.

We now relate LS to the polynomials from terminating branch of the Askey scheme.

Let $\Phi = (A, E_i, A^*, E_i^*)$ denote LS on V

PA $(\theta_i, \theta_i^*, \varphi_i, \phi_i)$

In L193 we saw some polys $\{p_i\}_{i=0}^N$ in $\mathbb{F}[x]$

s.t

$$E_i^* V = p_i(A) E_0^* V \qquad 0 \le i \le N$$

For $0 \le i \le N$ p_i defined up to nonzero scalar mult

WLOG p_i monic

Def p_{N+1} = (monic) min poly of A

It is conv to work with a certain normalization of p_i called u_i, v_i

The standard basis

Recall from L193 that

$$E_0^* \text{ is normalizing}$$

Sim

$$E_0 \text{ is normalizing}$$

So for $0 \le i \le N$

$$E_i^* E_0 \neq 0$$

So

$$E_i^* E_0 V \neq 0$$

So

$$E_i^* E_0 V = E_i^* V$$

This gives:

LEM 234 Given $0 \neq u \in E_0 V$. For $0 \le i \le N$

$E_i^* u$ is basis for $E_i^* V$

Moreover $\{E_i^* u\}_{i=0}^N$ is basis for V. "Φ-standard basis" for V

$\square$

pf ✓

LEM 235 Given vectors $\{w_i\}_{i=0}^N$ in V, not all 0.

then $\{w_i\}_{i=0}^N$ is a Φ-stand basis for V iff both

(i) $w_i \in E_i^* V \quad 0 \le i \le N$

(ii) $\sum_{i=0}^N w_i \in E_0 V$

pf ex

LEM 236 Given basis $\{w_i\}_{i=0}^N$ for V

let B = mat in $\mathrm{Mat}_{N+1}(\mathbb{F})$ that reps A rel $\{w_i\}_{i=0}^N$

B^* = ...

then $\{w_i\}_{i=0}^N$ is Φ-st basis iff both

(i) B has const row sum θ_0

(ii) $B^* = \mathrm{diag}(\theta_i^*)_{i=0}^N$

pf ex

Def 237 $\forall X \in \operatorname{End} V$ let $X^\flat$ denote matrix in $\operatorname{Mat}_{N+1}(\mathbb{F})$ that reps X rel E-stand basis for V

Obs

$$\flat : \begin{array}{ccc} \operatorname{End} V & \to & \operatorname{Mat}_{N+1}(\mathbb{F}) \\ X & \to & X^\flat \end{array}$$

is iso of $\mathbb{F}$-algebras.

— · —

By constr and L 236

$$A^\flat = \text{irred tridiag with const row sum } \theta_0$$

$$A^{*\flat} = \operatorname{diag}(\theta_i^*)_{i=0}^{N}$$

$$E_i^{*\flat} = \begin{array}{c c} & i \\ i & \begin{pmatrix} 0 \ddots & & 0 \\ & 1 & \\ 0 & & 0 \ddots \end{pmatrix} \end{array} \qquad 0 \le i \le N$$

Write

$$A^\flat = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & c_N & a_N \end{pmatrix}$$

Notation $c_i, b_i \in \mathbb{F}$

For $0 \le i \le N$

$$a_i := \operatorname{tr} A E_i^*$$

is same as Def 184

Obs

$$c_i + a_i + b_i = \theta_0 \qquad 0 \le i \le N \qquad *$$

$$c_0 = 0, \quad b_N = 0$$

Def $\{v_i\}_{i=0}^N$ in $\mathbb{F}[x]$ by

$$x v_i = b_{i-1} v_{i-1} + a_i v_i + c_{i+1} v_{i+1} \qquad 0 \le i \le N-1$$

$$v_0 = 1, \qquad v_{-1} = 0$$

For $0 \le i \le N$

v_i has deg i

leading coeff is $\dfrac{1}{c_1 \cdots c_i}$

Let $\{w_i\}_{i=0}^N$ denote $\mathbb{F}$-st basis for V

By constr

$$v_i(A) w_0 = w_i \qquad 0 \le i \le N$$

So

$$v_i \in \mathbb{F} \, p_i \qquad 0 \le i \le N$$

Comparing leading coeffs

$$v_i = \frac{p_i}{c_1 c_2 \cdots c_i} \qquad 0 \le i \le N$$

Def $\{u_i\}_{i=0}^N$ in $\mathbb{F}[x]$ by

$$x u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad 0 \le i \le N-1$$

$$u_0 = 1, \qquad u_{-1} = 0$$

For $0 \le i \le N$ $\quad u_i$ deg i

leading coef is $\dfrac{1}{b_0 b_1 \cdots b_{i-1}}$

One checks

$$v_i = u_i k_i \qquad 0 \le i \le N$$

where $\quad k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$

So $\quad u_i \in \mathbb{F} p_i \qquad 0 \le i \le N$

So $\quad u_i = \dfrac{p_i}{b_0 b_1 \cdots b_{i-1}} \qquad 0 \le i \le N$

Using the def of $\{u_i\}_{i=0}^{N}$ and $(*)$ get

$$u_i(\theta_0) = 1 \qquad 0 \le i \le N$$

So $\quad p_i(\theta_0) = b_0 b_1 \cdots b_{i-1} \qquad 0 \le i \le N$

thm 238 For $0 \le i \le N$

$$u_i = \sum_{h=0}^{i} \frac{(x-\theta_0)(x-\theta_1)\cdots(x-\theta_{h-1})(\theta_i^*-\theta_0^*)(\theta_i^*-\theta_1^*)\cdots(\theta_i^*-\theta_{h-1}^*)}{\varphi_1\varphi_2\cdots\varphi_h}$$

$$\text{''}\ \frac{\tau_h(x)\,\tau_h^*(\theta_i^*)}{\varphi_1\cdots\varphi_h}$$

pf Since u_i deg i

$\exists\ \alpha_0, \alpha_1, \ldots, \alpha_i \in \mathbb{F}$ s.t

$$u_i = \sum_{h=0}^{i} \alpha_h \tau_h \qquad *$$

Show

$$\alpha_h = \frac{\tau_h^*(\theta_i^*)}{\varphi_1\cdots\varphi_h} \qquad 0 \le h \le i$$

To do this, show

$$\alpha_0 = 1,$$

$$\alpha_h(\theta_i^* - \theta_h^*) = \alpha_{h+1}\varphi_{h+1} \qquad 0 \le h \le i-1 \qquad **$$

To get $\alpha_0 = 1$ apply $*$ to $x = \theta_0$

$$\begin{aligned} 1 &= u_i(\theta_0) \\ &= \sum_{h=0}^{i} \alpha_h \underbrace{\tau_h(\theta_0)}_{\begin{cases} 1 & \text{if } h=0 \\ 0 & \text{if } h\ne 0 \end{cases}} \\ &= \alpha_0 \end{aligned}$$

pf of $**$: Fix $0 \ne v \in E_0^* V$

Recall $u_i(A)v \in E_i^* V$

$E_i^* V$ is eigspace for A^* eigval θ_i^*

So

$$\left(A^{*}-\theta_i^{*} I\right) u_i(A) v = 0$$

We saw earlier that wrt basis

$$\{ T_j(A) v \}_{j=0}^{N}$$

matrix rep A^{*} is

$$\begin{pmatrix} \theta_0^{*} & \varphi_1 & & & 0 \\ & \theta_1^{*} & \varphi_2 & & \\ & & \ddots & \ddots & \\ & 0 & & & \varphi_N \\ & & & & \theta_N^{*} \end{pmatrix}$$

So

$$\begin{aligned} 0 &= \left(A^{*}-\theta_i^{*} I\right) u_i(A) v \\ &= \left(A^{*}-\theta_i^{*} I\right) \sum_{h=0}^{i} \alpha_h T_h(A) v \\ &= \sum_{h=0}^{i} \alpha_h \Big((\theta_h^{*}-\theta_i^{*}) T_h(A) v + \varphi_h T_{h-1}(A) v \Big) \\ &= \sum_{h=0}^{i-1} T_h(A) v \Big(\underbrace{\alpha_{h+1} (\varphi_{h+1}) - \alpha_h (\theta_i^{*}-\theta_h^{*})}_{\text{must be } 0} \Big) \end{aligned}$$

$\square$

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Lem 239 For $0 \le i \le N$

$$p_i(\theta_0) = \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\tau_i^*(\theta_i^*)}$$

pf We saw the leading coef of u_i is

$$\frac{1}{b_0 b_1 \cdots b_{i-1}} = \frac{1}{p_i(\theta_0)}$$

By th 238 the leading coef of u_i is

$$\frac{\tau_i^*(\theta_i^*)}{\varphi_1 \varphi_2 \cdots \varphi_i}$$

□

thm 240 For $0 \le i \le N$

$$p_i = \sum_{h=0}^{i} \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_1 \varphi_2 \cdots \varphi_h} \frac{\tau_h^*(\theta_i^*)}{\tau_i^*(\theta_i^*)} \tau_h$$

pf

$$p_i = \underset{\text{use L239}}{p_i(\theta_0)} \; \underset{\text{use th 238}}{u_i}$$

$\square$

thm 241 For $0 \le i \le N$

$$p_i = \sum_{h=0}^{i} \frac{\phi_1 \phi_2 \cdots \phi_i}{\phi_1 \phi_2 \cdots \phi_h} \frac{\tau_h^*(\theta_i^*)}{\tau_i^*(\theta_i^*)} \eta_h$$

pf Obs p_i is inv if we replace Φ by $\Phi^{\Downarrow}$:

By def p_i is monic deg i and

$$p_i(A) E_0^* V = E_i^* V$$

For $0 \le j \le N$ $E_j^*(\Phi^{\Downarrow}) = E_j^*$.

Now apply thm 240 to $\Phi^{\Downarrow}$ and use

$$\Phi \to \Phi^{\Downarrow}$$

$$\varphi_i \to \phi_i$$

$$\tau_i \to \eta_i$$

$$\tau_i^* \to \tau_i^*$$

$$\theta_i^* \to \theta_i^*$$

$\square$

thm 242 Fn $0 \le i \le N$

$$u_i = \frac{\phi_1 \cdots \phi_i}{\varphi_1 \cdots \varphi_i} \sum_{h=0}^{i} \frac{(x-\theta_N)(x-\theta_{N-1}) \cdots (x-\theta_{N-h+1})(\theta_i^*-\theta_0^*)(\theta_i^*-\theta_1^*) \cdots (\theta_i^*-\theta_{h-1}^*)}{\phi_1 \phi_2 \cdots \phi_h}$$

pf

$$u_i = \frac{p_i}{p_i(\theta_0)}$$

p_i ← use th 241

$p_i(\theta_0)$ ← use L239

$\square$

For our LS Φ we defined some polys $\{u_i\}_{i=0}^N$

let $\{u_i^*\}_{i=0}^N$ denote corresp polys for Φ^*

thm 243 For $0 \le i, j \le N$

$$u_i(\theta_j) = u_j^*(\theta_i^*)$$

"Askey-Wilson duality"

pf By lm 238

$$u_i(\theta_j) = \sum_{h=0}^{i} \frac{\tau_h(\theta_j)\,\tau_h^*(\theta_i^*)}{\varphi_1 \cdots \varphi_h} = \sum_{h=0}^{d} \frac{\tau_h(\theta_j)\,\tau_h^*(\theta_i^*)}{\varphi_1 \cdots \varphi_h}$$

Applying this to Φ^* get

$$u_j^*(\theta_i^*) = \sum_{h=0}^{d} \frac{\tau_h(\theta_j)\,\tau_h^*(\theta_i^*)}{\varphi_1^* \cdots \varphi_h^*}$$

But $\varphi_h^* = \varphi_h \qquad 1 \le h \le N$

by Prop 221.

Result follows.

$\square$

We now link the polys $\{u_i\}_{i=0}^N$ to the Askey scheme.

Start with sp case of Krawtchouk type.

LEM 244 Assume

$$\theta_i = i \qquad 0 \le i \le N$$

$$\theta_i^* = i$$

[so char $\mathbb{F} = 0$ or $> N$ by PA2]

then $\exists\ p \in \mathbb{F}$ $(p \ne 0,\ p \ne 1)$ s/t

$$\varphi_i = -p\,i\,(N-i+1) \qquad 1 \le i \le N$$

$$\phi_i = (1-p)\,i\,(N-i+1)$$

pf For $1 \le i \le N$

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} = \sum_{h=0}^{i-1} \frac{h-(N-h)}{0-N}$$

$$= \frac{i\,(N-i+1)}{N}$$

Def $p = -\dfrac{\varphi_1}{N}$ so $\varphi_1 = -pN$

By thm PA4, for $1 \le i \le N$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + (\theta_i^* - \theta_0^*)(\theta_{N-i+1} - \theta_0)$$

$$= -pN\,\frac{i(N-i+1)}{N} + i\,(N-i+1)$$

$$= (1-p)\,i\,(N-i+1)$$

so $\phi_1 = (1-p)N$

By PA3, for $1 \le i \le N$

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_N)$$

$$= (1-p)N \frac{i(N-i+1)}{N} + i(i-1-N)$$

$$= -p\,i\,(N-i+1)$$

$\square$

Thm 245 Assume

$$\theta_i = i, \qquad \theta_i^* = i \qquad 0 \le i \le N$$

Then

$$u_i(x) = K_i(x; p, N) \qquad 0 \le i \le N$$

↑ Krawtchouk

where p is from L 244

pf In Th 238 eval $u_i(x)$ using

$\theta_i = i$

$\theta_i^* = i$

$\varphi_i = -pi(N-i+1)$

gets

$$u_i(x) = {}_2F_1\left({-i \;\; -x \atop -N} \Big| \frac{1}{p} \right)$$

$$= K_i(x; p, N)$$

□

Note: Ref to Th 245.

Since $\theta_i = \theta_i^*$ $0 \le i \le N$

Φ, Φ^* have same PA

so $u_i(x) = u_i^*(x)$ $0 \le i \le N$

So AW duality Th 243 asserts

$$u_i(\theta_j) = u_j(\theta_i) \qquad 0 \le i, j \le N$$

$$\begin{array}{cc} \| & \| \\ K_i(j; p, N) & K_j(i; p, N) \\ \| & \| \end{array}$$

$${}_2F_1\left({-i \;\; -j \atop -N} \Big| \frac{1}{p} \right)$$

In the handout we list all the parameter arrays over $\mathbb{F}$

For each array $(\theta_i, \theta_i^*, \varphi_i, \phi_i)$ we give

$$u_i(\theta_j) \qquad 0 \le i, j \le N$$

The resulting formula shows the $\{u_i\}_{i=0}^N$ are from Askey scheme

Also, every poly sequence from term branch of Askey scheme is realized as $\{u_i\}_{i=0}^N$ for some PA.

Theorem 34.14 *[51, Section 10] Assume $\mathbb{K}$ is algebraically closed. Let q denote a nonzero scalar in $\mathbb{K}$ that is not a root of unity. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, A^* denote a tridiagonal pair on V that has q-geometric type. Then there exists an irreducible $\boxtimes_q$-module structure on V such that A acts as a scalar multiple of x_{01} and A^* acts as a scalar multiple of x_{23}. Conversely, let V denote a finite dimensional irreducible $\boxtimes_q$-module. Then the generators x_{01}, x_{23} act on V as a tridiagonal pair of q-geometric type.*

We end this section with a conjecture.

Conjecture 34.15 *Assume $\mathbb{K}$ is algebraically closed. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. To avoid degenerate situations we assume q is not a root of unity, where $\beta = q^2 + q^{-2}$, and where β is from Theorem 34.8. Then referring to Definition 34.12, there exists an irreducible $\boxtimes_q$-module structure on V such that A acts as a linear combination of x_{01}, x_{12}, I and A^* acts as a linear combination of x_{23}, x_{30}, I.*

35 Appendix: List of parameter arrays

In this section we display all the parameter arrays over $\mathbb{K}$. We will use the following notatation.

Definition 35.1 Let $p = (\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ denote a parameter array over $\mathbb{K}$. For $0 \leq i \leq d$ we let u_i denote the following polynomial in $\mathbb{K}[\lambda]$.

$$u_i = \sum_{n=0}^{i} \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n}. \tag{134}$$

We call $u_0, u_1, \ldots, u_d$ *the polynomials that correspond to* p.

We now display all the parameter arrays over $\mathbb{K}$. For each displayed array $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ we present $u_i(\theta_j)$ for $0 \leq i, j \leq d$, where $u_0, u_1, \ldots, u_d$ are the corresponding polynomials. Our presentation is organized as follows. In each of Example 35.2–35.14 below we give a family of parameter arrays over $\mathbb{K}$. In Theorem 35.15 we show every parameter array over $\mathbb{K}$ is contained in at least one of these families.

In each of Example 35.2–35.14 below the following implicit assumptions apply: d denotes a nonnegative integer, the scalars $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ are contained in $\mathbb{K}$, and the scalars $q, h, h^* \ldots$ are contained in the algebraic closure of $\mathbb{K}$.

Example 35.2 *(q-Racah) Assume*

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \tag{135}$$
$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i} \tag{136}$$

for $0 \leq i \leq d$ and

$$\varphi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1q^i)(1 - r_2q^i), \tag{137}$$
$$\phi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^* \tag{138}$$

for $1 \leq i \leq d$. Assume $h, h^*, q, s, s^*, r_1, r_2$ are nonzero and $r_1 r_2 = ss^* q^{d+1}$. Assume none of $q^i, r_1 q^i, r_2 q^i, s^* q^i/r_1, s^* q^i/r_2$ is equal to 1 for $1 \leq i \leq d$ and that neither of $sq^i, s^* q^i$ is equal to 1 for $2 \leq i \leq 2d$. Then $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_4\phi_3\left(\begin{matrix} q^{-i}, \; s^* q^{i+1}, \; q^{-j}, \; sq^{j+1} \\ r_1 q, \; r_2 q, \; q^{-d} \end{matrix} \middle| \; q, \; q \right)$$

for $0 \leq i, j \leq d$. These u_i are the q-Racah polynomials.

Example 35.3 *(q-Hahn) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + h(1 - q^i)q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)(1 - s^* q^{i+1})q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= hh^* q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - rq^i), \\ \phi_i &= -hh^* q^{1-i}(1 - q^i)(1 - q^{i-d-1})(r - s^* q^i) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s^*, r are nonzero. Assume none of $q^i, rq^i, s^* q^i/r$ is equal to 1 for $1 \leq i \leq d$ and that $s^* q^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, \; s^* q^{i+1}, \; q^{-j} \\ rq, \; q^{-d} \end{matrix} \middle| \; q, \; q \right)$$

for $0 \leq i, j \leq d$. These u_i are the q-Hahn polynomials.

Example 35.4 *(Dual q-Hahn) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \\ \theta_i^* &= \theta_0^* + h^*(1 - q^i)q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= hh^* q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - rq^i), \\ \phi_i &= hh^* q^{d+2-2i}(1 - q^i)(1 - q^{i-d-1})(s - rq^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, r, s are nonzero. Assume none of $q^i, rq^i, sq^i/r$ is equal to 1 for $1 \leq i \leq d$ and that $sq^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i}, \; q^{-j}, \; sq^{j+1} \\ rq, \; q^{-d} \end{matrix} \middle| \; q, \; q \right)$$

for $0 \leq i, j \leq d$. These u_i are the dual q-Hahn polynomials.

Example 35.5 *(Quantum q-Krawtchouk) Assume*

$$\begin{aligned} \theta_i &= \theta_0 - sq(1-q^i), \\ \theta^*_i &= \theta^*_0 + h^*(1-q^i)q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ *and*

$$\begin{aligned} \varphi_i &= -rh^*q^{1-i}(1-q^i)(1-q^{i-d-1}), \\ \phi_i &= h^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$. *Assume* h^*, q, r, s *are nonzero. Assume neither of* $q^i, sq^i/r$ *is equal to* 1 *for* $1 \leq i \leq d$. *Then the sequence* $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ *is a parameter array over* $\mathbb{K}$. *The corresponding polynomials* u_i *satisfy*

$$u_i(\theta_j) = {}_2\phi_1\left(\begin{matrix} q^{-i},\ q^{-j} \\ q^{-d} \end{matrix} \,\middle|\, q,\ sr^{-1}q^{j+1}\right)$$

for $0 \leq i, j \leq d$. *These* u_i *are the quantum q-Krawtchouk polynomials.*

Example 35.6 *(q-Krawtchouk) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + h(1-q^i)q^{-i}, \\ \theta^*_i &= \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ *and*

$$\begin{aligned} \varphi_i &= hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1}), \\ \phi_i &= hh^*s^*q(1-q^i)(1-q^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$. *Assume* h, h^*, q, s^* *are nonzero. Assume* $q^i \neq 1$ *for* $1 \leq i \leq d$ *and that* $s^*q^i \neq 1$ *for* $2 \leq i \leq 2d$. *Then the sequence* $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ *is a parameter array over* $\mathbb{K}$. *The corresponding polynomials* u_i *satisfy*

$$u_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i},\ s^*q^{i+1},\ q^{-j} \\ 0,\ q^{-d} \end{matrix} \,\middle|\, q,\ q\right)$$

for $0 \leq i, j \leq d$. *These* u_i *are the q-Krawtchouk polynomials.*

Example 35.7 *(Affine q-Krawtchouk) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + h(1-q^i)q^{-i}, \\ \theta^*_i &= \theta^*_0 + h^*(1-q^i)q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ *and*

$$\begin{aligned} \varphi_i &= hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i), \\ \phi_i &= -hh^*rq^{1-i}(1-q^i)(1-q^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, r are nonzero. Assume neither of q^i, rq^i is equal to 1 for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i},\ 0,\ q^{-j} \\ rq,\ q^{-d} \end{matrix} \,\middle|\, q,\ q \right),$$

for $0 \leq i, j \leq d$. These u_i are the affine q-Krawtchouk polynomials.

Example 35.8 *(Dual q-Krawtchouk) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \\ \theta^*_i &= \theta^*_0 + h^*(1 - q^i)q^{-i} \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= hh^* q^{1-2i}(1 - q^i)(1 - q^{i-d-1}), \\ \phi_i &= hh^* sq^{d+2-2i}(1 - q^i)(1 - q^{i-d-1}) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^*, q, s are nonzero. Assume $q^i \neq 1$ for $1 \leq i \leq d$ and $sq^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_3\phi_2\left(\begin{matrix} q^{-i},\ q^{-j},\ sq^{j+1} \\ 0,\ q^{-d} \end{matrix} \,\middle|\, q,\ q \right)$$

for $0 \leq i, j \leq d$. These u_i are the dual q-Krawtchouk polynomials.

Example 35.9 *(Racah) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + hi(i + 1 + s), && (139) \\ \theta^*_i &= \theta^*_0 + h^* i(i + 1 + s^*) && (140) \end{aligned}$$

for $0 \leq i \leq d$ and

$$\begin{aligned} \varphi_i &= hh^* i(i - d - 1)(i + r_1)(i + r_2), && (141) \\ \phi_i &= hh^* i(i - d - 1)(i + s^* - r_1)(i + s^* - r_2) && (142) \end{aligned}$$

for $1 \leq i \leq d$. Assume h, h^* are nonzero and that $r_1 + r_2 = s + s^* + d + 1$. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than d. Assume none of $r_1, r_2, s^* - r_1,\ s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$ and that neither of s, s^* is equal to $-i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_4F_3\left(\begin{matrix} -i,\ i + 1 + s^*,\ -j,\ j + 1 + s \\ r_1 + 1,\ r_2 + 1,\ -d \end{matrix} \,\middle|\, 1 \right)$$

for $0 \leq i, j \leq d$. These u_i are the Racah polynomials.

Example 35.10 *(Hahn) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + si, \\ \theta^*_i &= \theta^*_0 + h^* i(i+1+s^*) \end{aligned}$$

for $0 \leq i \leq d$ *and*

$$\begin{aligned} \varphi_i &= h^* si(i-d-1)(i+r), \\ \phi_i &= -h^* si(i-d-1)(i+s^*-r) \end{aligned}$$

for $1 \leq i \leq d$. *Assume* h^*, s *are nonzero. Assume the characteristic of* $\mathbb{K}$ *is* 0 *or a prime greater than* d. *Assume neither of* r, s^*-r *is equal to* $-i$ *for* $1 \leq i \leq d$ *and that* $s^* \neq -i$ *for* $2 \leq i \leq 2d$. *Then the sequence* $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ *is a parameter array over* $\mathbb{K}$. *The corresponding polynomials* u_i *satisfy*

$$u_i(\theta_j) = {}_3F_2\left(\begin{matrix} -i, \ i+1+s^*, \ -j \\ r+1, \ -d \end{matrix} \,\middle|\, 1 \right)$$

for $0 \leq i, j \leq d$. *These* u_i *are the Hahn polynomials.*

Example 35.11 *(Dual Hahn) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + hi(i+1+s), \\ \theta^*_i &= \theta^*_0 + s^* i \end{aligned}$$

for $0 \leq i \leq d$ *and*

$$\begin{aligned} \varphi_i &= hs^* i(i-d-1)(i+r), \\ \phi_i &= hs^* i(i-d-1)(i+r-s-d-1) \end{aligned}$$

for $1 \leq i \leq d$. *Assume* h, s^* *are nonzero. Assume the characteristic of* $\mathbb{K}$ *is* 0 *or a prime greater than* d. *Assume neither of* $r, s-r$ *is equal to* $-i$ *for* $1 \leq i \leq d$ *and that* $s \neq -i$ *for* $2 \leq i \leq 2d$. *Then the sequence* $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ *is a parameter array over* $\mathbb{K}$. *The corresponding polynomials* u_i *satisfy*

$$u_i(\theta_j) = {}_3F_2\left(\begin{matrix} -i, \ -j, \ j+1+s \\ r+1, \ -d \end{matrix} \,\middle|\, 1 \right)$$

for $0 \leq i, j \leq d$. *These* u_i *are the dual Hahn polynomials.*

Example 35.12 *(Krawtchouk) Assume*

$$\begin{aligned} \theta_i &= \theta_0 + si, \\ \theta^*_i &= \theta^*_0 + s^* i \end{aligned}$$

for $0 \leq i \leq d$ *and*

$$\begin{aligned} \varphi_i &= ri(i-d-1) \\ \phi_i &= (r-ss^*)i(i-d-1) \end{aligned}$$

for $1 \leq i \leq d$. Assume r, s, s^* are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than d. Assume $r \neq ss^*$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials u_i satisfy

$$u_i(\theta_j) = {}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \,\middle|\, r^{-1}ss^* \right) ,$$

for $0 \leq i, j \leq d$. These u_i are the Krawtchouk polynomials.

Example 35.13 *(Bannai/Ito) Assume*

$$\theta_i = \theta_0 + h(s - 1 + (1 - s + 2i)(-1)^i), \tag{143}$$
$$\theta^*_i = \theta^*_0 + h^*(s^* - 1 + (1 - s^* + 2i)(-1)^i) \tag{144}$$

for $0 \leq i \leq d$ and

$$\varphi_i = \begin{cases} -4hh^*i(i + r_1), & \text{if } i \text{ even, } d \text{ even;} \\ -4hh^*(i - d - 1)(i + r_2), & \text{if } i \text{ odd, } d \text{ even;} \\ -4hh^*i(i - d - 1), & \text{if } i \text{ even, } d \text{ odd;} \\ -4hh^*(i + r_1)(i + r_2), & \text{if } i \text{ odd, } d \text{ odd,} \end{cases} \tag{145}$$

$$\phi_i = \begin{cases} 4hh^*i(i - s^* - r_1), & \text{if } i \text{ even, } d \text{ even;} \\ 4hh^*(i - d - 1)(i - s^* - r_2), & \text{if } i \text{ odd, } d \text{ even;} \\ -4hh^*i(i - d - 1), & \text{if } i \text{ even, } d \text{ odd;} \\ -4hh^*(i - s^* - r_1)(i - s^* - r_2), & \text{if } i \text{ odd, } d \text{ odd} \end{cases} \tag{146}$$

for $1 \leq i \leq d$. Assume h, h^* are nonzero and that $r_1 + r_2 = -s - s^* + d + 1$. Assume the characteristic of $\mathbb{K}$ is either 0 or an odd prime greater than $d/2$. Assume neither of $r_1, -s^* - r_1$ is equal to $-i$ for $1 \leq i \leq d$, $d - i$ even. Assume neither of $r_2, -s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$, i odd. Assume neither of s, s^* is equal to $2i$ for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. We call the corresponding polynomials from Definition 35.1 the Bannai/Ito polynomials [11, p. 260].

Example 35.14 *(Orphan) For this example assume $\mathbb{K}$ has characteristic 2. For notational convenience we define some scalars $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $\mathbb{K}$. We define $\gamma_i = 0$ for $i \in \{0, 3\}$ and $\gamma_i = 1$ for $i \in \{1, 2\}$. Assume*

$$\theta_i = \theta_0 + h(si + \gamma_i), \tag{147}$$
$$\theta^*_i = \theta^*_0 + h^*(s^*i + \gamma_i) \tag{148}$$

for $0 \leq i \leq 3$. Assume $\varphi_1 = hh^*r$, $\varphi_2 = hh^*$, $\varphi_3 = hh^*(r+s+s^*)$ and $\phi_1 = hh^*(r+s(1+s^*))$, $\phi_2 = hh^*$, $\phi_3 = hh^*(r + s^*(1 + s))$. Assume each of h, h^*, s, s^*, r is nonzero. Assume neither of s, s^* is equal to 1 and that r is equal to none of $s + s^*$, $s(1 + s^*)$, $s^*(1 + s)$. Then the sequence $(\theta_i, \theta^*_i, i = 0..3; \varphi_j, \phi_j, j = 1..3)$ is a parameter array over $\mathbb{K}$ which has diameter 3. We call the corresponding polynomials from Definition 35.1 the orphan polynomials.

Theorem 35.15 *Every parameter array over $\mathbb{K}$ is listed in at least one of the Examples 35.2–35.14.*

Proof: Let $p := (\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ denote a parameter array over $\mathbb{K}$. We show this array is given in at least one of the Examples 35.2–35.14. We assume $d \geq 1$; otherwise the result is trivial. For notational convenience let $\tilde{K}$ denote the algebraic closure of $\mathbb{K}$. We define a scalar $q \in \tilde{\mathbb{K}}$ as follows. For $d \geq 3$, we let q denote a nonzero scalar in $\tilde{\mathbb{K}}$ such that $q + q^{-1} + 1$ is equal to the common value of (82). For $d < 3$ we let q denote a nonzero scalar in $\tilde{\mathbb{K}}$ such that $q \neq 1$ and $q \neq -1$. By PA5, both

$$\begin{aligned} \theta_{i-2} - \xi\theta_{i-1} + \xi\theta_i - \theta_{i+1} &= 0, &(149)\\ \theta^*_{i-2} - \xi\theta^*_{i-1} + \xi\theta^*_i - \theta^*_{i+1} &= 0 &(150)\end{aligned}$$

for $2 \leq i \leq d-1$, where $\xi = q + q^{-1} + 1$. We divide the argument into the following four cases. (I) $q \neq 1$, $q \neq -1$; (II) $q = 1$ and $\operatorname{char}(\mathbb{K}) \neq 2$; (III) $q = -1$ and $\operatorname{char}(\mathbb{K}) \neq 2$; (IV) $q = 1$ and $\operatorname{char}(\mathbb{K}) = 2$.

Case I: $q \neq 1$, $q \neq -1$.
By (149) there exist scalars η, μ, h in $\tilde{\mathbb{K}}$ such that

$$\theta_i = \eta + \mu q^i + h q^{-i} \qquad (0 \leq i \leq d). \qquad (151)$$

By (150) there exist scalars η^*, μ^*, h^* in $\tilde{\mathbb{K}}$ such that

$$\theta^*_i = \eta^* + \mu^* q^i + h^* q^{-i} \qquad (0 \leq i \leq d). \qquad (152)$$

Observe μ, h are not both 0; otherwise $\theta_1 = \theta_0$ by (151). Similarly μ^*, h^* are not both 0. For $1 \leq i \leq d$ we have $q^i \neq 1$; otherwise $\theta_i = \theta_0$ by (151). Setting $i = 0$ in (151), (152) we obtain

$$\begin{aligned} \theta_0 &= \eta + \mu + h, &(153)\\ \theta^*_0 &= \eta^* + \mu^* + h^*. &(154)\end{aligned}$$

We claim there exists $\tau \in \tilde{\mathbb{K}}$ such that both

$$\begin{aligned} \varphi_i &= (q^i - 1)(q^{d-i+1} - 1)(\tau - \mu\mu^* q^{i-1} - hh^* q^{-i-d}), &(155)\\ \phi_i &= (q^i - 1)(q^{d-i+1} - 1)(\tau - h\mu^* q^{i-d-1} - \mu h^* q^{-i}) &(156)\end{aligned}$$

for $1 \leq i \leq d$. Since $q \neq 1$ and $q^d \neq 1$ there exists $\tau \in \tilde{\mathbb{K}}$ such that (155) holds for $i = 1$. In the equation of PA4, we eliminate φ_1 using (155) at $i = 1$, and evaluate the result using (151), (152) in order to obtain (156) for $1 \leq i \leq d$. In the equation of PA3, we eliminate ϕ_1 using (156) at $i = 1$, and evaluate the result using (151), (152) in order to obtain (155) for $1 \leq i \leq d$. We have now proved the claim. We now break the argument into subcases. For each subcase our argument is similar. We will discuss the first subcase in detail in order to give the idea; for the remaining subcases we give the essentials only.

Subcase q-Racah: $\mu \neq 0, \mu^* \neq 0, h \neq 0, h^* \neq 0$. We show p is listed in Example 35.2. Define

$$s := \mu h^{-1} q^{-1}, \qquad s^* := \mu^* h^{*-1} q^{-1}. \qquad (157)$$

Eliminating η in (151) using (153) and eliminating μ in the result using the equation on the left in (157), we obtain (135) for $0 \leq i \leq d$. Similarly we obtain (136) for $0 \leq i \leq d$. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains scalars r_1, r_2 such that both

$$r_1 r_2 = s s^* q^{d+1}, \qquad r_1 + r_2 = \tau h^{-1} h^{*-1} q^d. \qquad (158)$$

Eliminating μ, μ^*, τ in (155), (156) using (157) and the equation on the right in (158), and evaluating the result using the equation on the left in (158), we obtain (137), (138) for $1 \leq i \leq d$. By the construction each of h, h^*, q, s, s^* is nonzero. Each of r_1, r_2 is nonzero by the equation on the left in (158). The remaining inequalities mentioned below (138) follow from PA1, PA2 and (135)–(138). We have now shown p is listed in Example 35.2.

We now give the remaining subcases of Case I. We list the essentials only.

Subcase q-Hahn: $\mu = 0, \mu^* \neq 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definitions:

$$s^* := \mu^* h^{*-1} q^{-1}, \qquad r := \tau h^{-1} h^{*-1} q^d.$$

Subcase dual q-Hahn: $\mu \neq 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definitions:

$$s := \mu h^{-1} q^{-1}, \qquad r := \tau h^{-1} h^{*-1} q^d.$$

Subcase quantum q-Krawtchouk: $\mu \neq 0, \mu^* = 0, h = 0, h^* \neq 0, \tau \neq 0$. Definitions:

$$s := \mu q^{-1}, \qquad r := \tau h^{*-1} q^d.$$

Subcase q-Krawtchouk: $\mu = 0, \mu^* \neq 0, h \neq 0, h^* \neq 0, \tau = 0$. Definition:

$$s^* := \mu^* h^{*-1} q^{-1}.$$

Subcase affine q-Krawtchouk: $\mu = 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definition:

$$r := \tau h^{-1} h^{*-1} q^d.$$

Subcase dual q-Krawtchouk: $\mu \neq 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau = 0$. Definition:

$$s := \mu h^{-1} q^{-1}.$$

We have a few more comments concerning Case I. Earlier we mentioned that μ, h are not both 0 and that μ^*, h^* are not both 0. Suppose one of μ, h is 0 and one of μ^*, h^* is 0. Then $\tau \neq 0$; otherwise $\varphi_1 = 0$ by (155) or $\phi_1 = 0$ by (156). Suppose $\mu^* \neq 0$, $h^* = 0$. Replacing q by q^{-1} we obtain $\mu^* = 0$, $h^* \neq 0$. Suppose $\mu^* \neq 0$, $h^* \neq 0$, $\mu \neq 0$, $h = 0$. Replacing q by q^{-1} we obtain $\mu^* \neq 0$, $h^* \neq 0$, $\mu = 0$, $h \neq 0$. By these comments we find that after replacing q by q^{-1} if necessary, one of the above subcases holds. This completes our argument for Case I.

Case II: $q = 1$ and $\mathrm{char}(\mathbb{K}) \neq 2$.
By (149) and since $\mathrm{char}(\mathbb{K}) \neq 2$, there exist scalars η, μ, h in $\tilde{\mathbb{K}}$ such that

$$\theta_i = \eta + (\mu + h)i + hi^2 \qquad (0 \leq i \leq d). \tag{159}$$

Similarly there exist scalars η^*, μ^*, h^* in $\tilde{\mathbb{K}}$ such that

$$\theta^*_i = \eta^* + (\mu^* + h^*)i + h^*i^2 \qquad (0 \leq i \leq d). \tag{160}$$

Observe μ, h are not both 0; otherwise $\theta_1 = \theta_0$. Similarly μ^*, h^* are not both 0. For any prime i such that $i \le d$ we have $\mathrm{char}(\mathbb{K}) \ne i$; otherwise $\theta_i = \theta_0$ by (159). Therefore $\mathrm{char}(\mathbb{K})$ is 0 or a prime greater than d. Setting $i = 0$ in (159), (160) we obtain

$$\theta_0 = \eta, \qquad \theta^*_0 = \eta^*. \tag{161}$$

We claim there exists $\tau \in \tilde{\mathbb{K}}$ such that both

$$\varphi_i = i(d-i+1)(\tau - (\mu h^* + h\mu^*)i - hh^*i(i+d+1)), \tag{162}$$
$$\phi_i = i(d-i+1)(\tau + \mu\mu^* + h\mu^*(1+d) + (\mu h^* - h\mu^*)i + hh^*i(d-i+1)) \tag{163}$$

for $1 \le i \le d$. There exists $\tau \in \tilde{\mathbb{K}}$ such that (162) holds for $i = 1$. In the equation of PA4, we eliminate φ_1 using (162) at $i = 1$, and evaluate the result using (159), (160) in order to obtain (163) for $1 \le i \le d$. In the equation of PA3, we eliminate ϕ_1 using (163) at $i = 1$, and evaluate the result using (159), (160) in order to obtain (162) for $1 \le i \le d$. We have now proved the claim. We now break the argument into subcases.

Subcase Racah: $h \ne 0$, $h^* \ne 0$. We show p is listed in Example 35.9. Define

$$s := \mu h^{-1}, \qquad s^* := \mu^* h^{*-1}. \tag{164}$$

Eliminating η, μ in (159) using (161), (164) we obtain (139) for $0 \le i \le d$. Eliminating η^*, μ^* in (160) using (161), (164) we obtain (140) for $0 \le i \le d$. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains scalars r_1, r_2 such that both

$$r_1 r_2 = -\tau h^{-1} h^{*-1}, \qquad r_1 + r_2 = s + s^* + d + 1. \tag{165}$$

Eliminating μ, μ^*, τ in (162), (163) using (164) and the equation on the left in (165) we obtain (141), (142) for $1 \le i \le d$. By the construction each of h, h^* is nonzero. The remaining inequalities mentioned below (142) follow from PA1, PA2 and (139)–(142). We have now shown p is listed in Example 35.9.

We now give the remaining subcases of Case II. We list the essentials only.

Subcase Hahn: $h = 0$, $h^* \ne 0$. Definitions:

$$s = \mu, \qquad s^* := \mu^* h^{*-1}, \qquad r := -\tau \mu^{-1} h^{*-1}.$$

Subcase dual Hahn: $h \ne 0, h^* = 0$. Definitions:

$$s := \mu h^{-1}, \qquad s^* = \mu^*, \qquad r := -\tau h^{-1} \mu^{*-1}.$$

Subcase Krawtchouk: $h = 0, h^* = 0$. Definitions:

$$s := \mu, \qquad s^* := \mu^*, \qquad r := -\tau.$$

Case III: $q = -1$ and $\mathrm{char}(\mathbb{K}) \ne 2$.
We show p is listed in Example 35.13. By (149) and since $\mathrm{char}(\mathbb{K}) \ne 2$, there exist scalars η, μ, h in $\mathbb{K}$ such that

$$\theta_i = \eta + \mu(-1)^i + 2hi(-1)^i \qquad (0 \le i \le d). \tag{166}$$

Similarly there exist scalars η^*, μ^*, h^* in $\tilde{\mathbb{K}}$ such that

$$\theta^*_i \quad = \quad \eta^* + \mu^*(-1)^i + 2h^*i(-1)^i \qquad (0 \leq i \leq d). \tag{167}$$

Observe $h \neq 0$; otherwise $\theta_2 = \theta_0$ by (166). Similarly $h^* \neq 0$. For any prime i such that $i \leq d/2$ we have $\mathrm{char}(\mathbb{K}) \neq i$; otherwise $\theta_{2i} = \theta_0$ by (166). By this and since $\mathrm{char}(\mathbb{K}) \neq 2$ we find $\mathrm{char}(\mathbb{K})$ is either 0 or an odd prime greater than $d/2$. Setting $i = 0$ in (166), (167) we obtain

$$\theta_0 \quad = \quad \eta + \mu, \qquad \theta^*_0 = \eta^* + \mu^*. \tag{168}$$

We define

$$s := 1 - \mu h^{-1}, \qquad s^* = 1 - \mu^* h^{*-1}. \tag{169}$$

Eliminating η in (166) using (168) and eliminating μ in the result using (169) we find (143) holds for $0 \leq i \leq d$. Similarly we find (144) holds for $0 \leq i \leq d$. We now define r_1, r_2. First assume d is odd. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains r_1, r_2 such that

$$r_1 + r_2 = -s - s^* + d + 1 \tag{170}$$

and such that

$$4hh^*(1 + r_1)(1 + r_2) = -\varphi_1. \tag{171}$$

Next assume d is even. Define

$$r_2 := -1 + \frac{\varphi_1}{4hh^*d} \tag{172}$$

and define r_1 so that (170) holds. We have now defined r_1, r_2 for either parity of d. In the equation of PA4, we eliminate φ_1 using (171) or (172), and evaluate the result using (143), (144) in order to obtain (146) for $1 \leq i \leq d$. In the equation of PA3, we eliminate ϕ_1 using (146) at $i = 1$, and evaluate the result using (143), (144) in order to obtain (145) for $1 \leq i \leq d$. We mentioned each of h, h^* is nonzero. The remaining inequalities mentioned below (146) follow from PA1, PA2 and (143)–(146). We have now shown p is listed in Example 35.13.

Case IV: $q = 1$ and $\mathrm{char}(\mathbb{K}) = 2$.
We show p is listed in Example 35.14. We first show $d = 3$. Recall $d \geq 3$ since $q = 1$. Suppose $d \geq 4$. By (149) we have $\sum_{j=0}^{3} \theta_j = 0$ and $\sum_{j=1}^{4} \theta_j = 0$. Adding these sums we find $\theta_0 = \theta_4$ which contradicts PA1. Therefore $d = 3$. We claim there exist nonzero scalars h, s in $\mathbb{K}$ such that (147) holds for $0 \leq i \leq 3$. Define $h = \theta_0 + \theta_2$. Observe $h \neq 0$; otherwise $\theta_0 = \theta_2$. Define $s = (\theta_0 + \theta_3)h^{-1}$. Observe $s \neq 0$; otherwise $\theta_0 = \theta_3$. Using these values for h, s we find (147) holds for $i = 0, 2, 3$. By this and $\sum_{j=0}^{3} \theta_j = 0$ we find (147) holds for $i = 1$. We have now proved our claim. Similarly there exist nonzero scalars h^*, s^* in $\mathbb{K}$ such that (148) holds for $0 \leq i \leq 3$. Define $r := \varphi_1 h^{-1} h^{*-1}$. Observe $r \neq 0$ and that $\varphi_1 = hh^*r$. In the equation of PA4, we eliminate φ_1 using $\varphi_1 = hh^*r$ and evaluate the result using (147), (148) in order to obtain $\phi_1 = hh^*(r + s(1 + s^*))$, $\phi_2 = hh^*$, $\phi_3 = hh^*(r + s^*(1 + s))$. In the equation of

PA3, we eliminate ϕ_1 using $\phi_1 = hh^*(r+s(1+s^*))$ and evaluate the result using (147), (148) in order to obtain $\varphi_2 = hh^*$, $\varphi_3 = hh^*(r+s+s^*)$. We mentioned each of h, h^*, s, s^*, r is nonzero. Observe $s \neq 1$; otherwise $\theta_1 = \theta_0$. Similarly $s^* \neq 1$. Observe $r \neq s+s^*$; otherwise $\varphi_3 = 0$. Observe $r \neq s(1+s^*)$; otherwise $\phi_1 = 0$. Observe $r \neq s^*(1+s)$; otherwise $\phi_3 = 0$. We have now shown p is listed in Example 35.14. We are done with Case IV and the proof is complete. □

36 Suggestions for further research

In this section we give some suggestions for further research.

Problem 36.1 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. Let $\alpha, \alpha^*, \beta, \beta^*$ denote scalars in $\mathbb{K}$ with α, α^* nonzero, and note that the pair $\alpha A + \beta I$, $\alpha^* A^* + \beta^* I$ is a tridiagonal pair on V. Find necessary and sufficient conditions for this tridiagonal pair to be isomorphic to the tridiagonal pair A, A^*. Also, find necessary and sufficient conditions for this tridiagonal pair to be isomorphic to the tridiagonal pair A^*, A. This problem has been solved for Leonard pairs [81].

Problem 36.2 Assume $\mathbb{K} = \mathbb{R}$. With reference to Definition 15.1, find a necessary and sufficient condition on the parameter array of Φ, for the bilinear form $\langle\, , \rangle$ to be positive definite. By definition the form $\langle\, , \rangle$ is positive definite whenever $\|u\|^2 > 0$ for all nonzero $u \in V$.

In order to motivate the next problem we make a definition.

Definition 36.3 Let Φ denote the Leonard system from Definition 3.2. For $0 \leq i \leq d$ we define $A_i = v_i(A)$, where the polynomial v_i is from Definition 13.1. Observe that there exist scalars $p^h_{ij} \in \mathbb{K}$ $(0 \leq h, i, j \leq d)$ such that

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \qquad (0 \leq i, j \leq d).$$

We call the p^h_{ij} the *intersection numbers* of Φ.

Problem 36.4 Let Φ denote the Leonard system from Definition 3.2. For each of the Examples 35.2–35.14, if possible express each intersection number as a hypergeometric series or a basic hypergeometric series. Also for $\mathbb{K} = \mathbb{R}$, determine those Φ for which the intersection numbers are all nonnegative.

Problem 36.5 Assume $\mathbb{K} = \mathbb{R}$ and let Φ denote the Leonard system from Definition 3.2. Determine those Φ for which the intersection numbers of each of Φ, $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$, $\Phi^{\downarrow\Downarrow}$ are all nonnegative. Also, determine those Φ for which the intersection numbers of each relative of Φ are all nonnegative.

Next goal Find the b_i, c_i in terms of PA

Given LS

$$\Phi = (A, E_i, A^*, E_i^*)$$

with PA

$$(\theta_i, \theta_i^*, \varphi_i, \phi_i)$$

Recall $c_i, b_i \in \mathbb{F}$ satisfy

$$x u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad 0 \le i \le N-1$$

Also

$$x u_N - c_N u_{N-1} - a_N u_N \in \mathbb{F} p_{N+1}$$

p_{N+1} = min poly of A

LEM 24.6

(i) $$b_i = \varphi_{i+1} \frac{\tau_i^*(\theta_i^*)}{\tau_{i+1}^*(\theta_{i+1}^*)} \qquad 0 \le i \le N-1$$

(ii) $$c_i = \phi_i \frac{\eta_{N-i}^*(\theta_i^*)}{\eta_{N-i+1}^*(\theta_{i-1}^*)} \qquad 1 \le i \le N$$

pf (i) We have seen

$$b_0 b_1 \cdots b_{i-1} = p_i(\theta_0)$$

$$= \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\tau_i^*(\theta_i^*)}$$

Result follows by induction on i.

(ii) Apply (i) to $\Phi^{\Downarrow}$ and note

$$c_i(\Phi^{\Downarrow}) = b_{N-i}(\Phi) \qquad \square$$

Another formula for b_i, c_i

(Assume $N \geq 1$ to avoid trivialities)

LEM 247 For $N \geq 1$

$$c_i(\theta^*_{i-1} - \theta^*_i) - b_i(\theta^*_i - \theta^*_{i+1}) = (\theta_1 - \theta_0)(\theta^*_i - \theta^*_0) + \varphi_1$$

$$0 \leq i \leq N \qquad (\theta^*_{-1}, \theta^*_{N+1} \text{ indet})$$

pf In th 238 set $x = \theta_1$ to get

$$u_i(\theta_1) = 1 + \frac{(\theta_1 - \theta_0)(\theta^*_i - \theta^*_0)}{\varphi_1} \qquad 0 \leq i \leq N \qquad *$$

In the 3-term rec

$$\theta_1 u_i(\theta_1) = c_i u_{i-1}(\theta_1) + a_i u_i(\theta_1) + b_i u_{i+1}(\theta_1) \qquad 0 \leq i \leq N$$

elim $u_j(\theta_1)$ using $*$ and a_i using

$$a_i = \theta_0 - b_i - c_i$$

$\square$

thm 248 For $N \geq 1$

(i) $$b_0 = \frac{\varphi_1}{\theta_1^* - \theta_0^*}$$

(ii) $$b_i = \frac{(\theta_0 - a_i)(\theta_i^* - \theta_{i-1}^*) + (\theta_0 - \theta_1)(\theta_0^* - \theta_i^*) + \varphi_1}{\theta_{i+1}^* - \theta_{i-1}^*}$$

$1 \leq i \leq N-1$

(iii) $$c_i = \frac{(\theta_0 - a_i)(\theta_i^* - \theta_{i+1}^*) + (\theta_0 - \theta_1)(\theta_0^* - \theta_i^*) + \varphi_1}{\theta_{i-1}^* - \theta_{i+1}^*}$$

$1 \leq i \leq N-1$

(iv) $$c_N = \frac{\phi_N}{\theta_{N-1}^* - \theta_N^*}$$

[Recall a_i given in L210]

pf (i) Set $i=0$ in L246

(ii), (iii) Solve the system

$$\left\{ \begin{array}{l} c_i + b_i = \theta_0 - a_i \\ \text{eq in L247} \end{array} \right\}$$

for c_i, b_i

(iv) Set $i=N$ in L246 $\square$

Ex 249 Assume Φ has Krawtchouk type ie

$$\theta_i = i \qquad \theta^*_i = i \qquad 0 \le i \le N$$

then

(i) $b_i = (i-N)p \qquad 0 \le i \le N$

(ii) $c_i = i(p-1) \qquad 0 \le i \le N$

(iii) $a_i = i(1-p) + (N-i)p \qquad 0 \le i \le N$
(p from L244)

" Same data as from Th 99 "

pf PA given in L244

Use either L246 and

$$c_i + a_i + b_i = \theta_0 \qquad 0 \le i \le N$$

or L248 and L210

$\square$

Lecture 34 Wednesday Dec 1 12/1/10

1

Next goal:

For the terminating branch of Askey scheme,

a uniform treatment of orthogonality using corresp LS.

Until further notice fix LS

$$\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N) \quad \text{on } V$$

Par array

$$(\{\theta_i\}_{i=0}^N, \{\theta_i^*\}_{i=0}^N, \{\varphi_i\}_{i=1}^N, \{\phi_i\}_{i=1}^N)$$

Def 250 Def

$$m_i = tr(E_i E_0^*) \qquad 0 \le i \le N$$

LEM 251

(i) $\quad E_i E_0^* E_i = m_i E_i \qquad 0 \le i \le N$

(ii) $\quad E_0^* E_i E_0^* = m_i E_0^* \qquad 0 \le i \le N$

(iii) $\quad m_i \neq 0 \qquad 0 \le i \le N$

(iv) $\quad \sum_{i=0}^{N} m_i = 1$

(v) $\quad m_0 = m_0^*$

pf (i) Since E_i is rank 1 idempotent

$$E_i \mathcal{A} E_i = \mathbb{F} E_i \qquad \mathcal{A} = \operatorname{End} V$$

So $\exists\ \alpha_i \in \mathbb{F}$ s.t

$$E_i E_0^* E_i = \alpha_i E_i$$

In this eq take trace to get $\alpha_i = m_i$

(ii) Sim to (i)

(iii) E_0^* is normalizing so $E_i E_0^* \neq 0$

Apply $\dagger$ to get $E_0^* E_i \neq 0$
so $E_0^* E_i V = E_0^* V$

Now

$$E_i E_0^* E_i \neq 0 \quad \text{since}$$

$$E_i E_0^* E_i V = E_i E_0^* V \neq 0$$

Now $m_i \neq 0$ by (i)

(iv) $\quad \sum_{i=0}^{N} m_i = \operatorname{tr} \sum_{i=0}^{N} E_i E_0^* = \operatorname{tr} E_0^* = 1$

$E_i E_0^*$, $E_0^* E_i$ have same trace.

Def 252 Put

$$\nu = \frac{1}{m_0} = \frac{1}{m_0^*}$$

So

$$\nu^{-1} = \operatorname{tr}(E_0 E_0^*)$$

Obs

$$\nu = \nu^*$$

LEM 253 We have

(i) $\nu E_0^* E_0 E_0^* = E_0^*$

(ii) $\nu E_0 E_0^* E_0 = E_0$

pf Set $i=0$ in L251 (i), (ii) $\square$

thm 254

$$\nu = \frac{\tau_N(\theta_0)\,\tau_N^*(\theta_0^*)}{\phi_1\phi_2\cdots\phi_N}$$

$$= \frac{(\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_N)\,(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_N^*)}{\phi_1\phi_2\cdots\phi_N}$$

pf Pick $0 \neq v \in E_0^*V$

Recall

$$\{\tau_i(A)v\}_{i=0}^N$$

is basis for V. Rel this basis

$$A: \begin{pmatrix} \theta_N & & & 0 \\ 1 & \theta_{N-1} & & \\ & 1 & \ddots & \\ 0 & & 1 & \theta_0 \end{pmatrix} \qquad A^*: \begin{pmatrix} \theta_0^* & \phi_1 & & 0 \\ & \theta_1^* & \phi_2 & \\ & & \ddots & \phi_N \\ 0 & & & \theta_N^* \end{pmatrix}$$

Rep E_0, E_0^* the basis.

Form is

$$E_0: \begin{pmatrix} & 0 & & \\ \hline \alpha & \cdots & * & 1 \end{pmatrix} \qquad E_0^*: \begin{pmatrix} 1 & * & \cdots & \beta \\ \hline & & 0 & \end{pmatrix}$$

$$E_0E_0^*: \begin{pmatrix} & 0 & & \\ \hline * & * & \cdots & \alpha\beta \end{pmatrix}$$

$\nu^{-1} = \operatorname{tr} E_0E_0^* = \alpha\beta$

Applying L206 to Φ^* $\quad \alpha = \dfrac{1}{\tau_N(\theta_0)}$

L208 " $\quad \beta = \phi_1\cdots\phi_N$

Q

Ex 255 Assume Φ has Krawtchouk type s

$$\theta_i = i, \qquad \theta_i^* = i \qquad 0 \le i \le N$$

then

$$\nu = (1-\rho)^{-N}$$

where ρ is from L 244

pf Eval th 254 using L 244 $\square$

Recall

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad 0 \le i \le N$$

LEM 256 Fn $0 \le i \le N$

$$k_i = \nu\, tr (E_i^* E_i)$$

$$(= \nu\, m_i^*)$$

pf Pick $0 \ne u \in E_0 V$

Recall $\bar{E}$ - stand basis fn V

$$E_i^* u \qquad 0 \le i \le N$$

Recall polys $\{v_i\}_{i=0}^N$ sat

$$v_i(A)\, E_0^* u = E_i^* u \qquad 0 \le i \le N$$

Obs

$$\begin{aligned}
m_i^* u &= m_i^* E_0 u \\
&= E_0 E_i^* E_0 u \\
&= E_0 E_i^* u \\
&= E_0 v_i(A) E_0^* u \\
&= v_i(\theta_0)\, E_0 E_0^* u \\
&= v_i(\theta_0)\, \underbrace{E_0 E_0^* E_0}_{\nu^{-1} E_0} u
\end{aligned}$$

$$\left[\begin{array}{ll} v_i = k_i u_i & u_i(\theta_0) = 1 \\ v_i(\theta_0) = k_i & \end{array} \right]$$

$$\begin{aligned}
&= k_i \nu^{-1} E_0 u \\
&= k_i \nu^{-1} u \qquad \text{so} \qquad m_i^* = k_i \nu^{-1}
\end{aligned}$$

□

We def a bil form

$$\langle\, , \rangle : V \times V \to \mathbb{F}$$

as follows.

Pick $0 \neq u \in E_0 V$

Recall Φ - stand basis for V:

$$E_i^* u \qquad 0 \le i \le N \qquad *$$

We def $\langle\, , \rangle$ on $*$

Pick $0 \neq f \in \mathbb{F}$. Put

$$\langle E_i^* u, E_j^* u \rangle = \delta_{ij} k_i f \qquad 0 \le i, j \le N$$

So matrix rep $\langle\, , \rangle$ rel $*$ is

$$K = f \operatorname{diag}(k_i)_{i=0}^N$$

Obs

$\langle\, , \rangle$ is sym

$\langle\, , \rangle$ is non deg since K^{-1} exists

Obs $k_0 = 1$ so

$$\| E_0^* u \|^2 = f$$

so $$\langle E_i^* u, E_j^* u \rangle = \delta_{ij} k_i \| E_0^* u \|^2 \qquad 0 \le i, j \le N$$

LEM 257 $\forall X \in \operatorname{End} V$ $\forall v, w \in V$

$$\langle Xv, w \rangle = \langle v, X^{\dagger} w \rangle$$

where $\dagger$ is antiaut of $\operatorname{End} V$ from L 188

pf By L 187

$$A, E_0^* \text{ gen } \operatorname{End} V$$

WLOG $X = A$ or $X = E_0^*$

Recall $A^{\dagger} = A$, $E_0^{*\dagger} = E_0^*$

Pick $0 \neq u \in E_0 V$ Consider Φ-st basis for V:

$$E_i^* u \qquad 0 \le i \le N \qquad *$$

WLOG $v, w \in *$

Rel $*$ matrix rep A, E_0^* is

$$A: \underbrace{\begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & & b_{N-1} \\ & & & c_N & a_N \end{pmatrix}}_{B} \qquad E_0^*: \operatorname{diag}(1, 0, 0, \ldots 0)$$

Case $X = A$

$$\langle A E_i^* u, E_j^* u \rangle \stackrel{?}{=} \langle E_i^* u, A E_j^* u \rangle \qquad 0 \le i, j \le N$$

Holds since

$$KB = B^t K$$

Case $X = E_0^*$: routine $\square$

LEM 258

(i) For $u \in E_0V$

$$\| E_0^* u \|^2 = \nu^{-1} \| u \|^2$$

(ii) For $v \in E_0^*V$

$$\| E_0 v \|^2 = \nu^{-1} \| v \|^2$$

pf (i) Obs $E_0 u = u$

$$\| E_0^* u \|^2 = \langle E_0^* u, \; E_0^* u \rangle = \langle E_0^* E_0 u, \; E_0^* E_0 u \rangle$$

$$= \langle u, \; \underbrace{E_0 E_0^*}_{\nu^{-1} E_0} E_0 u \rangle$$

$$= \nu^{-1} \langle u, u \rangle$$

(ii) Sim

$\square$

Pick $0 \neq v \in E_0^* V$

Consider Φ^* - stand basis for V:

$$E_i v \qquad 0 \le i \le N$$

LEM 259 With above notation

$$\langle E_i v, E_j v \rangle = \delta_{ij} \, k_i^* \, \| E_0 v \|^2 \qquad 0 \le i, j \le N$$

pf

$$\text{LHS} = \langle E_i E_0^* v, E_j E_0^* v \rangle$$

$$= \langle v, \underbrace{E_0^* E_i E_j E_0^*}_{\delta_{ij} m_i E_0^*} v \rangle$$

$$= \delta_{ij} m_i \| v \|^2$$

$$= \delta_{ij} \underbrace{m_i \nu}_{= k_i^* \text{ by L 255}} \| E_0 v \|^2$$

$\square$

LEM 260 Pick $0 \neq u \in E_0 V$ and $0 \neq v \in E_0^* V$

(i) Each of $\|u\|^2$, $\|v\|^2$, $\langle u, v \rangle$ is nonzero

(ii) $$E_0^* u = \frac{\langle u, v \rangle}{\|v\|^2} v$$

(iii) $$E_0 v = \frac{\langle u, v \rangle}{\|u\|^2} u$$

(iv) $$\nu \langle u, v \rangle^2 = \|u\|^2 \|v\|^2$$

pf (ii) v is basis for $E_0^* V$ so $\exists\, \alpha \in \mathbb{F}$

$$E_0^* u = \alpha v$$

so

$$\begin{array}{ccc} \langle E_0^* u, v \rangle & = & \langle \alpha v, v \rangle \\ \| & & \| \\ \langle u, E_0^* v \rangle & & \alpha \|v\|^2 \\ \| & & \\ \langle u, v \rangle & & \end{array}$$

$$\alpha = \frac{\langle u, v \rangle}{\|v\|^2}$$

(iii) Sim

(i) Since $E_0^* u \neq 0$

(iv)
$$\begin{aligned} \nu^{-1} \langle u, v \rangle &= \langle u, E_0^* E_0 E_0^* v \rangle \\ &= \langle E_0^* u, E_0 v \rangle \\ &= \frac{\langle u, v \rangle^2}{\|u\|^2 \|v\|^2} \underbrace{\langle v, u \rangle}_{\langle u, v \rangle} \end{aligned}$$

$\square$

Thm 261 For $0 \neq u \in E_0 V$ and $0 \neq v \in E_0^* V$

$$\langle E_i^* u, E_j v \rangle = \nu^{-1} k_i k_j^* u_i(\theta_j) \langle u, v \rangle \qquad 0 \le i, j \le N$$

pf

$$\langle E_i^* u, E_j v \rangle = \langle v_i(A) E_0^* u, E_j v \rangle$$

$$= \langle E_0^* u, \underbrace{v_i(A) E_j v}_{\parallel \atop v_i(\theta_j) E_j v} \rangle$$

$$= v_i(\theta_j) \langle E_0^* u, \underbrace{E_j v}_{\parallel \atop v_j^*(A^*) E_0 v} \rangle$$

$$= v_i(\theta_j) \langle \underbrace{v_j^*(A^*) E_0^* u}_{\parallel \atop {v_j^*(\theta_0^*) E_0^* u \atop {\parallel \atop k_j^*}}}, E_0 v \rangle$$

$$= k_j^* \underbrace{v_i(\theta_j)}_{\parallel \atop k_i u_i(\theta_j)} \underbrace{\langle E_0^* u, E_0 v \rangle}_{\parallel \ L260 \atop \nu^{-1} \langle u, v \rangle}$$

$\square$

thm 262 For $0 \neq u \in E_0 V$ and $0 \neq v \in E_0^* V$

(i) $$E_i^* u = \frac{\langle u, v \rangle}{\|v\|^2} \sum_{j=0}^{N} v_i(\theta_j) \, E_j v \qquad 0 \le i \le N$$

(ii) $$E_i v = \frac{\langle u, v \rangle}{\|u\|^2} \sum_{j=0}^{N} v_i^*(\theta_j^*) \, E_j^* u \qquad 0 \le i \le N$$

pf (i) $$E_i^* u = v_i(A) E_0^* u$$

$$= \left(\sum_{j=0}^{N} E_j \right) v_i(A) E_0^* u$$

$$= \sum_{j=0}^{N} v_i(\theta_j) \, E_j \underbrace{E_0^* u}_{\overset{\text{L 260}}{=} \frac{\langle u, v \rangle}{\|v\|^2} v}$$

(ii) Sim

□

We now give the orthogonality rel for the $\{v_i\}_{i=0}^N$

thm 263

(i) For $0 \le i,j \le N$

$$\sum_{r=0}^{N} v_i(\theta_r)\, v_j(\theta_r)\, k_r^* = \delta_{ij}\, \nu\, k_i$$

(ii) For $0 \le r,s \le N$

$$\sum_{i=0}^{N} v_i(\theta_r)\, v_i(\theta_s)\, k_i^{-1} = \delta_{rs}\, \nu\, k_r^{*-1}$$

pf (i) Pick $0 \ne u \in E_0V$ $\quad 0 \ne v \in E_0^*V$

$$\langle E_i^* u, E_j^* u \rangle \overset{\text{thm 262}}{=} \frac{\langle u, v\rangle^2}{\|v\|^4} \sum_{r=0}^{N} v_i(\theta_r)\, v_j(\theta_r)\, \underbrace{\|E_r v\|^2}$$

$$\|E_r v\|^2 \overset{L259}{=} k_r^* \|E_0 v\|^2 \overset{L258}{=} \nu^{-1} \|v\|^2$$

$$\langle E_i^* u, E_j^* u \rangle \overset{\text{def of } \langle\,,\rangle}{=} \delta_{ij} k_i \|E_0^* u\|^2 \overset{L258}{=} \delta_{ij} k_i \nu^{-1} \|u\|^2$$

simplify using L260 (iv)

(ii) Apply (i) to Φ^* use Askey-Wilson duality

$\square$

We now give the orthogonality for the $\{u_i\}_{i=0}^{N}$

Th 264

(i) For $0 \le i, j \le N$

$$\sum_{r=0}^{N} u_i(\theta_r)\, u_j(\theta_r)\, k_r^{*} = \delta_{ij}\, \nu\, k_i^{-1}$$

(ii) For $0 \le r, s \le N$

$$\sum_{i=0}^{N} u_i(\theta_r)\, u_i(\theta_s)\, k_i = \delta_{rs}\, \nu\, k_r^{*-1}$$

pf Eval Th 263 using

$v_h = u_h k_h \qquad 0 \le h \le N$

$\square$

Next goal: the difference equation for $\{u_i\}_{i=0}^N$

(analog of Th 108)

To motivate, recall 3-term rec:

For $0 \le i, j \le N$

$$\theta_j u_i(\theta_j) = c_i u_{i-1}(\theta_j) + a_i u_i(\theta_j) + b_i u_{i+1}(\theta_j) \qquad \ast$$

Th 265 For $0 \le i, j \le N$

$$\theta_i^* u_i(\theta_j) = c_j^* u_i(\theta_{j-1}) + a_j^* u_i(\theta_j) + b_j^* u_i(\theta_{j+1})$$

$c_0^* = 0, \quad b_N^* = 0,$

$\theta_{-1}, \quad \theta_{N+1}$ indet

Pf Apply $\ast$ to Φ^* and use Askey–Wilson duality $\square$

Lecture 35 Friday Dec 3

Next goal: Express our results on (Leonard systems / Term branch of Askey scheme) in terms of matrices

DEF 266 For our given LS Φ we define:

Matrix Name	matrix entries
U	$\left(u_i(\theta_j)\right)_{0 \le i,j \le N}$
B	$\begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ \mathbf{0} & & & c_N & a_N \end{pmatrix}$
D	$\mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_N)$
K	$\mathrm{diag}(k_0, k_1 \ldots k_N)$ $k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad 0 \le i \le N$

U^*, B^*, D^*, K^* are corresp matrices for Φ^*

thm 267 With ref to Def 266

Result	meaning	ref
$U^t = U^*$	Askey-Wilson duality	th 243
$B^t = KBK^{-1}$	✓	
$UD = BU$	3-term rec	Above thm 238
$D^*U = UB^{*t}$	difference equation	thm 265
$\nu^{-1} U K^* U^t K = I$	orthogonality	thm 264 (i)
$\nu^{-1} U^t K U K^* = I$		thm 264 (ii)

"compare with th 116"

For the Krawtchouk polys we defined a matrix P above Th 117.

We now define P for any LS

DEF 268 Def $P \in Mat_{N+1}(\mathbb{F})$ s.t.

$$P_{ij} = v_j(\theta_i) \qquad 0 \le i, j \le N$$

Obs

$$P = U^t K$$
$$= U^* K$$

$$U = K^{-1} P^t$$
$$= P^* K^{*-1}$$

thm 269 With ref to Def 266 and Def 268

$$P^t = K P^* K^{*-1}$$

$$B^t = K B K^{-1}$$

"compare with th 117"

$$P B^* = B^* P$$

$$P B = B P$$

$$P P^* = \nu I$$

pf In th 267 elim $\mathcal{U}$ using Def 268 □

Next goal: Φ- dual standard basis

Def 27.0 Given $0 \neq u \in E_0V$ obs

$$\frac{E_i^* u}{k_i} \qquad 0 \le i \le N$$

is basis for V. Call this a

" Φ- dual standard basis "

LEM 27.1. Fix $0 \neq u \in E_0 V$ Consider Φ - stand basis

$$E_i^* u \qquad 0 \le i \le N \qquad *$$

With respect to $\langle , \rangle$ the basis for V dual to $*$ is

$$\frac{\nu}{\|u\|^2} \frac{E_i^* u}{k_i} \qquad 0 \le i \le N \qquad **$$

Moreover $**$ is a Φ- dual standard basis

pf Recall

$$\langle E_i^* u, E_j^* u \rangle = \delta_{ij} k_i \|E_0^* u\|^2 \qquad 0 \le i, j \le N$$

and $\|E_0^* u\|^2 = \nu^{-1} \|u\|^2$

LEM 272 Let $\{w_i\}_{i=0}^{N}$ denote a

Φ-dual standard basis for V

Then the poly $\{u_i\}_{i=0}^{N}$ sat

$$u_i(A)\, w_0 = w_i \qquad 0 \le i \le N$$

pf $\exists\, 0 \ne u \in E_0 V$ s.t.

$$w_i = \frac{E_i^* u}{k_i} \qquad 0 \le i \le N$$

Recall

$$E_i^* u = v_i(A)\, E_0^* u \qquad 0 \le i \le N$$

$$v_i = k_i u_i \qquad 0 \le i \le N$$

$$k_0 = 1$$

$\square$

LEM 273 let $\{w_i\}_{i=0}^N$ denote a Φ-dual stand basis for V
$\{w_i^*\}_{i=0}^N$ Φ^*-dual stand basis

$$\langle w_i, w_j^* \rangle = u_i(\theta_j) \langle w_0, w_0^* \rangle \qquad 0 \le i, j \le N$$

pf $\exists\ 0 \ne u \in E_0 V$ s.t

$$w_i = \frac{E_i^* u}{k_i} \qquad 0 \le i \le N$$

$\exists\ 0 \ne v \in E_0^* V$ s.t

$$w_i^* = \frac{E_i v}{k_i^*} \qquad 0 \le i \le N$$

By th 261

$$\langle E_i^* u, E_j v \rangle = \nu^{-1} k_i k_j^* u_i(\theta_j) \langle u, v \rangle \qquad 0 \le i, j \le N$$

so

$$\langle E_0^* u, E_0 v \rangle = \nu^{-1} \langle u, v \rangle$$

so

$$\langle E_i^* u, E_j v \rangle = k_i k_j^* u_i(\theta_j) \langle E_0^* u, E_0 v \rangle$$

Result follows.

$\square$

Fix $0 \neq u \in E_0V$ and $0 \neq v \in E_0^*V$. Using u, v we get 4 bases for V

basis	description	
Φ - standard	$E_i^* u$	$0 \le i \le N$
Φ - dual standard	$\frac{\nu}{\lVert u \rVert^2} \frac{E_i^* u}{k_i}$	$0 \le i \le N$
Φ^* - standard	$E_i v$	$0 \le i \le N$
Φ^* - dual standard	$\frac{\nu}{\lVert v \rVert^2} \frac{E_i v}{k_i^*}$	$0 \le i \le N$

Matrices that represent A and A^*

Basis	A	A^*
Φ - standard	B	D^*
Φ - dual standard	B^t	D^*
Φ^* - standard	D	B^*
Φ^* - dual standard	D	B^{*t}

Transition Matrices

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\left\{ \frac{\nu}{\|u\|^2} \frac{E_i^* u}{k_i} \right\}_{i=0}^N \arrow[r, shift left, "K \frac{\|u\|^2}{\nu}"] \arrow[d, shift left, "KU \frac{\langle u,v\rangle}{\|v\|^2}"] \arrow[dr, shift left, "KUK^* \frac{\langle u,v\rangle}{\nu}"] & \{ E_i^* u \}_{i=0}^N \arrow[l, shift left, "K^{-1} \frac{\nu}{\|u\|^2}"] \arrow[d, shift left, "UK^* \frac{\langle u,v\rangle}{\|u\|^2}"] \arrow[dl, shift left, "U \frac{1}{\langle u,v\rangle}"] \\
\left\{ \frac{\nu}{\|v\|^2} \frac{E_i v}{k_i^*} \right\}_{i=0}^N \arrow[r, shift left, "K^* \frac{\|v\|^2}{\nu}"] \arrow[u, shift left, "K^*U^* \frac{\langle u,v\rangle}{\|u\|^2}"] \arrow[ur, shift left, "K^*U^*K \frac{\langle u,v\rangle}{\nu}"] & \{ E_i v \}_{i=0}^N \arrow[l, shift left, "(K^*)^{-1} \frac{\nu}{\|v\|^2}"] \arrow[u, shift left, "U^*K \frac{\langle u,v\rangle}{\|v\|^2}"] \arrow[ul, shift left, "U^* \frac{1}{\langle u,v\rangle}"]
\end{tikzcd}$$

key: $\{u_i\}_{i=0}^N \xrightarrow{M} \{v_i\}_{i=0}^N$

means

$$v_j = \sum_{i=0}^N M_{ij} u_i \qquad j=0,1,\dots,N$$

Matrices that represent $\langle\ ,\ \rangle$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\left\{\frac{\nu}{\|u\|^2}\frac{E_i^* u}{k_i}\right\}_{i=0}^N \arrow[loop above, "K^{-1}\frac{\nu}{\|u\|^2}"] \arrow[r, dash, "I"] \arrow[d, shift right, "U\frac{1}{\langle u,v\rangle}"'] \arrow[dr, shift left, "UK^*\frac{\langle u,v\rangle}{\|u\|^2}"] & \left\{E_i^* u\right\}_{i=0}^N \arrow[loop above, "K\frac{\|u\|^2}{\nu}"] \arrow[d, shift left, "KUK^*\frac{\langle u,v\rangle}{\nu}"] \arrow[dl, shift left, "KU\frac{\langle u,v\rangle}{\|v\|^2}"] \\
\left\{\frac{\nu}{\|v\|^2}\frac{E_i v}{k_i^*}\right\}_{i=0}^N \arrow[loop below, "(K^*)^{-1}\frac{\nu}{\|v\|^2}"] \arrow[r, dash, "I"'] \arrow[u, shift right, "U^*\frac{1}{\langle u,v\rangle}"'] \arrow[ur, shift left, "U^*K\frac{\langle u,v\rangle}{\|v\|^2}"] & \left\{E_i v\right\}_{i=0}^N \arrow[loop below, "K^*\frac{\|v\|^2}{\nu}"] \arrow[u, shift left, "K^*U^*K\frac{\langle u,v\rangle}{\nu}"] \arrow[ul, shift left, "K^*U^*\frac{\langle u,v\rangle}{\|u\|^2}"]
\end{tikzcd}
$$

Key: $\{u_i\}_{i=0}^N \xrightarrow{M} \{v_i\}_{j=0}^N$

means $M_{ij} = \langle u_i, v_j\rangle \quad (0 \le i,j \le N)$

Note If we pick $\langle\,,\,\rangle$ such that

$$\|u\|^2 = 1$$

and pick v such that

$$\langle u, v\rangle = 1$$

Then

$$\|v\|^2 = \nu$$

by L 260 (iv)

In this case above diagrams match the ones above Th 158

$\square$

topics

- Given a_i, b_i, c_i
  $(a_i + b_i + c_i = \theta_0)$
  find PA

  go thru DRGs
  and find all
  PA
  focus on strange
  examples / quotients

- qRac case → Mock spin model W
  $\mathbb{Z}_3$-sym version →
  → $\rho_i = u_i(\theta_0)$
  ρ_i / ρ_{i-1} sat quad rel
  Ψ, $\Delta = \exp_q(\Psi)$

- Algebras
  AW(3) ($\mathbb{Z}_3$-sym ver)
  TD (q-Onsager)
  $U_q sl_2$ (Left and right embedding)
  $\boxtimes_q$
  DAHA

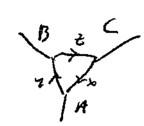
$\beta \boxtimes$
$B^2, \beta K, KB, K^2, \beta, K, I$
lin ~~dep~~ dep

Kraw → $sl_2 \otimes \mathbb{F}[\rho, \frac{1}{\rho}, \frac{1}{\rho - 1}]$

sl_2 $e f h$ → sl_2 $x y z$
keep track of iso param
→ Drinfeld.

- Rosengren ~~rep~~ model

Bip / dual bip case
$\mathbb{Z}_3$-sym AW
spin model W

- aut$^\sigma$ of $U_q sl_2$ order 2
- is it inner?
- W, W^+, W^ε
- A, z is LP of Krawtype
- 

- what is subalg of $U_q sl_2$ gen by A, B, C

- b_i, c_i form
  $x, y, z, \sigma(y), \sigma(z)$ gives $\boxtimes \to \boxtimes_q$