

# Leonard systems and their ...

Leonard pairs and the terminating branch of the Askey Scheme

We saw that the Krawtchouk polys  $\{K_i(x, p, N)\}_{i=0}^N$  corresp to LS whose equal reqs and dual equal req is in arith progression.

We now classify all the LS  $\mathbb{F}$ .

Each sol  $\mathbb{F}$  corresponds to a poly sequence  $\{P_i\}_{i=0}^N$

from the Askey Scheme, yielding the

"terminating branch of Askey Scheme"

$q$ -Racah	Racah
$q$ -Hahn	Hahn
dual $q$ -Hahn	dual Hahn
$q$ -Krawtchouk	Kraw
dual $q$ -Krawtchouk	Bannai/Ito
quantum $q$ -Kraw	orphans (char $\mathbb{F}=2$ $N=3$ only)
affine $q$ -Kraw	

If  $\mathbb{F}$  has equal req  $\{\theta_i\}_{i=0}^N$  and dual equal req  $\{\theta_i^*\}_{i=0}^N$

then

$$P_i \in \mathbb{F} \sum_{j=0}^i \frac{(x - \theta_0) \dots (x - \theta_{i-1}) (\theta_1^* - \theta_0^*) \dots (\theta_1^* - \theta_{i-1}^*)}{\varphi_1 \varphi_2 \dots \varphi_i}$$

0575N

For some non-zero  $\{\varphi_i\}_{i=1}^N$  called the 1st split sequence

Also

$$p_j \in \mathbb{F} \sum_{i=0}^j \frac{(x - \alpha_N) \cdots (x - \alpha_{N-i+1})(\theta_j^* - \theta_0^*) \cdots (\theta_j^* - \theta_{i-1}^*)}{\phi_i \phi_2 \cdots \phi_i}$$

057EN

For some non 0 scalars  $\{\phi_i\}_{i=1}^N$  called 2nd split req.

# Classification of LS Outline

## I Basics

- the  $a_i, a_i^*$
- the anti-ant  $\dagger$
- Normalizing idempotents

## II the split decomp and parameter array

## III Recurrent sequences

## IV the tridiag relations

## V Conclusion

# I Basics

Throughout this section assume:

field  $\mathbb{F}$  arb

$N =$  non neg integer

$V =$  vector space over  $\mathbb{F}$  of dim  $N+1$

Given

$$(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$$

s.t.

- Each of  $A, A^*$  is MF el of  $\text{End } V$
- $\{E_i\}_{i=0}^N$  is ordering of prim ids of  $A$
- $\{E_i^*\}_{i=0}^N \dots A^*$

For  $0 \leq i \leq N$  let

$\theta_i =$  eigen of  $A$  for  $E_i$

$\theta_i^* =$  eigen of  $A^*$  for  $E_i^*$

let

$\mathcal{D} =$  subalg of  $\text{End } V$  gen by  $A$

$\mathcal{D}^* = \dots A^*$

For  $0 \leq i \leq N$

$$\gamma_i = (x - \theta_0) \dots (x - \theta_{i-1})$$

$$\gamma_i = (x - \theta_0) \dots (x - \theta_{N-i})$$

$$\gamma_i^* = (x - \theta_0^*) \dots (x - \theta_{i-1}^*)$$

$$\gamma_i^* = (x - \theta_0^*) \dots (x - \theta_{N-i}^*)$$

Def 182 By a decomposition of  $V$ , we mean

a sequence  $\{V_i\}_{i=0}^N$  of subspaces of  $V$  s.t.

$$\dim V_i = 1 \quad i=0, 1, \dots, N$$

$$V = \sum_{i=0}^N V_i \quad (ds)$$

For not conv put

$$V_{-1} = 0$$

$$V_{N+1} = 0$$

Ex 183 Each of

$$\{E_i V\}_{i=0}^N$$

$$\{E_i^* V\}_{i=0}^N$$

is a decomp of  $V$

Recall

$$\text{tr } E_i = 1$$

$$\text{tr } E_i^* = 1$$

$0 \leq i \leq N$

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Def 184

define

$$a_i = \text{tr} (A E_i^*)$$

$0 \leq i \leq N$

$$a_i^* = \text{tr} (A^* E_i)$$

LEM 185

$$(i) \quad \sum_{i=0}^N \theta_i = \sum_{i=0}^N a_i$$

$$(ii) \quad \sum_{i=0}^N \theta_i^* = \sum_{i=0}^N a_i^*$$

pt (i)

$$\text{LHS} = \text{tr}(A)$$

$$= \text{tr} \left( A \sum_{i=0}^N E_i^* \right)$$

$$= \text{RHS}$$

Cor Sim

□

LEM For  $0 \leq i \leq N$

$$(i) \quad E_i^* A E_i^* = a_i E_i^*$$

$$(ii) \quad E_i A^* E_i = a_i^* E_i$$

PROOF: Routine using Def 184

LEM 186 Assume

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad 0 \leq i, j \leq N$$

then

$$A^i E_0^* A^j \quad 0 \leq i, j \leq N$$

is a basis for  $\text{End } V$

pf Like Prop 174

□

LEM 187 Ref to Lem 186

each of following is gen set for End V

Cor A,  $E_0^*$

Cor A,  $A^*$

pf Like Cor 175

□



An antiauto of  $\text{End} V$  is an iso of

$\mathbb{F}$ -vector spaces  $\sigma: \text{End} V \rightarrow \text{End} V$  s.t.

$$(xy)^\sigma = y^\sigma x^\sigma \quad \forall x, y \in \text{End} V$$

LEM 188 Ref to LEM 186

$\exists$  unique antiauto  $\dagger$  of  $\text{End} V$  s.t.

$$A^\dagger = A \quad A^{*\dagger} = A^* \quad (*)$$

Moreover

$$(X^\dagger)^\dagger = X \quad \forall X \in \text{End} V$$

pf Pick

$$\text{of } v_i \in E_i^{\times} V \quad 0 \in V \in N$$

So  $\{v_i\}_{i=0}^N$  is basis for  $V$

$\forall X \in \text{End} V$  let

$$X^b = \text{matrix rep } X \text{ rel } \{v_i\}_{i=0}^N$$

$$b: \text{End} V \rightarrow \text{Mat}_{N+1}(\mathbb{F})$$

$$X \rightarrow X^b$$

is  $\mathbb{F}$ -alg iso

Write

$$B = A^b$$

$$B^* = A^{*b}$$

$B =$  irred tri diag

$$B^* = \text{diag} (\theta_i^{*b})_{i=0}^N$$

For basis  $N$  def

$$k_i = \frac{B_{0i} \ B_{1i} \ \dots \ B_{i-1,i}}{B_{i0} \ B_{i1} \ \dots \ B_{ii}}$$

def

$$K = \text{diag} (k_i)_{i=0}^N$$

Recall

$$B^t = K B K^{-1}$$

so

$$K^{-1} B^t K = B$$

Define

$$\begin{aligned} \gamma: \text{Mat}_{N+1}(\mathbb{F}) &\rightarrow \text{Mat}_{N+1}(\mathbb{F}) \\ X &\rightarrow K^{-1} X^t K \end{aligned}$$

$\gamma$  is anti aut that fixes each of  $B, B^*$

Now Composition

$$\begin{aligned} \dagger: \text{End } V &\rightarrow \text{Mat}_{N+1}(\mathbb{F}) \xrightarrow{\gamma} \text{Mat}_{N+1}(\mathbb{F}) \rightarrow \text{End } V \\ &\quad b \qquad \qquad \qquad \gamma \qquad \qquad \qquad b^{-1} \end{aligned}$$

is anti aut of  $\text{End } V$  that fixes  $A, A^*$

Uniqueness: Suppose  $\sigma$  is anti-ant of  $\text{End } V$

that fixes  $A, A^*$

Then  $\tau \circ \sigma^{-1}$  is anti of  $\text{End } V$  that fixes  $A, A^*$

$A, A^*$  gen  $\text{End } V$  so  $\tau \circ \sigma^{-1} = 1$  so  $\tau = \sigma$

To get (\*) note

$\tau \circ \tau$  is anti of  $\text{End } V$  that fixes  $A, A^*$

so  $\tau \circ \tau = 1$

□

Thm 189 (3/4 Theorem)

Consider the following four assumptions

(i)  $E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \quad 0 \leq i, j \leq N$

(ii)  $E_i^* A E_j^* = \begin{cases} 0 & \text{if } j-i > 1 \\ \neq 0 & \text{if } j-i = 1 \end{cases} \quad "$

(iii)  $E_i A^* E_j = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \quad "$

(iv)  $E_i A^* E_j = \begin{cases} 0 & \text{if } j-i > 1 \\ \neq 0 & \text{if } j-i = 1 \end{cases} \quad "$

Assume at least 3 of (i) - (iv) hold

Then they all hold i.e.

$(A, E_i, A^*, E_j^*)$  is LS mV

pt Interchanging  $A, A^*$  if nec wlog (i), (ii) hold.

Now by L 188  $\exists$  antiinv  $\dagger$  of  $\text{End} V$  that

$A^\dagger = A, \quad A^{*\dagger} = A^*$

obs  $E_i^\dagger = E_i, \quad (E_i^*)^\dagger = E_i^* \quad 0 \leq i \leq N$

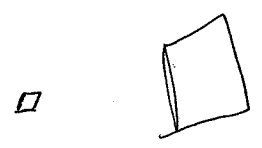
For  $0 \leq i, j \leq N$

$(E_i A^* E_j)^\dagger = E_j A^* E_i$

So  $E_i A^* E_j = 0 \iff E_j A^* E_i = 0$

So (iii)  $\Leftrightarrow$  (iv)

Result follows.



## I. Basics, cont.

Def 190 The idempotent  $E_0^*$  is called  
normalizing whenever

$$E_i E_0^* = 0 \quad \forall i \in N$$

LEM 191 TFAE

(i)  $E_0^x$  is normalizing(ii)  $\mathcal{D} E_0^x$  has dim  $NH$ (iii)  $\{A^i E_0^x\}_{i=0}^N$  are lin indep(iv)  $\forall x \in \mathcal{D}$ 

$$x E_0^x = 0 \rightarrow x = 0$$

(v)  $\mathcal{D} E_0^x V = V$ (vi)  $\forall 0 \neq v \in V$  the map

$$\mathcal{D} \rightarrow V$$

$$x \rightarrow xv$$

is a bijection

pf Routine

□

LEM 192 Assume  $E_0^*$  is normalizing

and define

$$u_i = T_i(A) E_0^* v \quad 0 \leq i \leq N$$

Then

(i)  $\{u_i\}_{i=0}^N$  is a dec of  $V$   
( )  $(A - \theta_i I) u_i = u_{i+1} \quad 0 \leq i \leq N$

(ii)  $u_0 + u_1 + \dots + u_i = E_0^* v + A E_0^* v + \dots + A^i E_0^* v \quad 0 \leq i \leq N$

(iii)  $u_0 + u_1 + \dots + u_N = E_0^* v + E_1^* v + \dots + E_N^* v \quad 0 \leq i \leq N$

pf (i) Each  $u_i$  has dim 1 by L191 (iv)

$$V = \sum_{i=0}^N u_i \text{ by L191 (v)}$$

(ii) Recall  $\deg T_i^* = i \quad 0 \leq i \leq N$

(iii) Both sides dim  $N+1$

So to show  $\subseteq$

$$\text{RHS} = \{ w \in V \mid (A - \theta_0 I) \dots (A - \theta_N I) w = 0 \}$$

$$\subseteq \text{LHS}$$

□

LEM 193 TFAE

(i)  $E_i^x A E_j^x = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \quad 0 \leq i, j \leq N$

(ii)  $\exists$  poly sequence  $\{p_i\}_{i=0}^N$  in  $\mathbb{F}[x]$  s.t.

$E_i^x V = p_i(A) E_0^x V \quad 0 \leq i \leq N$

(iii) FA  $0 \leq i \leq N$

$E_0^x V + \dots + E_i^x V = E_0^x V + A E_0^x V + \dots + A^i E_0^x V$

Suppose (i) - (iii) then  $E_0^x$  is normalizing

pf

(i)  $\rightarrow$  (ii) Fix

$0 \neq v_i \in E_i^x V \quad 0 \leq i \leq N$

So  $\{v_i\}_{i=0}^N$  is basis for  $V$

Let  $B \in \text{Mat}_{N+1}(\mathbb{F})$  rep  $A$  w.r.t  $\{v_i\}_{i=0}^N$

$B_{ij} = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \quad 0 \leq i, j \leq N$

"upper Hessenberg"

Define  $\{p_i\}_{i=0}^N$  in  $\mathbb{F}[x]$  by  $p_0 = 1$  and

$x p_i = \sum_{j=0}^{i+1} B_{ij} p_j \quad 0 \leq i \leq N-1$

then  $p_i$  has deg exactly  $i$  for  $0 \leq i \leq N$

By const

$v_i^x = p_i(A) v_0 \quad 0 \leq i \leq N$

So

$E_i^x V = p_i(A) E_0^x V \quad 0 \leq i \leq N$



(ii)  $\rightarrow$  (iii)  $E_0^x$  normalizing

Both sides dim  $n+1$

Suffices to show  $\subseteq$

Follows since  $\deg p_2 = 1$  for  $0 \leq j \leq n$

(iii)  $\rightarrow$  (i) Part

$$\begin{aligned} V_i &= E_0^x V + E_1^x V + \dots + E_i^x V && 0 \leq i \leq n \\ &= E_0^x V + A E_0^x V + \dots + A^i E_0^x V \end{aligned}$$

obs  $E_i^x V_j = \begin{cases} 0 & \text{if } i > j \\ \neq 0 & \text{if } i \leq j \end{cases}$  0  $\leq i, j \leq n$

obs  $V_{2n} = V_n + A V_n$  0  $\leq j \leq n$

Given 0  $\leq i, j \leq n$

For  $n-j > 1$

$$\begin{aligned} E_i^x A E_j^x V &\subseteq E_i^x A V_j \\ &\subseteq E_i^x V_{2n} \\ &= 0 \end{aligned}$$

so  $E_i^x A E_j^x = 0$

For  $n-j=1$  Suppose

$$E_i^x A E_j^x = 0$$

Then  $E_i^x A E_n^x = 0$  0  $\leq i \leq n-1$

so  $E_i^x A V_{2n} = 0$

Also  $E_i^x V_{2n} = 0$

Now  $E_i^x V_i = E_i^x (V_{2n} + A V_{2n})$   
 $= 0$  cont

so  $E_i^x A E_j^x \neq 0$

Suppose (i) - (iii) Set  $i = n$  in (iii) to find  $\bigoplus E_0^x V = V$

□

LEM 194 Given any dec  $\{E_i\}_{i=0}^N \neq V$  TRUE

(i) For  $0 \leq i \leq N$  both

$$V_0 + V_1 + \dots + V_i = E_0^* V + \dots + E_i^* V$$

$$V_i + V_{i+1} + \dots + V_N = E_i V + \dots + E_N V$$

(ii) For  $0 \leq i \leq N$  both

$$(A - \theta_i I) V_i \leq V_{i+1} + \dots + V_N \quad *$$

$$(A^* - \theta_i^* I) V_i \leq V_0 + \dots + V_{i-1} \quad **$$

Suppose (i), (ii) true

$$V_i = (E_0^* V + \dots + E_i^* V) \wedge (E_i V + \dots + E_N V)$$

pf (i)  $\rightarrow$  (ii) concerning  $*$ :

$$(A - \theta_i I) V_i \leq (A - \theta_i I) (V_{i+1} + \dots + V_N)$$

$$= (A - \theta_i I) (E_i V + \dots + E_N V)$$

$$= E_{i+1} V + \dots + E_N V$$

$$= V_{i+1} + \dots + V_N$$

\*\* is sum

$(i) \rightarrow (i)$  —

Iterate in \*

$$(A - \alpha_i I) \dots (A - \alpha_N I) v_i = 0$$

so

$$v_i \subseteq \{ w \in V \mid (A - \alpha_i I) \dots (A - \alpha_N I) w = 0 \}$$

$$= E_i V + \dots + E_N V$$

so

$$v_i + \dots + v_N \subseteq E_i V + \dots + E_N V$$

Both sides have dim  $N - i$  so =

Other eq is sim

□



LEM 195 Suppose  $\{v_i\}_{i=0}^N$  is a basis of  $V$  with which

$$A = \begin{pmatrix} a_{00} & a_{01} & \dots & 0 \\ * & & & \\ & & & a_{N-1,N} \\ & & & & a_{NN} \end{pmatrix} \quad A^* = \begin{pmatrix} a_{00}^* & a_{01}^* & \dots & * \\ & & & \\ & 0 & & a_{N-1,N}^* \\ & & & & a_{NN}^* \end{pmatrix}$$

"  $B$  "  $B^*$

then for  $0 \leq r < s \leq N$  TFAE

(i) the  $(i,j)$ -entry of  $B$  is 0 for  $s \leq i < j \leq N$  and  $0 \leq j < s$

(ii)  $E_i^* A E_j^* = 0$  for  $s \leq i < j \leq N$  and  $0 \leq j < s$

pf Put

$$V_i = \sum_{j=0}^N v_j$$

(i)  $\rightarrow$  (ii)

$$\begin{aligned} E_i^* A E_j^* V &\leq E_i^* A (E_0^* V + \dots + E_r^* V) \\ &= E_i^* A (V_0 + \dots + V_r) && \text{by L194} \\ &\leq E_i^* (V_0 + \dots + V_{s-1}) && \text{by (i)} \\ &= E_i^* (E_0^* V + \dots + E_{s-1}^* V) && \text{by L194} \\ &= 0 && \text{since } i \geq s \end{aligned}$$

(ii)  $\rightarrow$  (i)

$$\begin{aligned} AV_j &\leq A (V_0 + \dots + V_r) \\ &= A (E_0^* V + \dots + E_r^* V) \\ &= (E_0^* + \dots + E_r^*) A (E_0^* V + \dots + E_r^* V) \\ &= (E_0^* + \dots + E_r^*) A (E_0^* V + \dots + E_r^* V) \\ &\leq E_0^* V + \dots + E_r^* V \\ &= V_0 + \dots + V_r \end{aligned}$$

□

for  $0 \leq r < \infty$  TFAE

(i) the (i)st-entry of  $B^r$  is 0 for  $0 \leq i < r$  and  $r \leq j \leq n$

(ii)  $E_i A^r E_j = 0$  for  $0 \leq i < r$  and  $r \leq j \leq n$

pt Sim to pt of L195

□



## I Basics, cont.

We rephrase L195, 196 in terms of decomp's

LEM 197 Given a decomp  $\{V_i\}_{i=0}^N$  of  $V$  that satisfies the equiv cond (i), (ii) of L194

Then for  $0 \leq r < \lambda \leq N$  TFAE

$$(i) \quad (A - \alpha I)V_j \subseteq V_{j+1} + V_{j-1} \quad 0 \leq j \leq r$$

$$(ii) \quad E_i^* A E_j^* = 0 \quad \text{for } \lambda \leq i \leq N \text{ and } 0 \leq j \leq r$$

pf By L195 and disc above it. □

LEM 198 Given decomp  $\{V_i\}_{i=0}^N$  of  $V$  that  
satisfies the equiv conds (i), (ii) of L194

Then for  $0 \leq r < s \leq N$  TFAE

$$(i) \quad (A^* - e_j^* I) V_r \leq V_{r+1} + \dots + V_{j-1} \quad s \leq j \leq N$$

$$(ii) \quad E_i A^* E_j = 0 \quad \text{for } 0 \leq i \leq r \text{ and } s \leq j \leq N$$

pf By L196 and disc above L195. □



Prop 199 Assume

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad 0 \leq i, j \leq N$$

and put

$$u_i = T_i(A) E_0^* V \quad 0 \leq i \leq N$$

then

(i)  $\{u_i\}_{i=0}^N$  is dec of  $V$

(ii)  $(A - \theta_i I) u_i = u_{i+1} \quad 0 \leq i \leq N$

(iii)  $(A^* - \theta_i^* I) u_i \leq u_0 + \dots + u_{i-1} \quad 0 \leq i \leq N$

pf  $E_0^*$  is norm by L193

(i) By L192 (i)

(ii) By def of  $T_i$

(iii) For  $0 \leq i \leq N$

$$\begin{aligned} u_0 + u_1 + \dots + u_i &= E_0^* V + A E_0^* V + \dots + A^i E_0^* V && \text{by L192 (ii)} \\ &= E_0^* V + E_1^* V + \dots + E_i^* V && \text{by L193 (iii)} \end{aligned}$$

Also  $u_i + u_{i+1} + \dots + u_N = E_i^* V + \dots + E_N V \quad \text{by L192 (iii)}$

Now by L194

$$(A^* - \theta_i^* I) u_i \leq u_0 + u_1 + \dots + u_{i-1} \quad 0 \leq i \leq N$$

□

Here is converse to Prop 199

Prop 200 Assume  $\exists$  decomp  $\{u_i\}_{i=0}^N$  of  $V$  such that

both

$$(A - \theta_i I) u_i = u_{i+1} \quad 0 \leq i < N$$

$$(A^* - \theta_i^* I) u_i \subseteq u_0 + u_1 + \dots + u_{i-1} \quad 0 \leq i \leq N$$

Then

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad 0 \leq i, j \leq N$$

Moreover

$$u_i = \tau_i(A) E_0^* v \quad 0 \leq i \leq N$$

pf The decomp  $\{u_i\}_{i=0}^N$  satisfies the conds of L194

so L197 applies.

Result \* follows from L197

show \*\*:

$$(A^* - \theta_0^* I) u_0 = 0$$

$$\rightarrow u_0 \in E_0^* V$$

Now \*\* follows since

$$(A - \theta_i I) u_i = u_{i+1} \quad 0 \leq i < N$$

□

Prop 201 Assume both

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad 0 \leq i, j \leq N$$

$$E_i A^* E_j = 0 \quad \text{if } |j-i| > 1 \quad 0 \leq i, j \leq N$$

Put  $U_i = \pi_i(A) E_0^* V \quad 0 \leq i \leq N$

then

(i)  $\{U_i\}_{i=0}^N$  is dec of  $V$

(ii)  $(A - \theta_i I) U_i = U_{i+1} \quad 0 \leq i \leq N$

(iii)  $(A^* - \theta_i^* I) U_i = U_{i-1} \quad 0 \leq i \leq N$

pf (i), (ii) By Prop 199

(iii) By Prop 199 (ii), (iii) dec  $\{U_i\}_{i=0}^N$

satisfies conds of L199

Now apply L198



Here is a converse to Prop 201

6

Prop 202 Assume  $\exists$  decomp  $\{u_i\}_{i=0}^N$  of  $V$  such that

both

$$(A - \theta_i I) u_i = u_{i-1} \quad 0 \leq i \leq N,$$

$$(A^* - \theta_i^* I) u_i \subseteq u_{i-1} \quad 0 \leq i \leq N$$

Then both

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \quad 0 \leq i, j \leq N, \quad *$$

$$E_i A^* E_j = 0 \quad \text{if } j-i > 1 \quad 0 \leq i, j \leq N \quad **$$

Moreover

$$u_i = \pi_i(A) E_0^* v \quad 0 \leq i \leq N$$

pf

The decomp  $\{u_i\}_{i=0}^N$  satisfies the conds of L194

So L197, L198 apply.

Result follows.  $\square$

\*\*\* follows from Prop 200

In sert section

$I + \frac{1}{2}$  Some formula

Assumes:  $A$  in  $I_1$  and also both

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad 0 \leq i, j \leq N$$

$$E_i A^* E_j = 0 \quad \text{if } |j-i| > 1 \quad 0 \leq i, j \leq N$$

Recall  $E_0^*$  normalizing by L193

Consider dec  $\{u_i\}_{i=0}^N$  of  $V$  from Prop 201

For  $1 \leq i \leq N$

$$(A^* - \theta_i^* I) u_i \in U_{i-1}$$

$$(A - \theta_{i-1} I) u_{i-1} = u_i$$

So  $u_i$  is invar under

$$(A - \theta_{i-1} I)(A^* - \theta_i^* I)$$

Let

$$\psi_i = \text{eigval}$$

[caution: poss  $\psi_i = 0$ ]

For not conv  $\psi_0 = 0$   $\psi_N = 0$   
Fix  $0 \neq v \in E_0^* V$

For  $0 \leq i \leq N$  def

$$u_i = T_i(A) v$$

obs

$$0 \neq u_i \in U_i$$

So

$\{u_i\}_{i=0}^N$  is basis for  $V$

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Def 203  $\forall X \in \text{End } V$  let

$X^{\mathcal{U}}$  = matrix in  $\text{Mat}_{N+1}(\mathbb{F})$  that reps  $X$  w.r.t  $\{u_i\}_{i=0}^N$

Obs

$\mathcal{U}: \text{End } V \rightarrow \text{Mat}_{N+1}(\mathbb{F})$

is  $\mathbb{F}$ -alg iso

LEM 204 We have

$$A^{\mathcal{U}} = \begin{pmatrix} \theta_0 & & & & 0 \\ 1 & \theta_1 & & & \\ & 1 & \ddots & & \\ 0 & & & \ddots & \\ & & & & 1 & \theta_N \end{pmatrix}$$

$$A^{*\mathcal{U}} = \begin{pmatrix} \theta_0^* & & & & 0 \\ \varphi_1 & \theta_1^* & & & \\ & \varphi_2 & \ddots & & \\ 0 & & & \ddots & \\ & & & & \varphi_N & \theta_N^* \end{pmatrix}$$

LEM 205  $F_n$   $1 \leq j \leq N$  TFAE

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$$(i) \quad \varphi_j \neq 0$$

$$(ii) \quad E_{j^*} A^* E_j \neq 0$$

$$(iii) \quad (A^* - \theta_j^* I) u_j \neq u_{j^*}$$

pf (i)  $\Leftrightarrow$  (ii) Use L196 and L209

(i)  $\Leftrightarrow$  (iii) by Def of  $\varphi_j$

□

LEM 206 For  $0 \leq r \leq N$

the matrix  $E_r$  is lower triangular with entries

$$\frac{T_j(\theta_r) z_{j-1}(\theta_r)}{T_r(\theta_r) z_{N-r}(\theta_r)}$$

for  $0 \leq j \leq r \leq N$

pf Use L 104 and

$$E_r = \prod_{\substack{0 \leq \lambda \leq N \\ \lambda \neq r}} \frac{A - \theta_\lambda I}{\theta_r - \theta_\lambda}$$

□



Ex 2 of For N=2

$$E_0^4 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\theta_0 - \theta_1} & 0 & 0 \\ \frac{1}{(\theta_0 - \theta_1)(\theta_0 - \theta_2)} & 0 & 0 \end{pmatrix}$$

$$E_1^4 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\theta_1 - \theta_0} & 1 & 0 \\ \frac{1}{(\theta_1 - \theta_0)(\theta_1 - \theta_2)} & \frac{1}{\theta_1 - \theta_2} & 0 \end{pmatrix}$$

$$E_2^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{(\theta_2 - \theta_0)(\theta_2 - \theta_1)} & \frac{1}{\theta_2 - \theta_1} & 1 \end{pmatrix}$$

LEM 208 For  $0 \leq r \leq N$

the matrix  $(E_r^*)^{\dagger}$  is upper triangular with entries

$$\frac{\psi_{i_1} \cdots \psi_{i_j} \tau_i^*(\theta_r^*) \gamma_{N-j}^*(\theta_r^*)}{\tau_r^*(\theta_r^*) \gamma_{N-r}^*(\theta_r^*)}$$

for  $0 \leq i \leq j \leq N$

pf Use L 104 and

$$E_r^* = \prod_{\substack{0 \leq i \leq N \\ i \neq r}} \frac{A - \theta_i^* I}{\theta_r^* - \theta_i^*}$$

□

Next goal: Find  $a_i, a_i^*$  in terms of the

$$\theta_j, \theta_j^*, \varphi_j$$

Suppose  $N=0$  then  $A = \theta_0 I, A^* = \theta_0^* I$

So 
$$a_0 = \theta_0 \quad a_0^* = \theta_0^*$$

LEM 210 Assume  $N \geq 1$

(i)

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*}$$

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*}$$

$1 \leq i \leq N-1$

$$a_N = \theta_N + \frac{\varphi_N}{\theta_N^* - \theta_{N-1}^*}$$

(ii)

$$a_0^* = \theta_0^* + \frac{\varphi_1}{\theta_0 - \theta_1}$$

$$a_i^* = \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}}$$

$1 \leq i \leq N-1$

$$a_N^* = \theta_N^* + \frac{\varphi_N}{\theta_N - \theta_{N-1}}$$

pf (i) For  $0 \leq i \leq N$

$$a_i = \text{tr}(A E_i^*) = \text{tr}(A^T (E_i^*)^T)$$

Compute this using L204, L208

(ii) Sim

□

Ex 209

Fn N=2

$$E_0^{*4} = \begin{pmatrix} 1 & \frac{\varphi_1}{\theta_0^* - \theta_1^*} & \frac{\varphi_1 \varphi_2}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_1^{*4} = \begin{pmatrix} 0 & \frac{\varphi_1}{\theta_1^* - \theta_0^*} & \frac{\varphi_1 \varphi_2}{(\theta_1^* - \theta_0^*)(\theta_1^* - \theta_2^*)} \\ 0 & 1 & \frac{\varphi_2}{\theta_1^* - \theta_2^*} \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_2^{*4} = \begin{pmatrix} 0 & 0 & \frac{\varphi_1 \varphi_2}{(\theta_2^* - \theta_0^*)(\theta_2^* - \theta_1^*)} \\ 0 & 0 & \frac{\varphi_2}{\theta_2^* - \theta_1^*} \\ 0 & 0 & 1 \end{pmatrix}$$

LEM 211 For  $1 \leq i \leq N$

$\varphi_i$  is equal to each of the following:

$$(i) \quad (\theta_i^* - \theta_{i-1}^*) \sum_{h=0}^{i-1} (\theta_h - a_h)$$

$$(ii) \quad (\theta_{i-1}^* - \theta_i^*) \sum_{h=i}^N (\theta_h - a_h)$$

$$(iii) \quad (\theta_i - \theta_{i-1}) \sum_{h=0}^{i-1} (\theta_h^* - a_h^*)$$

$$(iv) \quad (\theta_{i-1} - \theta_i) \sum_{h=i}^N (\theta_h^* - a_h^*)$$

pf Assume  $N \geq 1$  else triv.

(i) use L210

(ii) Use (i) and  $\theta_0 + \dots + \theta_N = a_0 + \dots + a_N$

(iii), (iv) Sim.

□



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1

I + 1/2 cont.

Def 2.12 Define

$$J_0 = \varphi_0 - (\theta_i^x - \theta_0^x) (\theta_{i+1} - \theta_n)$$

$i \in N$

$$J_0 = 0$$

$$J_{NH} = 0$$

thm 2.13 Assume  $N \geq 2$ . Then

$$(i) \sum_{i=0}^{N-2} E_N A^* E_i E_0^* (\theta_i - \theta_{N-1}) = (\beta_1 - \beta_N) E_N E_0^*$$

$$(ii) \sum_{i=2}^N E_0^* A E_i^* E_N (\theta_i^* - \theta_1^*) = (\beta_1 - \beta_N) E_0^* E_N$$

pf (i) Recall

$$I = \sum_{i=0}^N E_i^*$$

Mult each term on right by  $A E_0^*$

Simplify using

$$E_0^* A E_0^* = a_0 E_0^*$$

$$E_i^* A E_0^* = 0 \quad 2 \leq i \leq N$$

Get

$$A E_0^* = a_0 E_0^* + E_1^* A E_0^* \tag{1}$$

In (1) mult each term on left by  $A^*$

Get

$$A^* A E_0^* = a_0 \theta_0^* E_0^* + \theta_1^* E_1^* A E_0^* \tag{2}$$

Recall

$$I = \sum_{i=0}^N E_i$$

Mult each term on left by  $E_N A^*$

Simplify using

$$E_N A^* E_N = a_N^* E_N$$

Get

$$E_N A^* = a_N^* E_N + \sum_{i=0}^{N-1} E_N A^* E_i \quad (3)$$

In (3) mult each term on right by A

Get

$$E_N A^* A = \theta_N a_N^* E_N + \sum_{i=0}^{N-1} \theta_i E_N A^* E_i \quad (4)$$

Consider eq which is

$$\theta_i^* E_N (1) - E_N (2) - \theta_{N-1} (3) E_0^* + (4) E_0^*$$

Get

$$E_N E_0^* \left( (\theta_0^* - \theta_i^*) a_0 + (\theta_{N-1} - \theta_N) a_N^* + \theta_N \theta_i^* - \theta_{N-1} \theta_0^* \right) = \sum_{i=0}^{N-2} E_N A^* E_i E_0^* (\theta_i^* - \theta_{N-1})$$

Simplify the  $E_N E_0^*$  coef using

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_i^*}$$

$$a_N^* = \theta_N^* + \frac{\varphi_N}{\theta_N - \theta_{N-1}}$$

$$\varphi_1 = g_1 + (\theta_i^* - \theta_0^*) (\theta_0 - \theta_N)$$

$$\varphi_N = g_N + (\theta_N^* - \theta_0^*) (\theta_{N-1} - \theta_N)$$

to get

$$g_1 - g_N$$

(1.1) Sim

□



## II the split decomp and the parameter array

Assumptions:

Field  $\mathbb{F}$  arb

$N =$  nonneg integer

$V =$  v.s. over  $\mathbb{F}$   $\dim N+1$

Given LS

$$\Phi = (A, \{E_i\}_{i=0}^N, A^* \{E_i^*\}_{i=0}^N)$$

on  $V$  with equal seq  $\{e_i\}_{i=0}^N$  and dual equal  
seq  $\{e_i^*\}_{i=0}^N$



By the  $\Phi$ -split decomp of  $V$  we mean

the decomp  $\{U_i\}_{i=0}^N$  of  $V$  from Prop 2.01

We have

$$U_i = T_i(A) E_0^* V \quad 0 \leq i < N$$

$$U_i = T_{N-i}^*(A^*) E_N V \quad \dots$$

$$U_i = (E_0^* V + \dots + E_i^* V) \cap (E_i V + \dots + E_N V) \quad \dots$$

$$U_0 + \dots + U_i = E_0^* V + \dots + E_i^* V \quad \dots$$

$$U_i + \dots + U_N = E_i V + \dots + E_N V \quad \dots$$

$$(A - \theta_i I) U_i = U_{i+1} \quad 0 \leq i < N$$

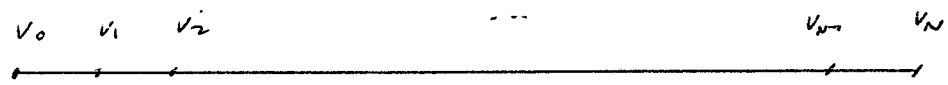
$$(A^* - \theta_i^* I) U_i = U_{i-1} \quad 0 \leq i < N$$

Obs  $\{U_{N-i}\}_{i=0}^N$  is split decomp for LS

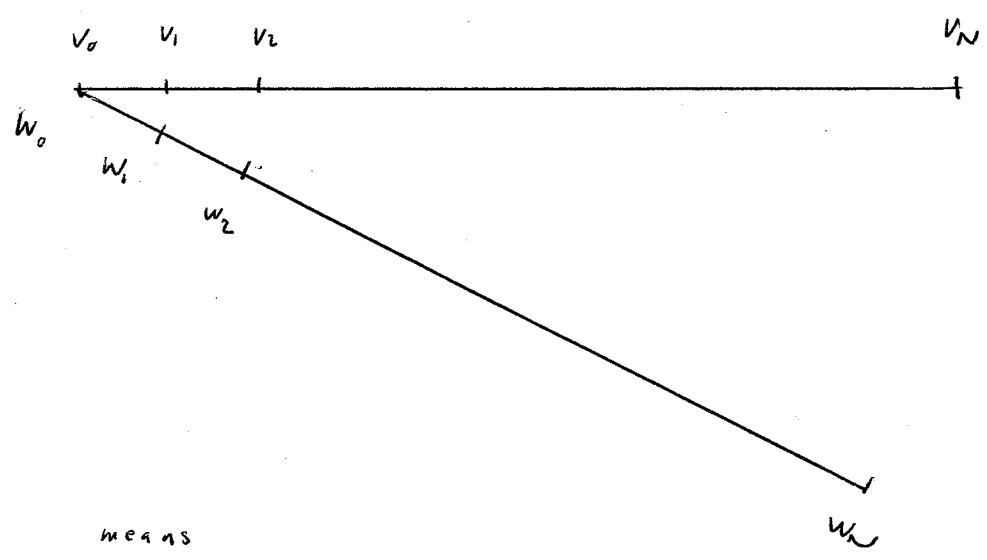
$$(A^*, \{E_{N-i}^*\}_{i=0}^N, A, \{E_{N-i}\}_{i=0}^N) = \Phi^{* \downarrow \downarrow}$$

Let us represent a decomp  $\{v_i\}_{i=0}^N$  of  $V$

by a Line segment



Given decomp  $\{w_i\}_{i=0}^N$  of  $V$

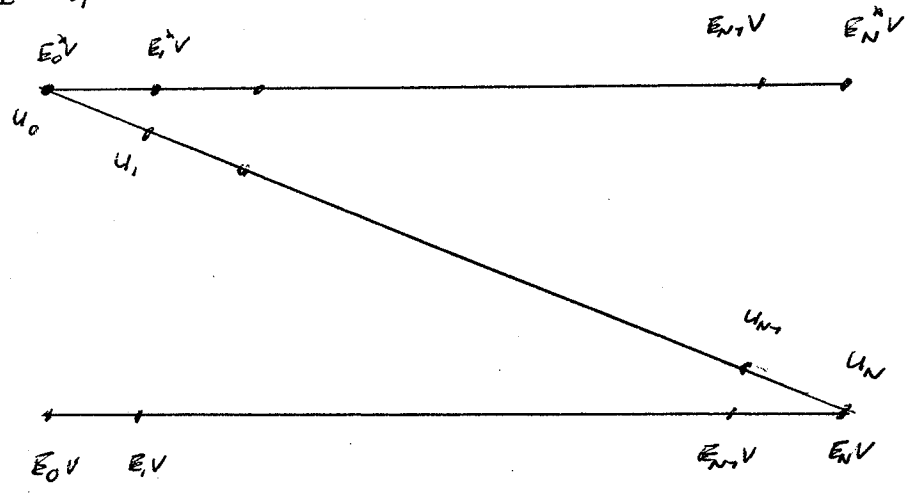


means

$$v_0 + v_1 + \dots + v_i = w_0 + w_1 + \dots + w_i$$

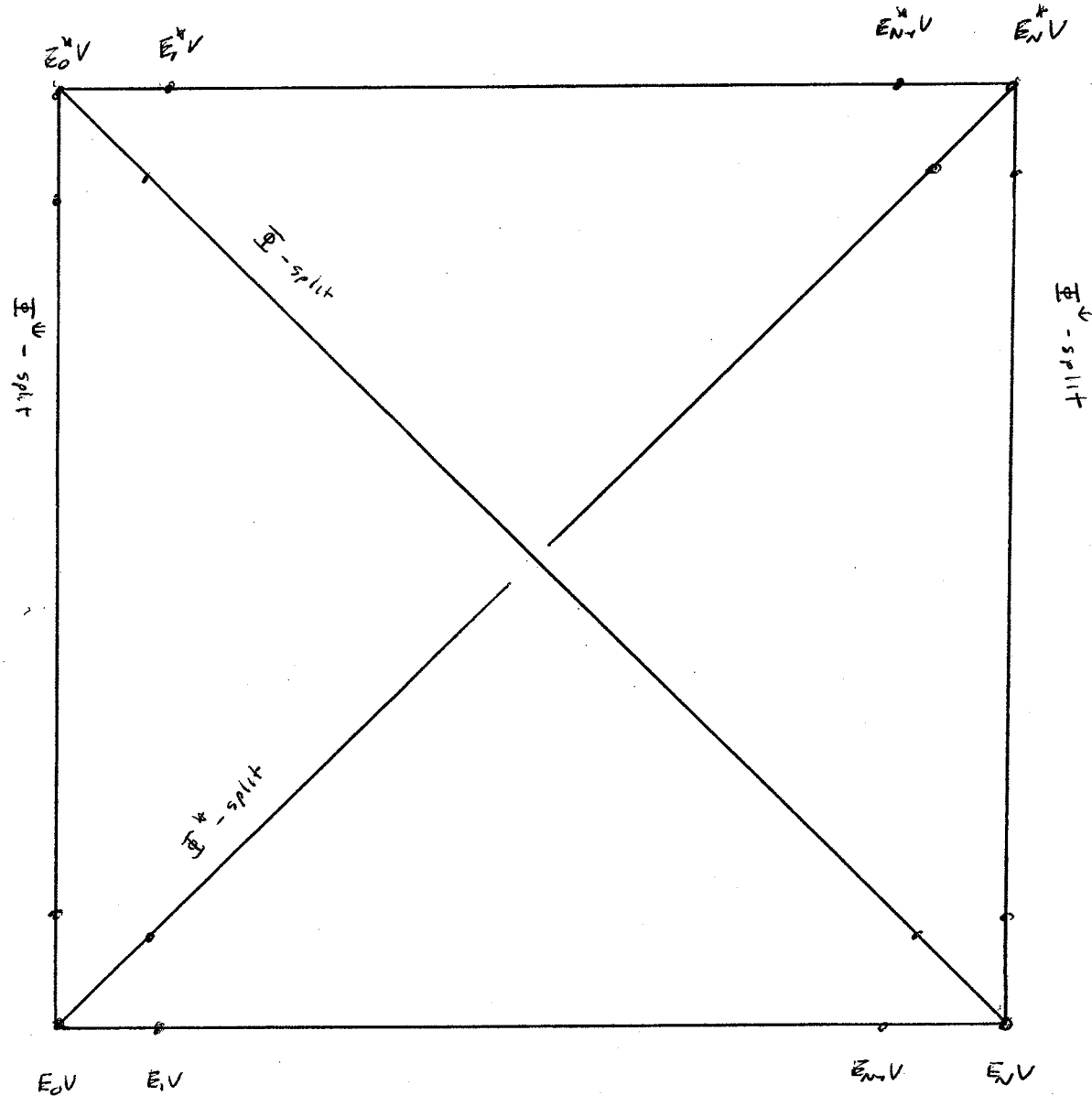
$0 \leq i \leq N$

$\Phi$ -split decomp  $\{u_i\}_{i=0}^N$  satisfies



We also have split decomp's for

$$\mathbb{F}^n, \mathbb{F}^u, \mathbb{F}^w$$



DEF 214 By the 1st split sequence for  $\Phi$

we mean  $\{\varphi_i\}_{i=1}^N$  from Section I 1/2

By L205

$$\varphi_i \neq 0 \quad 1 \leq i \leq N$$

Def 215 Let  $\{\varphi_i\}_{i=1}^N$  denote the 1st split sequence for  $\Phi^\psi$ . Call  $\{\varphi_i\}_{i=1}^N$  the 2nd split sequence for  $\Phi$ . Obs

$$\varphi_i \neq 0 \quad 1 \leq i \leq N$$

For not conv  $\varphi_0 = 0 \quad \varphi_{N+1} = 0$

Note 216 Given  $0 \neq v \in E_0^* V$  Obs

$\{\gamma_i(A)v\}_{i=0}^N$  is basis for  $V$

Rel the basis

$$A: \begin{pmatrix} \varphi_N & & & 0 \\ 1 & \varphi_{N-1} & & \\ & 1 & \ddots & \\ 0 & & 1 & \varphi_0 \end{pmatrix} \quad A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & 0 \\ \theta_1^* & \varphi_2 & & \\ & \ddots & \ddots & \\ 0 & & & \varphi_N \\ & & & \theta_N^* \end{pmatrix}$$

Notation Given iso of v.s.

$$\sigma: V \rightarrow V'$$

Abb

$$X^\sigma = \sigma X \sigma^{-1} \quad \forall X \in \text{End} V$$

So

$$\begin{aligned} \text{End} V &\rightarrow \text{End} V' \\ X &\rightarrow X^\sigma \end{aligned}$$

is iso of  $\mathbb{F}$ -algebras

For our LS  $\mathbb{E} = (A, \{E_i\}_{i=0}^N, A^* \{E_i^*\}_{i=0}^N)$  on  $V$

$$\mathbb{E}^\sigma := (A^\sigma, \{E_i^\sigma\}_{i=0}^N, A^{*\sigma} \{E_i^{*\sigma}\}_{i=0}^N)$$

is LS on  $V'$

Given LS  $\mathbb{E}'$  on  $V'$  By an iso of LS from  $\mathbb{E}$  to  $\mathbb{E}'$

we mean an iso of v.s.  $\sigma: V \rightarrow V'$  st  $\mathbb{E}^\sigma = \mathbb{E}'$

Call  $\mathbb{E}, \mathbb{E}'$  isomorphic whenever  $\exists$  iso of LS from  $\mathbb{E}$  to  $\mathbb{E}'$

LEM 217 TFAE

(i)  $\Phi, \Phi'$  are iso(ii)  $\Phi, \Phi'$  have same equal req, dual equal req, 1st split req(iii)  $\dots$   $\dots$  2nd split reqpf (i)  $\Leftrightarrow$  (ii) By L204(ii)  $\Leftrightarrow$  (iii) Apply L204 to  $\Phi^{\downarrow}, \Phi'^{\downarrow}$ 

□

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By the parameter array of  $\Phi$  we mean  
the sequence

$$\left( \begin{array}{cccc} \{\theta_i\}_{i=0}^N & \{\theta_i^*\}_{i=0}^N & \{\psi_i\}_{i=1}^N & \{\phi_i\}_{i=1}^N \end{array} \right)$$

|
|
|
|

equal
dual equal
1st split
2nd split

req.
req.
req.
req.

Cor 219 Two LS over  $\mathbb{F}$  are iso iff they have  
the same parameter array.





II contL 220  $F_n \quad 1 \leq i \leq n$  $\phi_i$  is equal to each of the following

$$(i) \quad (\theta_i^* - \theta_{i-1}^*) \sum_{h=0}^{i-1} (\theta_{N-h} - a_h)$$

$$(ii) \quad (\theta_{i-1}^* - \theta_i^*) \sum_{h=i}^N (\theta_{N-h} - a_h)$$

$$(iii) \quad (\theta_{N-i} - \theta_{N-i+1}) \sum_{h=0}^{i-1} (\theta_h^* - a_{N-h}^*)$$

$$(iv) \quad (\theta_{N-i+1} - \theta_{N-i}) \sum_{h=i}^N (\theta_h^* - a_{N-h}^*)$$

pf Recall  $\{\phi_i\}_{i=1}^N$  is 1st split rep for  $\Phi^{\psi}$

 $F_n \quad 0 \leq i \leq n$ 

$$\theta_0(\Phi^{\psi}) = \theta_{N-1}$$

$$\theta_1^*(\Phi^{\psi}) = \theta_1^*$$

$$a_2(\Phi^{\psi}) = a_2$$

$$a_3^*(\Phi^{\psi}) = a_{N-3}^*$$

Now apply L 211 to  $\Phi^{\psi}$

Prop 221 The parameter arrays of  
 $\Phi, \Phi^{\downarrow}, \Phi^*, \Phi^{\downarrow}$

are related as follows

LS	PA
$\Phi$	$(\theta_i, \theta_i^*, \varphi_i, \phi_i)$
$\Phi^{\downarrow}$	$(\theta_{N-i}, \theta_i^*, \phi_i, \varphi_i)$
$\Phi^*$	$(\theta_i^*, \theta_i, \varphi_i, \phi_{N-i})$
$\Phi^{\downarrow}$	$(\theta_i, \theta_{N-i}^*, \phi_{N-i}, \varphi_{N-i})$

pf Use L211, L220 and Def 184

$\Phi^{\downarrow}$  : By def of  $\phi_i, \varphi_i$  ✓

$\Phi^*$  : Compare L211 (i), (iii) to get  
 $\varphi_i(\Phi^*) = \varphi_i$   
 Compare L220 (i), (iii) to get  
 $\phi_i(\Phi^*) = \phi_{N-i}$

$\Phi^{\downarrow}$  : Use above data and  $\Phi^{\downarrow} = ((\Phi^*)^{\downarrow})^*$

□