Leonard pairs and the terminating branch of the Askey Scheme

We saw that the Krawtchouk polys $\{K_i(x,p,N)\}_{i=0}^{N}$ corresp to LS whose eigval sequence and dual eigval seq is in arith progression.

We now classify all the LS Φ.

Each sol Φ corresponds to a poly sequence $\{p_j\}_{j=0}^{N}$ from the Askey Scheme, yielding the

"terminating branch of Askey Scheme"

q-Racah	Racah
q-Hahn	Hahn
dual q-Hahn	dual Hahn
q-Krawtchouk	Kraw
dual q-Krawtchouk	Bannai/Ito
quantum q-Kraw	orphans (char $\mathbb{F}=2$ $N=3$ only)
affine q-Kraw	

If Φ has eigval seq $\{\theta_i\}_{i=0}^{N}$ and dual eigval seq $\{\theta_i^*\}_{i=0}^{N}$ then

$$p_j \in \mathbb{F} \sum_{i=0}^{j} \frac{(x-\theta_0)\cdots(x-\theta_{i-1})(\theta_j^*-\theta_0^*)\cdots(\theta_j^*-\theta_{i-1}^*)}{\varphi_1\varphi_2\cdots\varphi_i} \qquad 0 \le j \le N$$

For some nonzero scalars $\{\varphi_i\}_{i=1}^{N}$ called the 1st split sequence

Also

$$\rho_j \in \mathbb{F} \sum_{i=0}^{j} \frac{(x-\theta_N)\cdots(x-\theta_{N-i+1})(\theta_j^*-\theta_0^*)\cdots(\theta_j^*-\theta_{i-1}^*)}{\phi_1\phi_2\cdots\phi_i} \qquad 0 \le j \le N$$

For some non 0 scalars $\{\phi_i\}_{i=1}^N$ called 2nd split seq

Classification of LS Outline

I Basics

- the a_i, a_i^*
- the antiaut $\dagger$
- Normalizing idempotents

II the split decomp and parameter array

III Recurrent sequences

IV the tridiag relations

V Conclusion

I Basics

Throughout this section assume:

field $\mathbb{F}$ arb

N = non neg integer

V = vector space over $\mathbb{F}$ of dim $N+1$

Given

$$\left(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N\right)$$

s.t.

- Each of A, A^* is MF el of End V
- $\{E_i\}_{i=0}^N$ is ordering of prim ids of A
- $\{E_i^*\}_{i=0}^N$ . . . A^*

For $0 \le i \le N$ let

θ_i = eigval of A for E_i

θ_i^* = eigval of A^* for E_i^*

let

$\mathcal{D}$ = subalg of End V gen by A

$\mathcal{D}^*$ = . . . A^*

For $0 \le i \le N$

$\tau_i = (x - \theta_0) \cdots (x - \theta_{i-1})$ $\qquad$ $\tau_i^* = (x - \theta_0^*) \cdots (x - \theta_{i-1}^*)$

$\eta_i = (x - \theta_N) \cdots (x - \theta_{N-i+1})$ $\qquad$ $\eta_i^* = (x - \theta_N^*) \cdots (x - \theta_{N-i+1}^*)$

Def 182 By a <u>decomposition of V</u>. we mean a sequence $\{V_i\}_{i=0}^{N}$ of subspaces, of V s.t.

$$\dim V_i = 1 \qquad i=0,1,\ldots,N$$

$$V = \sum_{i=0}^{N} V_i \qquad (ds)$$

For not conv put

$$V_{-1} = 0 \qquad V_{N+1} = 0$$

Ex 183 Each of

$$\{E_i V\}_{i=0}^{N} \qquad \{E_i^* V\}_{i=0}^{N}$$

is a decomp of V

Recall $\operatorname{tr} E_i = 1$ $\quad \operatorname{tr} E_i^* = 1$ $\quad 0 \le i \le N$

Def 184 Define

$$a_i = \operatorname{tr}(A E_i^*) \qquad 0 \le i \le N$$

$$a_i^* = \operatorname{tr}(A^* E_i)$$

LEM 185

(i) $$\sum_{i=0}^{N} \theta_i = \sum_{i=0}^{N} a_i$$

(ii) $$\sum_{i=0}^{N} \theta_i^* = \sum_{i=0}^{N} a_i^*$$

pf (i)

$$\begin{aligned} LHS &= \operatorname{tr}(A) \\ &= \operatorname{tr}\left(A \sum_{i=0}^{N} E_i^*\right) \\ &= RHS \end{aligned}$$

(ii) Sim $\square$

LEM For $0 \le i \le N$

(i) $E_i^* A E_i^* = a_i E_i^*$

(ii) $E_i A^* E_i = a_i^* E_i$

PROOF: Routine using Def 184

LEM 18.6 Assume

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad 0 \leq i,j \leq N$$

then

$$A^i E_0^* A^j \qquad 0 \leq i,j \leq N$$

is a basis for $\operatorname{End} V$

pf Like Prop 17.4 $\square$

LEM 187 Ref to Lem 186

each of following is gen set for End V

(ii) A, E_0^*

(iii) A, A^*

pf Like Cor 175

$\square$

An <u>anti aut</u> of $\mathrm{End}V$ is an iso of

$\mathbb{F}$- vector spaces $\sigma: \mathrm{End}V \to \mathrm{End}V$ s.t.

$$(XY)^\sigma = Y^\sigma X^\sigma \qquad \forall X,Y \in \mathrm{End}V$$

LEM 188 Ref to LEM 186

$\exists$ unique anti aut $\dagger$ of $\mathrm{End}V$ s.t.

$$A^\dagger = A \qquad A^{*\dagger} = A^* \qquad (*)$$

Moreover
$$(X^\dagger)^\dagger = X \qquad \forall X \in \mathrm{End}V$$

pf Pick

$$0 \neq v_i \in E_i^* V \qquad 0 \le i \le N$$

So $\{v_i\}_{i=0}^N$ is basis for V

$\forall X \in \mathrm{End}V$ let

$$X^b = \text{matrix rep } X \text{ rel } \{v_i\}_{i=0}^N$$

$$b: \mathrm{End}V \longrightarrow \mathrm{Mat}_{N+1}(\mathbb{F})$$
$$X \to X^b$$

is $\mathbb{F}$-alg iso

Write

$$B = A^b \qquad B^* = A^{*b}$$

$B =$ irred tridiag

$B^* = \mathrm{diag}(\theta_i^*)_{i=0}^N$

For $0 \le i \le N$ def

$$k_i = \frac{B_{01} B_{12} \cdots B_{i-1,i}}{B_{10} B_{21} \cdots B_{i,i-1}}$$

def

$$K = \mathrm{diag}(k_i)_{i=0}^N$$

Recall

$$B^t = K B K^{-1}$$

so

$$K^{-1} B^t K = B$$

Define

$$\gamma : \quad \mathrm{Mat}_{N+1}(\mathbb{F}) \longrightarrow \mathrm{Mat}_{N+1}(\mathbb{F})$$

$$X \longrightarrow K^{-1} X^t K$$

γ is anti aut that fixes each of B, B^*

Now Composition

$$\dagger : \quad \mathrm{End}\,V \xrightarrow{\ b\ } \mathrm{Mat}_{N+1}(\mathbb{F}) \xrightarrow{\ \gamma\ } \mathrm{Mat}_{N+1}(\mathbb{F}) \xrightarrow{\ b^{-1}\ } \mathrm{End}\,V$$

is anti aut of End V that fixes A, A^*

Uniqueness : Suppose σ is antiaut of End V

that fixes A, A^*

then $\dagger \circ \sigma^{-1}$ is aut of End V that fixes A, A^*

A, A^* gen End V so $\dagger \circ \sigma^{-1} = 1$ so $\dagger = \sigma$

To get $(*)$ note

$\dagger \circ \dagger$ is aut of End V that fixes A, A^*

so $\dagger \circ \dagger = 1$ $\square$

thm 189 (3/4 theorem)

Consider the following four assumptions

(i) $$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j>1 \\ \neq 0 & \text{if } i-j=1 \end{cases} \qquad 0 \le i,j \le N$$

(ii) $$E_i^* A E_j^* = \begin{cases} 0 & \text{if } j-i>1 \\ \neq 0 & \text{if } j-i=1 \end{cases} \qquad \text{''}$$

(iii) $$E_i A^* E_j = \begin{cases} 0 & \text{if } i-j>1 \\ \neq 0 & \text{if } i-j=1 \end{cases} \qquad \text{''}$$

(iv) $$E_i A^* E_j = \begin{cases} 0 & \text{if } j-i>1 \\ \neq 0 & \text{if } j-i=1 \end{cases} \qquad \text{''}$$

Assume at least 3 of (i) – (iv) hold.

Then they all hold, ie

$$(A, E_i, A^*, E_i^*) \text{ is LS on } V$$

pf Interchanging A, A^* if nec WLOG (i), (ii) hold.

Now by L 188 $\exists$ antiaut $\dagger$ of End V that

$$A^\dagger = A, \qquad A^{*\dagger} = A^*$$

obs
$$E_i^\dagger = E_i \qquad (E_i^*)^\dagger = E_i^* \qquad 0 \le i \le N$$

For $0 \le i,j \le N$

$$(E_i A^* E_j)^\dagger = E_j A^* E_i$$

So
$$E_i A^* E_j = 0 \quad \text{iff} \quad E_j A^* E_i = 0$$

So
$$\text{(iii)} \iff \text{(iv)}$$

Result follows. $\square$

Lecture 26 Wed Nov 10

I. Basics, cont.

Def 190 The idempotent E_0^* is called <u>normalizing</u> whenever

$$E_i E_0^* \neq 0 \qquad 0 \le i \le N$$

LEM 191 TFAE

(i) E_0^* is normalizing

(ii) $\mathcal{D} E_0^*$ has dim $N+1$

(iii) $\{A^i E_0^*\}_{i=0}^N$ are lin indep

(iv) $\forall\, x \in \mathcal{D}$

$$x E_0^* = 0 \to x = 0$$

(v) $\mathcal{D} E_0^* V = V$

(vi) $\forall\, 0 \neq v \in V$ the map

$$\mathcal{D} \to V$$
$$x \to xv$$

is a bijection

pf Routine

$\square$

LEM 192 Assume E_0^* is normalizing

and define

$$U_i = T_i(A) E_0^* V \qquad 0 \le i \le N$$

then

(i) $\{U_i\}_{i=0}^N$ is a dec of V

( ) $(A - \theta_i I) U_i = U_{i+1}$ $\quad 0 \le i \le N$

(ii) $U_0 + U_1 + \cdots + U_i = E_0^* V + A E_0^* V + \cdots + A^i E_0^* V \qquad 0 \le i \le N$

(iii) $U_i + U_{i+1} + \cdots + U_N = E_i V + E_{i+1} V + \cdots + E_N V \qquad 0 \le i \le N$

pf (i) Each U_i has dim 1 by L191 (iv)

$$V = \sum_{i=0}^N U_i \text{ by L191 (v)}$$

(ii) Recall $\deg T_i = i \qquad 0 \le i \le N$

(iii) Both sides dim $N-i+1$

Suf to show $\supseteq$

$$RHS = \{ w \in V \mid (A - \theta_i I) \cdots (A - \theta_N I) w = 0 \}$$

$$\subseteq LHS$$

$\square$

LEM 193 TFAE

(i) $$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j>1 \\ \neq 0 & \text{if } i-j=1 \end{cases} \qquad 0 \le i,j \le N$$

(ii) $\exists$ poly sequence $\{p_i\}_{i=0}^N$ in $\mathbb{F}[x]$ s.t.

$$E_i^* V = p_i(A) E_0^* V \qquad 0 \le i \le N$$

(iii) For $0 \le i \le N$

$$E_0^* V + \cdots + E_i^* V = E_0^* V + A E_0^* V + \cdots + A^i E_0^* V$$

Suppose (i) – (iii) then E_0^* is normalizing

pf

(i) $\to$ (ii) Fix

$$0 \neq v_i \in E_i^* V \qquad 0 \le i \le N$$

So $\{v_i\}_{i=0}^N$ is basis for V

Let $B \in \mathrm{Mat}_{N+1}(\mathbb{F})$ rep A wrt $\{v_i\}_{i=0}^N$

$$B_{ij} = \begin{cases} 0 & \text{if } i-j>1 \\ \neq 0 & \text{if } i-j=1 \end{cases} \qquad 0 \le i,j \le N$$

"upper Hessenberg"

Define $\{p_i\}_{i=0}^N$ in $\mathbb{F}[x]$ by $p_0 = 1$ and

$$x p_j = \sum_{i=0}^{j+1} B_{ij} p_i \qquad 0 \le j \le N-1$$

then p_j has deg exactly j for $0 \le j \le N$

By const

$$v_i = p_i(A) v_0 \qquad 0 \le i \le N$$

So $$E_i^* V = p_i(A) E_0^* V \qquad 0 \le i \le N$$

(ii) → (iii) $\quad E_0^*$ normalizing

Both sides dim $i+1$

Suffices to show $\subseteq$

Follows since $\deg p_j = j$ for $0 \le j \le N$

(iii) → (i) $\quad$ Put

$$V_i = E_0^*V + E_1^*V + \cdots + E_i^*V \qquad 0 \le i \le N$$
$$= E_0^*V + AE_0^*V + \cdots + A^iE_0^*V$$

obs

$$E_i^* V_j = \begin{cases} 0 & \text{if } i > j \\ \neq 0 & \text{if } i \le j \end{cases} \qquad 0 \le i, j \le N$$

obs

$$V_{j+1} = V_j + AV_j \qquad 0 \le j \le N-1$$

Given $0 \le i, j \le N$

For $i - j > 1$

$$E_i^* A E_j^* V \subseteq E_i^* A V_j$$
$$\subseteq E_i^* V_{j+1}$$
$$= 0$$

so $\quad E_i^* A E_j^* = 0$

For $i - j = 1$ $\quad$ Suppose

$$E_i^* A E_j^* = 0$$

Then $\quad E_i^* A E_h^* = 0 \qquad 0 \le h \le i-1$

so $\quad E_i^* A V_{i-1} = 0$

Also $\quad E_i^* V_{i-1} = 0$

Now $\quad E_i^* V_i = E_i^* (V_{i-1} + AV_{i-1})$

$$= 0 \quad \text{cont}$$

So $\quad E_i^* A E_j^* \neq 0$

Suppose (i) – (iii) $\quad$ Set $i = N$ in (iii) to find $\mathcal{D} E_0^* V = V$

$\square$

LEM 19.4 Given any dec $\{V_i\}_{i=0}^N$ of V TFAE

(i) For $0 \le i \le N$ both

$$V_0 + V_1 + \cdots + V_i = E_0^* V + \cdots + E_i^* V$$

$$V_i + V_{i+1} + \cdots + V_N = E_i V + \cdots + E_N V$$

(ii) For $0 \le i \le N$ both

$$(A - \theta_i I) V_i \subseteq V_{i+1} + \cdots + V_N \qquad *$$

$$(A^* - \theta_i^* I) V_i \subseteq V_0 + \cdots + V_{i-1} \qquad **$$

Suppose (i), (ii) then

$$V_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_N V)$$

pf (i) → (ii) Concerning $*$:

$$(A - \theta_i I) V_i \subseteq (A - \theta_i I)(V_i + \cdots + V_N)$$

$$= (A - \theta_i I)(E_i V + \cdots + E_N V)$$

$$= E_{i+1} V + \cdots + E_N V$$

$$= V_{i+1} + \cdots + V_N$$

$**$ is sim

(ii) → (i) —

Iterate in *

$$(A - e_i I) \cdots (A - e_N I) V_i = 0$$

so

$$V_i \subseteq \{ w \in V \mid (A - e_i I) \cdots (A - e_N I) w = 0 \}$$

$$= E_i V + \cdots + E_N V$$

So

$$V_i + \cdots V_N \subseteq E_i V + \cdots + E_N V$$

Both sides have dim $N - i + 1$ so $=$

Other eq is sim

$\square$

Comments on L194

Given a decomp $\{V_i\}_{i=0}^N$ of V that sats

L194 (i), (ii)

Fix $0 \neq v_i \in V_i$ $\quad 0 \le i \le N$

So $\{v_i\}_{i=0}^N$ is basis for V

rel the basis

$$A: \begin{pmatrix} \theta_0 & & & 0 \\ & \theta_1 & & \\ & & \ddots & \\ * & & & \theta_N \end{pmatrix} \qquad A^*: \begin{pmatrix} \theta_0^* & & & * \\ & \theta_1^* & & \\ & & \ddots & \\ 0 & & & \theta_N^* \end{pmatrix} \qquad (*)$$

Conversely let $\{v_i\}_{i=0}^N$ denote any basis for V with respect to which A, A^* are rep by $*$.

Define $V_j = \mathbb{F} v_j \qquad 0 \le j \le N$

Then $\{V_j\}_{j=0}^N$ is dec of V that satisfies L194 (i), (ii)

LEM 19 5 Suppose $\exists$ basis $\{v_i\}_{i=0}^N$ of V wrt which

$$A: \underbrace{\begin{pmatrix} \theta_0 & & & 0 \\ & \theta_1 & & \\ & * & \ddots & \\ & & & \theta_N \end{pmatrix}}_{B} \qquad A^*: \underbrace{\begin{pmatrix} \theta_0^* & & & * \\ & \theta_1^* & & \\ & 0 & \ddots & \\ & & & \theta_N^* \end{pmatrix}}_{B^*}$$

then for $0 \le r < s \le N$ TFAE

(i) the (i,j)-entry of B is 0 for $s \le i \le N$ and $0 \le j \le r$

(ii) $E_i^* A E_j^* = 0$ for $s \le i \le N$ and $0 \le j \le r$

pf Put

$$V_i = \mathbb{F} v_i \qquad 0 \le i \le N$$

(i) → (ii)

$$\begin{aligned} E_i^* A E_j^* V &\subseteq E_i^* A (E_0^* V + \cdots + E_r^* V) \\ &= E_i^* A (V_0 + \cdots + V_r) && \text{by L194} \\ &\subseteq E_i^* (V_0 + \cdots + V_{s-1}) && \text{by (i)} \\ &= E_i^* (E_0^* V + \cdots + E_{s-1}^* V) && \text{by L194} \\ &= 0 && \text{since } i \ge s \end{aligned}$$

(ii) → (i)

$$\begin{aligned} A v_j &\subseteq A (V_0 + \cdots + V_r) \\ &= A (E_0^* V + \cdots + E_r^* V) \\ &= (E_0^* + \cdots + E_N^*) A (E_0^* V + \cdots + E_r^* V) \\ &= (E_0^* + \cdots + E_{s-1}^*) A (E_0^* V + \cdots + E_r^* V) \\ &\subseteq E_0^* V + \cdots + E_{s-1}^* V \\ &= V_0 + \cdots + V_{s-1} \end{aligned}$$

$\square$

LEM 196 Ref to L195

fn $0 \le r < s \le N$ TFAE

(i) the (i,j)-entry of B^* is 0 fn $0 \le i \le r$ and $s \le j \le N$

(ii) $E_i A^* E_j = 0$ fn $0 \le i \le r$ and $s \le j \le N$

pf Sim to pf of L195 $\square$

LEC 27 Fri Nov 12

I Basics, cont.

We rephrase L195, 196 in terms of decomps

LEM 197 Given a decomp $\{V_i\}_{i=0}^N$ of V that satisfies the equiv cond (i), (ii) of L194

Then for $0 \le r < s \le N$ TFAE

(i) $(A - \theta_j I)V_j \subseteq V_{j+1} + \cdots + V_{s-1}$ $\quad 0 \le j \le r$

(ii) $E_i^* A E_j^* = 0$ for $s \le i \le N$ and $0 \le j \le r$

$\square$

pf By L195 and disc above it.

LEM 198 Given decomp $\{V_i\}_{i=0}^{N}$ of V that satisfies the equiv conds (i), (ii) of L194

Then for $0 \le r < s \le N$ TFAE

(i) $(A^* - \theta_j^* I) V_j \subseteq V_{r+1} + \cdots + V_{j-1}$ $\quad s \le j \le N$

(ii) $E_i A^* E_j = 0$ for $0 \le i \le r$ and $s \le j \le N$

pf By L196 and disc above L195. $\square$

Prop 199 Assume

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j>1 \\ \neq 0 & \text{if } i-j=1 \end{cases} \qquad 0 \le i,j \le N$$

and put

$$U_i = \tau_i(A) E_0^* V \qquad 0 \le i \le N$$

then

(i) $\{U_i\}_{i=0}^N$ is dec of V

(ii) $(A - \theta_i I) U_i = U_{i+1} \qquad 0 \le i \le N$

(iii) $(A^* - \theta_i^* I) U_i \subseteq U_0 + \cdots + U_{i-1} \qquad 0 \le i \le N$

pf E_0^* is norm by L193

(i) By L192 (i)

(ii) By def of τ_i

(iii) For $0 \le i \le N$

$$\begin{aligned} U_0 + U_1 + \cdots + U_i &= E_0^* V + A E_0^* V + \cdots + A^i E_0^* V && \text{by L192 (ii)} \\ &= E_0^* V + E_1^* V + \cdots + E_i^* V && \text{by L193 (iii)} \end{aligned}$$

Also

$$U_i + U_{i+1} + \cdots + U_N = E_i V + \cdots + E_N V \qquad \text{by L192 (iii)}$$

Now by L194

$$(A^* - \theta_i^* I) U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \qquad 0 \le i \le N$$

□

Here is converse to Prop 199

Prop 200 Assume $\exists$ decomp $\{U_i\}_{i=0}^N$ of V such that

both

$$(A - \theta_i I) U_i = U_{i+1} \qquad 0 \le i \le N$$

$$(A^* - \theta_i^* I) U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \qquad 0 \le i \le N$$

Then

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \quad\; i-j = 1 \end{cases} \qquad 0 \le i, j \le N \qquad *$$

Moreover $\quad U_i = \tau_i(A) E_0^* V \qquad 0 \le i \le N \qquad **$

pf The decomp $\{U_i\}_{i=0}^N$ satisfies the conds of L 194

so L 197 applies.

Result $*$ follows from L 197

show $**$:

$$(A^* - \theta_0^* I) U_0 = 0$$

$$\to U_0 = E_0^* V$$

Now $**$ follows since

$$(A - \theta_i I) U_i = U_{i+1} \qquad 0 \le i \le N$$

$\square$

Prop 201 Assume both

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \qquad 0 \le i,j \le N$$

$$E_i A^* E_j = 0 \quad \text{if } j-i > 1 \qquad 0 \le i,j \le N$$

Put

$$U_i = \tau_i(A) E_0^* V \qquad 0 \le i \le N$$

Then

(i) $\{U_i\}_{i=0}^N$ is dec of V

(ii) $(A - \theta_i I) U_i = U_{i+1}$ $\quad 0 \le i \le N$

(iii) $(A^* - \theta_i^* I) U_i \subseteq U_{i-1}$ $\quad 0 \le i \le N$

pf (i), (ii) By Prop 199

(iii) By Prop 199 (ii), (iii) dec $\{U_i\}_{i=0}^N$

satisfies conds of L 194

Now apply L 198 $\square$

Here is a converse to Prop 201

Prop 202 Assume $\exists$ decomp $\{U_i\}_{i=0}^N$ of V such that both

$$(A - \theta_i I) U_i = U_{i+1} \qquad 0 \le i \le N,$$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1} \qquad 0 \le i \le N$$

Then both

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \qquad 0 \le i,j \le N, \qquad *$$

$$E_i A^* E_j = 0 \text{ if } j-i > 1 \qquad 0 \le i,j \le N \qquad **$$

Moreover $U_i = \tau_i(A) E_0^* V \qquad 0 \le i \le N$ ***

pf The decomp $\{U_i\}_{i=0}^N$ satisfies the conds of L194

So L197, L198 apply. Result *, ** follows. $\square$

*** follows from Prop 200

Insert section

$I + \frac{1}{2}$ Some formula

Assume: As in I_1 and also both

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } i-j > 1 \\ \neq 0 & \text{if } i-j = 1 \end{cases} \qquad 0 \le i,j \le N$$

$$E_i A^* E_j = 0 \quad \text{if} \quad j - i > 1 \qquad 0 \le i,j \le N$$

Recall

E_0^* normalizing by L 193

Consider dec $\{U_i\}_{i=0}^N$ of V from Prop 201

For $1 \le i \le N$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1}$$

$$(A - \theta_{i-1} I) U_{i-1} = U_i$$

So U_i is invar under

$$(A - \theta_{i-1} I)(A^* - \theta_i^* I)$$

Let

φ_i = eigval

[caution: poss $\varphi_i = 0$]

For not conv $\varphi_0 = 0$ $\varphi_{N+1} = 0$

Fix $0 \neq v \in E_0^* V$

For $0 \le i \le N$ def

$$u_i = \tau_i(A) v$$

obs

$$0 \neq u_i \in U_i$$

So $\{u_i\}_{i=0}^{N}$ is basis for V

Def 203 $\forall X \in \operatorname{End} V$ let

$X^{\natural}$ = matrix in $\operatorname{Mat}_{N+1}(\mathbb{F})$ that reps X rel $\{u_i\}_{i=0}^{N}$

Obs

$\natural : \operatorname{End} V \to \operatorname{Mat}_{N+1}(\mathbb{F})$

is $\mathbb{F}$-alg iso

LEM 204 We have

$$A^{\natural} = \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_N \end{pmatrix}$$

$$A^{*\natural} = \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varphi_N \\ \mathbf{0} & & & & \theta_N^* \end{pmatrix}$$

LEM 205 For $1 \le j \le N$ TFAE

(i) $\varphi_j \neq 0$

(ii) $E_{j-1} A^* E_j \neq 0$

(iii) $(A^* - \theta_j^* I)\, u_j = u_{j-1}$

pf (i)$\Leftrightarrow$(ii) Use L196 and L204

(i) $\Leftrightarrow$ (iii) By Def of φ_j $\square$

LEM 206 For $0 \le r \le N$

the matrix $E_r^{\natural}$ is lower triang with entries

$$\frac{\tau_j(\theta_r)\, \eta_{N-i}(\theta_r)}{\tau_r(\theta_r)\, \eta_{N-r}(\theta_r)}$$

for $0 \le j \le i \le N$

pf Use L 104 and

$$E_r = \prod_{\substack{0 \le s \le N \\ s \ne r}} \frac{A - \theta_s I}{\theta_r - \theta_s}$$

$\square$

Ex 2 07 For $N=2$

$$E_0^{\natural} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\theta_0-\theta_1} & 0 & 0 \\ \frac{1}{(\theta_0-\theta_1)(\theta_0-\theta_2)} & 0 & 0 \end{pmatrix}$$

$$E_1^{\natural} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\theta_1-\theta_0} & 1 & 0 \\ \frac{1}{(\theta_1-\theta_0)(\theta_1-\theta_2)} & \frac{1}{\theta_1-\theta_2} & 0 \end{pmatrix}$$

$$E_2^{\natural} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{(\theta_2-\theta_0)(\theta_2-\theta_1)} & \frac{1}{\theta_2-\theta_1} & 1 \end{pmatrix}$$

LEM 208 For $0 \le r \le N$

the matrix $(E^*_r)^\natural$ is upper triangular with entries

$$\frac{\varphi_{i+1} \cdots \varphi_j \ \tau^*_i(\theta^*_r)\ \eta^*_{N-j}(\theta^*_r)}{\tau^*_r(\theta^*_r)\ \eta^*_{N-r}(\theta^*_r)}$$

for $0 \le i \le j \le N$

pf Use L 104 and

$$E^*_r = \prod_{\substack{0 \le s \le N \\ s \ne r}} \frac{A^\natural - \theta^*_s I}{\theta^*_r - \theta^*_s}$$

$\square$

Next goal: Find a_i, a_i^* in terms of the

θ_j θ_j^*; φ_j

Suppose $N=0$ then $A = \theta_0 I$, $A^* = \theta_0^* I$

so $a_0 = \theta_0$ $a_0^* = \theta_0^*$

LEM 210 Assume $N \geq 1$

(i)
$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*}$$

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*}$$

$1 \leq i \leq N-1$

$$a_N = \theta_N + \frac{\varphi_N}{\theta_N^* - \theta_{N-1}^*}$$

(ii)
$$a_0^* = \theta_0^* + \frac{\varphi_1}{\theta_0 - \theta_1}$$

$$a_i^* = \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}}$$

$1 \leq i \leq N-1$

$$a_N^* = \theta_N^* + \frac{\varphi_N}{\theta_N - \theta_{N-1}}$$

pf (i) For $0 \leq i \leq N$

$a_i = tr(A E_i^*) = tr(A^\natural (E_i^*)^\natural)$

Compute this using L204, L208

(ii) Sim

$\square$

Ex 209 For $N=2$

$$E_0^{*q} = \begin{pmatrix} 1 & \dfrac{\varphi_1}{\theta_0^* - \theta_1^*} & \dfrac{\varphi_1 \varphi_2}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_1^{*q} = \begin{pmatrix} 0 & \dfrac{\varphi_1}{\theta_1^* - \theta_0^*} & \dfrac{\varphi_1 \varphi_2}{(\theta_1^* - \theta_0^*)(\theta_1^* - \theta_2^*)} \\ 0 & 1 & \dfrac{\varphi_2}{\theta_1^* - \theta_2^*} \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_2^{*q} = \begin{pmatrix} 0 & 0 & \dfrac{\varphi_1 \varphi_2}{(\theta_2^* - \theta_0^*)(\theta_2^* - \theta_1^*)} \\ 0 & 0 & \dfrac{\varphi_2}{\theta_2^* - \theta_1^*} \\ 0 & 0 & 1 \end{pmatrix}$$

LEM 211 For $1 \le i \le N$

φ_i is equal to each of the following:

(i) $$(\theta^*_i - \theta^*_{i-1}) \sum_{h=0}^{i-1} (\theta_h - a_h)$$

(ii) $$(\theta^*_{i-1} - \theta^*_i) \sum_{h=i}^{N} (\theta_h - a_h)$$

(iii) $$(\theta_i - \theta_{i-1}) \sum_{h=0}^{i-1} (\theta^*_h - a^*_h)$$

(iv) $$(\theta_{i-1} - \theta_i) \sum_{h=i}^{N} (\theta^*_h - a^*_h)$$

pf Assume $N \ge 1$ else triv.

(i) use L210

(ii) Use (i) and $\theta_0 + \cdots + \theta_N = a_0 + \cdots + a_N$

(iii), (iv) Sim $\square$

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I + 1/2 cont.

Def 2.12 Define

$$g_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_{i+1} - \theta_N)$$

$1 \le i \le N$

$g_0 = 0$

$g_{N+1} = 0$

thm 2B Assume $N \geq 2$. Then

(i) $$\sum_{i=0}^{N-2} E_N A^* E_i E_0^* (\theta_i - \theta_{N-1}) = (\gamma_1 - \gamma_N) E_N E_0^*$$

(ii) $$\sum_{i=2}^{N} E_0^* A E_i^* E_N (\theta_i^* - \theta_1^*) = (\gamma_1 - \gamma_N) E_0^* E_N$$

pf (i) Recall

$$I = \sum_{i=0}^{N} E_i^*$$

Mult each term on right by $A E_0^*$

Simplify using

$$E_0^* A E_0^* = a_0 E_0^*$$

$$E_i^* A E_0^* = 0 \qquad 2 \leq i \leq N$$

Get

$$A E_0^* = a_0 E_0^* + E_1^* A E_0^* \tag{1}$$

In (1) mult each term on left by A^*

Get

$$A^* A E_0^* = a_0 \theta_0^* E_0^* + \theta_1^* E_1^* A E_0^* \tag{2}$$

Recall

$$I = \sum_{i=0}^{N} E_i$$

Mult each term on left by $E_N A^*$

Simplify using

$$E_N A^* E_N = a_N^* E_N$$

Get

$$E_N A^* = a_N^* E_N + \sum_{i=0}^{N-1} E_N A^* E_i \qquad (3)$$

In (3) mult each term on right by A

Get

$$E_N A^* A = \theta_N a_N^* E_N + \sum_{i=0}^{N-1} \theta_i E_N A^* E_i \qquad (4)$$

Consider eq which is

$$\theta_1^* E_N (1) - E_N (2) - \theta_{N-1} (3) E_0^* + (4) E_0^*$$

Get

$$E_N E_0^* \left((\theta_0^* - \theta_1^*) a_0 + (\theta_{N-1} - \theta_N) a_N^* + \theta_N \theta_1^* - \theta_{N-1} \theta_0^* \right) = \sum_{i=0}^{N-2} E_N A^* E_i E_0^* (\theta_i^* - \theta_{N-1})$$

Simplify the $E_N E_0^*$ coef using

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*} \qquad a_N^* = \theta_N^* + \frac{\varphi_N}{\theta_N - \theta_{N-1}}$$

$$\varphi_1 = \phi_1 + (\theta_1^* - \theta_0^*)(\theta_0 - \theta_N) \qquad \varphi_N = \phi_N + (\theta_N^* - \theta_0^*)(\theta_{N-1} - \theta_N)$$

to get $\phi_1 - \phi_N$

(ii) Sim

II the split decomp and the parameter array

Assumptions :

Field $\mathbb{F}$ arb

N = nonneg integer

V = v.s. over $\mathbb{F}$ dim $N+1$

Given LS

$$\Phi = \left(A, \{E_i\}_{i=0}^{N}\ A^* \{E_i^*\}_{i=0}^{N}\right)$$

on V with eigenval seq $\{\theta_i\}_{i=0}^{N}$ and dual eigenval seq $\{\theta_i^*\}_{i=0}^{N}$

—

By the $\underline{\Phi\text{- split decomp}}$ of V we mean

the decomp $\{U_i\}_{i=0}^{N}$ of V from Prop 201

We have

$$U_i = \tau_i(A) E_0^* V \qquad 0 \le i \le N$$

$$U_i = \eta_{N-i}^*(A^*) E_N V \qquad ''$$

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_N V) \qquad ''$$

$$U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V \qquad ''$$

$$U_i + \cdots + U_N = E_i V + \cdots + E_N V \qquad ''$$

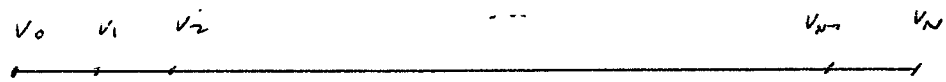
$$(A - \theta_i I) U_i = U_{i+1} \qquad 0 \le i \le N$$

$$(A^* - \theta_i^* I) U_i = U_{i-1} \qquad 0 \le i \le N$$

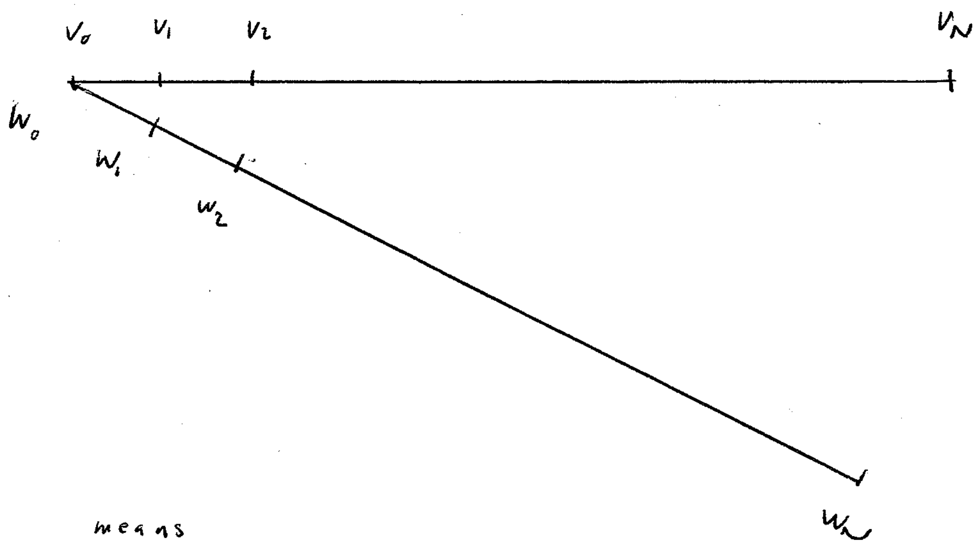
Obs $\{U_{N-i}\}_{i=0}^{N}$ is split decomp for LS

$$(A^*, \{E_{N-i}^*\}_{i=0}^{N}, A, \{E_{N-i}\}_{i=0}^{N}) = \Phi^{*\downarrow\Downarrow}$$

Let us represent a decomp $\{V_i\}_{i=0}^{N}$ of V by a line segment



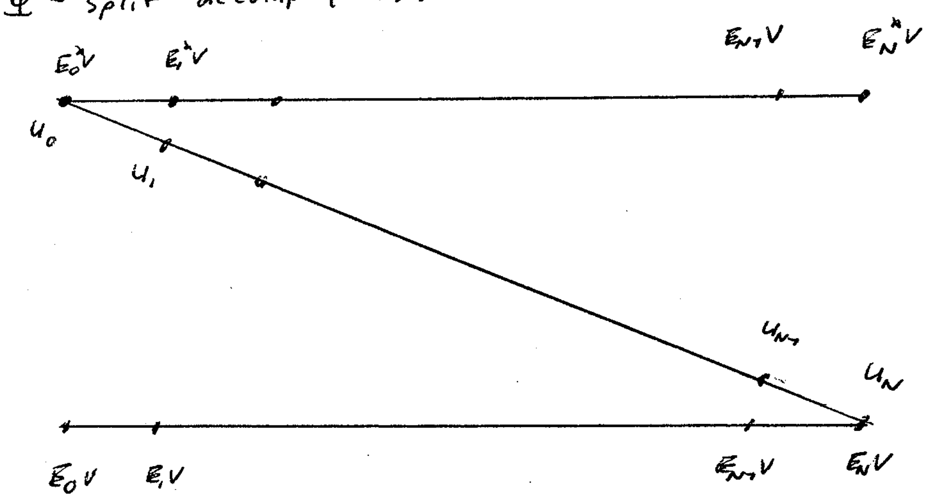
Given decomp $\{W_i\}_{i=0}^{N}$ of V



means

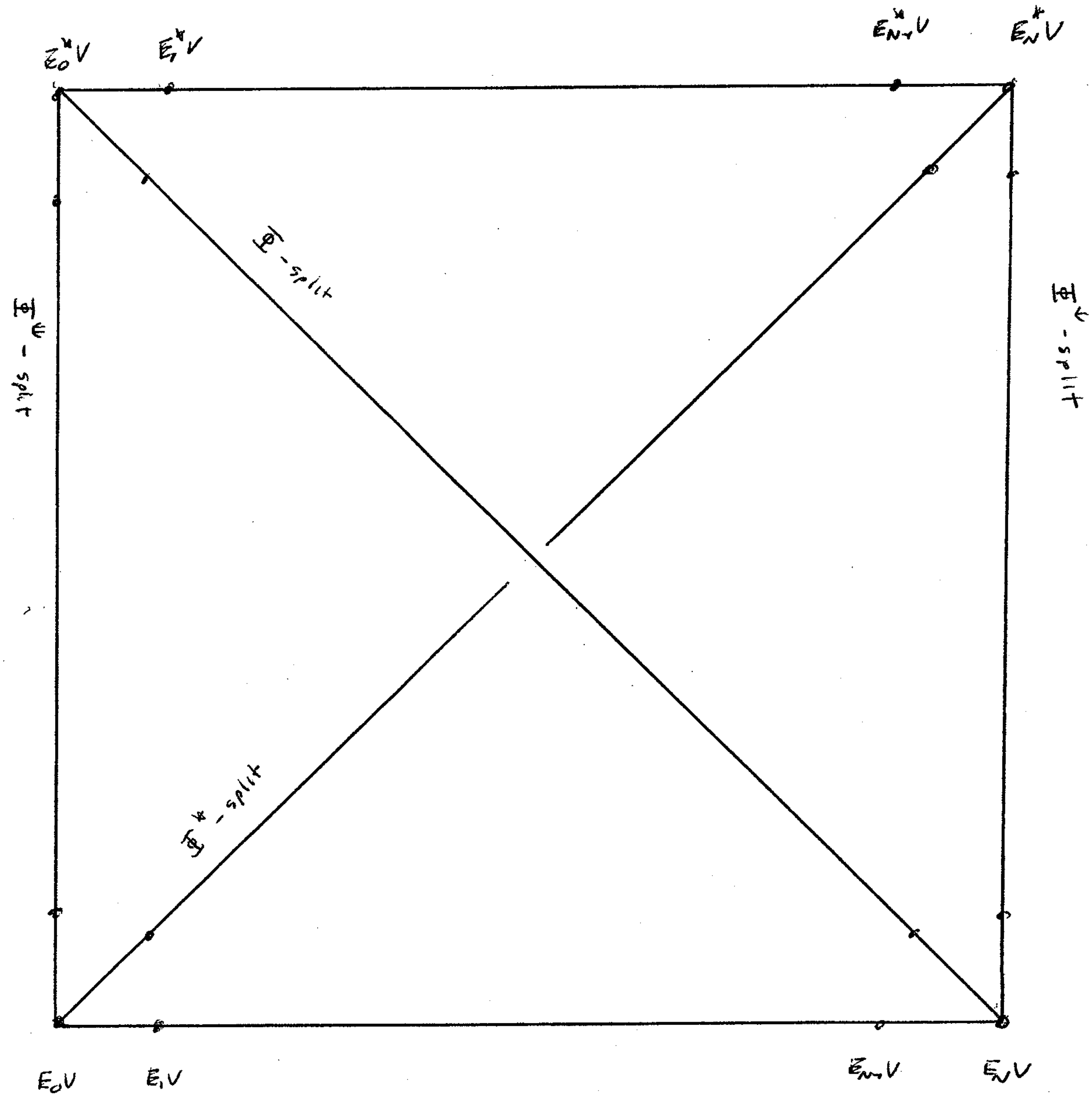
$$V_0 + V_1 + \cdots + V_i = W_0 + W_1 + \cdots + W_i \qquad 0 \le i \le N$$

Φ - split decomp $\{U_i\}_{i=0}^{N}$ satisfies



We also have split decomps for

Φ^{*}, $\Phi^{\downarrow}$, $\Phi^{\Downarrow}$:



DEF 214 By the <u>1st split sequence</u> for Φ we mean $\{\varphi_i\}_{i=1}^N$ from Section I ½

By L 205

$$\varphi_i \neq 0 \qquad 1 \le i \le N$$

Def 215 Let $\{\phi_i\}_{i=1}^N$ denote the 1st split sequence for $\Phi^\Downarrow$. Call $\{\phi_i\}_{i=1}^N$ the <u>2nd split sequence</u> for Φ. Obs

$$\phi_i \neq 0 \qquad 1 \le i \le N$$

For not conv $\phi_0 = 0 \qquad \phi_{N+1} = 0$

Note 216 Given $0 \neq v \in E_0^* V$ Obs

$$\{\eta_i(A) v\}_{i=0}^N \quad \text{is basis for } V$$

Rel this basis

$$A: \begin{pmatrix} \theta_N & & & 0 \\ 1 & \theta_{N-1} & & \\ & 1 & \ddots & \\ 0 & & \ddots & \\ & & 1 & \theta_0 \end{pmatrix} \qquad A^*: \begin{pmatrix} \theta_0^* & \phi_1 & & 0 \\ & \theta_1^* & \phi_2 & \\ & & \ddots & \ddots \\ 0 & & & \phi_N \\ & & & \theta_N^* \end{pmatrix}$$

Notation Given iso of v.s.

$$\sigma : V \longrightarrow V'$$

Abbr

$$X^\sigma = \sigma X \sigma^{-1} \qquad \forall X \in \mathrm{End} V$$

So

$$\mathrm{End} V \to \mathrm{End} V'$$

$$X \to X^\sigma$$

is iso of $\mathbb{F}$-algebras

For our LS $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ on V

$$\Phi^\sigma := (A^\sigma, \{E_i^\sigma\}_{i=0}^N, A^{*\sigma}, \{E_i^{*\sigma}\}_{i=0}^N)$$

is LS on V'

Given LS Φ' on V'. By an iso of LS from Φ to Φ'

we mean an iso of v.s. $\sigma : V \to V'$ s.t. $\Phi^\sigma = \Phi'$

Call Φ, Φ' isomorphic whenever $\exists$ iso of LS from Φ to Φ'

LEM 217 TFAE

(i) Φ, Φ' are iso

(ii) Φ, Φ' have same eigval seq, dual eigval seq, 1st split seq

(iii) ... ... 2nd split seq

pf (i) $\Leftrightarrow$ (ii) By L204

(i) $\Leftrightarrow$ (iii) Apply L204 to $\Phi^{\Downarrow}$, $\Phi'^{\Downarrow}$ $\square$

218 By the <u>parameter</u> <u>array</u> of Φ we mean the sequence

$$\Big(\underbrace{\{\theta_i\}_{i=0}^N}_{\text{eigval seq}} \quad \underbrace{\{\theta_i^*\}_{i=0}^N}_{\text{dual eigval seq}} \quad \underbrace{\{\varphi_i\}_{i=1}^N}_{\text{1st split seq}} \quad \underbrace{\{\phi_i\}_{i=1}^N}_{\text{2nd split seq.}} \Big)$$

Cor 219 Two LS over $\mathbb{F}$ are iso iff they have the same parameter array.

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II cont

L 220 For $1 \le i \le N$

ϕ_i is equal to each of the following

(i) $$(\theta_i^* - \theta_{i-1}^*) \sum_{h=0}^{i-1} (\theta_{N-h} - a_h)$$

(ii) $$(\theta_{i-1}^* - \theta_i^*) \sum_{h=i}^{N} (\theta_{N-h} - a_h)$$

(iii) $$(\theta_{N-i} - \theta_{N-i+1}) \sum_{h=0}^{i-1} (\theta_h^* - a_{N-h}^*)$$

(iv) $$(\theta_{N-i+1} - \theta_{N-i}) \sum_{h=i}^{N} (\theta_h^* - a_{N-h}^*)$$

pf Recall $\{\phi_i\}_{i=1}^N$ is 1st split seq for $\Phi^{\downarrow\Downarrow}$

For $0 \le j \le N$

$$\theta_j(\Phi^{\Downarrow}) = \theta_{N-j}$$

$$\theta_j^*(\Phi^{\Downarrow}) = \theta_j^*$$

$$a_j(\Phi^{\Downarrow}) = a_j$$

$$a_j^*(\Phi^{\Downarrow}) = a_{N-j}^*$$

Now apply L 211 to $\Phi^{\Downarrow}$

Prop 221 The parameter arrays of

$$\Phi, \quad \Phi^{\Downarrow}, \quad \Phi^{*}, \quad \Phi^{\downarrow}$$

are related as follows

LS	PA
Φ	$(\theta_i \quad \theta^*_i \quad \varphi_i \quad \phi_i)$
$\Phi^{\Downarrow}$	$(\theta_{N-i} \quad \theta^*_i \quad \phi_i \quad \varphi_i)$
Φ^{*}	$(\theta^*_i \quad \theta_i \quad \varphi_i \quad \phi_{N-i+1})$
$\Phi^{\downarrow}$	$(\theta_i \quad \theta^*_{N-i} \quad \phi_{N-i+1} \quad \varphi_{N-i+1})$

Pf Use L211, L220 and Def 184

$\Phi^{\Downarrow}$: By def of ϕ_i, φ_i ✓.

Φ^{*}; Compare L211(i), (iii) to get

$$\varphi_i(\Phi^*) = \varphi_i$$

Compare L220 (i), (iii) to get

$$\phi_i(\Phi^*) = \phi_{N-i+1}$$

$\Phi^{\downarrow}$: Use above data and $\Phi^{\downarrow} = ((\Phi^*)^{\Downarrow})^*$

□

We now state our classification thm for LS

thm 22.2 Given a sequence

$$\left(\{\theta_i\}_{i=0}^{N} \quad \{\theta_i^*\}_{i=0}^{N} \quad \{\varphi_i\}_{i=1}^{N} \quad \{\phi_i\}_{i=1}^{N} \right) \qquad (*)$$

taken from $\mathbb{F}$. Then $\exists$ LS Φ over $\mathbb{F}$ with par array $(*)$ iff the following hold:

(PA1) $\varphi_i \neq 0 \quad \phi_i \neq 0 \qquad (1 \le i \le N)$

(PA2) $\theta_i \neq \theta_j \quad \theta_i^* \neq \theta_j^*$ if $i \neq j \qquad (0 \le i, j \le N)$

(PA3) $$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_N) \qquad 1 \le i \le N$$

(PA4) $$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + (\theta_i^* - \theta_0^*)(\theta_{N-i+1} - \theta_0) \qquad 1 \le i \le N$$

(PA5) the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and indep of i for $2 \le i \le N-1$

Moreover if Φ exists then it is unique up to iso of LS. $\square$

III Rec. sequences

$\mathbb{F}$ arb $\qquad \overline{\mathbb{F}}$ = alg cl of $\mathbb{F}$

Assume: $\quad N = 0, 1, 2 \ldots$

$\{\theta_i\}_{i=0}^{N}$ any sequence from $\overline{\mathbb{F}}$

Fix $\beta \in \overline{\mathbb{F}}$

We consider when this sequence satisfies

3-term rec. See handout.

We use			Dont need
8.1	9.1	10.1	10.3
8.2	9.2	10.2	10.4
8.3	9.3	10.6	10.5
8.4	9.4	10.7	
8.5	9.5		

For notational convenience, we set $\phi_0 = 0$, $\phi_{d+1} = 0$.

Lemma 7.2. *With reference to Definition 4.1, let $V = \mathbb{K}^{d+1}$ denote the irreducible left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$, and let W denote a nonzero (A, A^*)-module in V. Then there exists an integer r $(0 \leqslant r \leqslant d)$ such that both*

$$W = \sum_{h=r}^{d} E_h^* V, \qquad W = \sum_{h=0}^{d-r} E_h V. \tag{80}$$

Moreover, the scalar ϕ_r from Definition 7.1 is 0.

Proof. Since W is nonzero and $A^*W \subseteq W$, there exists a nonempty subset S^* of $\{0, 1, \ldots, d\}$ such that $W = \sum_{i \in S^*} E_i^* V$. Recall by Lemma 4.9(ii) that $E_{i+1}^* A E_i^* \neq 0$ for $0 \leqslant i \leqslant d-1$. Combining this with Lemma 2.4(ii), we find $i \in S^*$ implies $i+1 \in S^*$ for $0 \leqslant i \leqslant d-1$. It follows $S^* = \{r, r+1, \ldots, d\}$ for some integer r $(0 \leqslant r \leqslant d)$. Since W is nonzero and $AW \subseteq W$, there exists a nonempty subset S of $\{0, 1, \ldots, d\}$ such that $W = \sum_{i \in S} E_i V$. Recall by Lemma 4.9(i) that $E_{i-1} A^* E_i \neq 0$ for $1 \leqslant i \leqslant d$. Combining this with Lemma 2.3(ii), we find $i \in S$ implies $i-1 \in S$ for $1 \leqslant i \leqslant d$. It follows $S = \{0, 1, \ldots, s\}$ for some integer s $(0 \leqslant s \leqslant d)$. Considering the dimension of W we find $|S| = |S^*|$, so $s = d - r$, and (80) follows. It remains to show $\phi_r = 0$. This holds by definition if $r = 0$, so assume $r \geqslant 1$. To get $\phi_r = 0$ in this case, we first show

$$a_r + a_{r+1} + \cdots + a_d = \theta_0 + \theta_1 + \cdots + \theta_{d-r}. \tag{81}$$

For convenience, we abbreviate $E = \sum_{h=0}^{d-r} E_h$ and $E^* = \sum_{h=r}^{d} E_h^*$. We show AE and AE^* have the same trace. To do this, we put $X = A(E - E^*)$, and show X has trace 0. In fact $X^2 = 0$. To see this, we show $XV \subseteq W$ and $XW = 0$. Each of EV, E^*V equals W by (80), so $(E - E^*)V \subseteq W$. Recall $AW \subseteq W$, so $XV \subseteq W$. Observe each of E, E^* acts as the identity on W, so $(E - E^*)W = 0$, and it follows $XW = 0$. We have now shown $X^2 = 0$, so X has trace 0, and AE, AE^* have the same trace. We now compute these traces. By (2) and since each E_h has trace 1, we find AE has trace $\sum_{h=0}^{d-r} \theta_h$. Using Definition 2.5, we routinely find AE^* has trace $\sum_{h=r}^{d} a_h$. We now have (81). Eliminating the left-hand side of (81) using the equation on the right-hand side in (70), we find $\phi_r = 0$. □

Theorem 7.3. *With reference to Definition 4.1, let $V = \mathbb{K}^{d+1}$ denote the irreducible left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$, and suppose the scalars $\phi_1, \phi_2, \ldots, \phi_d$ from (79) are all nonzero. Then V is irreducible as an (A, A^*)-module.*

Proof. Let W denote a nonzero (A, A^*)-module in V. We show $W = V$. Let r denote the integer associated with W from Lemma 7.2. From that lemma and our present assumption we find r is not one of $1, 2, \ldots, d$, so $r = 0$. Setting $r = 0$ in (80), we find $W = V$. □

8. Recurrent sequences

It is going to turn out that the eigenvalue sequence and dual eigenvalue sequence of a Leonard system each satisfy a certain recurrence. In this section, we set the stage by considering this recurrence from several points of views.

Definition 8.1. In this section, d will denote a nonnegative integer, and $\theta_0, \theta_1, \ldots, \theta_d$ will denote a sequence of scalars taken from $\mathbb{K}$.

Definition 8.2. With reference to Definition 8.1, let β, γ, ϱ denote scalars in $\mathbb{K}$.

(i) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be *recurrent* whenever $\theta_{i-1} \neq \theta_i$ for $2 \leqslant i \leqslant d-1$, and
$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \tag{82}$$
is independent of i for $2 \leqslant i \leqslant d-1$.

(ii) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be *β-recurrent* whenever
$$\theta_{i-2} - (\beta + 1)\theta_{i-1} + (\beta + 1)\theta_i - \theta_{i+1} \tag{83}$$
is 0 for $2 \leqslant i \leqslant d-1$.

(iii) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be *(β, γ)-recurrent* whenever
$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma \tag{84}$$
for $1 \leqslant i \leqslant d-1$.

(iv) The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is said to be *(β, γ, ϱ)-recurrent* whenever
$$\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) = \varrho \tag{85}$$
for $1 \leqslant i \leqslant d$.

Lemma 8.3. *With reference to Definition 8.1; the following are equivalent:*

(i) *The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent.*

(ii) *The scalars $\theta_{i-1} \neq \theta_i$ for $2 \leqslant i \leqslant d-1$, and there exists $\beta \in \mathbb{K}$ such that $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent.*

Suppose (i), (ii), *and that $d \geqslant 3$. Then the common value of* (82) *equals $\beta + 1$.*

Proof. Routine. □

Lemma 8.4. *With reference to Definition 8.1, the following are equivalent for all $\beta \in \mathbb{K}$:*

(i) *The sequence $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent.*

(ii) *There exists $\gamma \in \mathbb{K}$ such that $\theta_0, \theta_1, \ldots, \theta_d$ is (β, γ)-recurrent.*

Proof. (i) → (ii): For $2 \leqslant i \leqslant d-1$, expression (83) is 0 by assumption, so

$$\theta_{i-2} - \beta\theta_{i-1} + \theta_i = \theta_{i-1} - \beta\theta_i + \theta_{i+1}.$$

Apparently the left-hand side of (84) is independent of i, and the result follows.

(ii) $\to$ (i): Subtracting Eq. (84) at i from the corresponding equation obtained by replacing i by $i-1$, we find (83) is 0 for $2 \leqslant i \leqslant d-1$. □

Lemma 8.5. *With reference to Definition* 8.1, *the following* (i) *and* (ii) *hold for all* $\beta, \gamma \in \mathbb{K}$:

(i) *Suppose* $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ)*-recurrent. Then there exists* $\varrho \in \mathbb{K}$ *such that* $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ, ϱ)*-recurrent.*

(ii) *Suppose* $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ, ϱ)*-recurrent, and that* $\theta_{i-1} \neq \theta_{i+1}$ *for* $1 \leqslant i \leqslant d-1$. *Then* $\theta_0, \theta_1, \ldots, \theta_d$ *is* (β, γ)*-recurrent.*

Proof. Let p_i denote the expression on the left-hand side in (85), and observe

$$p_i - p_{i+1} = (\theta_{i-1} - \theta_{i+1})(\theta_{i-1} - \beta\theta_i + \theta_{i+1} - \gamma)$$

for $1 \leqslant i \leqslant d-1$. Assertions (i) and (ii) are both routine consequences of this. □

9. Recurrent sequences in closed form

In this section, we obtain some formula involving recurrent sequences.

Definition 9.1. In this section, d will denote a nonnegative integer, and $\beta, \theta_0, \theta_1, \ldots, \theta_d$ will denote scalars in $\mathbb{K}$ such that $\theta_0, \theta_1, \ldots, \theta_d$ is β-recurrent. We let $\mathbb{K}^{\mathrm{cl}}$ denote the algebraic closure of $\mathbb{K}$. For all $q \in \mathbb{K}^{\mathrm{cl}}$, we let $\mathbb{K}[q]$ denote the field extention of $\mathbb{K}$ generated by q.

Lemma 9.2. *With reference to Definition* 9.1, *the following* (i)–(iv) *hold:*

(i) *Suppose* $\beta \neq 2$, $\beta \neq -2$, *and pick* $q \in \mathbb{K}^{\mathrm{cl}}$ *such that* $q + q^{-1} = \beta$. *Then there exist scalars* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathbb{K}[q]$ *such that*

$$\theta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i} \quad (0 \leqslant i \leqslant d). \tag{86}$$

(ii) *Suppose* $\beta = 2$. *Then there exist* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathbb{K}$ *such that*

$$\theta_i = \alpha_1 + \alpha_2 i + \alpha_3 i(i-1)/2 \quad (0 \leqslant i \leqslant d). \tag{87}$$

(iii) *Suppose* $\beta = -2$ *and* $\mathrm{char}(\mathbb{K}) \neq 2$. *Then there exist* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathbb{K}$ *such that*

$$\theta_i = \alpha_1 + \alpha_2(-1)^i + \alpha_3 i(-1)^i \quad (0 \leqslant i \leqslant d). \tag{88}$$

Referring to case (ii) *above, if* $\mathrm{char}(\mathbb{K}) = 2$, *we interpret the expression* $i(i-1)/2$ *as* 0 *if* $i = 0$ *or* $i = 1$ (mod 4), *and as* 1 *if* $i = 2$ *or* $i = 3$ (mod 4).

Proof. (i) We assume $d \geqslant 2$; otherwise the result is trivial. Let q be given, and consider Eqs. (86) for $i = 0, 1, 2$. These equations are linear in $\alpha_1, \alpha_2, \alpha_3$. We routinely find the coefficient matrix is nonsingular, so there exist $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{K}[q]$ such that (86) holds for $i = 0, 1, 2$. Using these scalars, let ε_i denote the left-hand side of (86) minus the right-hand side of (86), for $0 \leqslant i \leqslant d$. On one hand, $\varepsilon_0, \varepsilon_1, \varepsilon_2$ are 0 from the construction. On the other hand, one readily checks

$$\varepsilon_{i-2} - (\beta+1)\varepsilon_{i-1} + (\beta+1)\varepsilon_i - \varepsilon_{i+1} = 0$$

for $2 \leqslant i \leqslant d-1$. Combining these facts, we find $\varepsilon_i = 0$ for $0 \leqslant i \leqslant d$, and the result follows.

(ii) and (iii) Similar to the proof of (i) above. □

Lemma 9.3. *With reference to Definition* 9.1, *assume* $\theta_0, \theta_1, \ldots, \theta_d$ *are distinct. Then* (i)–(iv) *hold below:*

(i) *Suppose* $\beta \neq 2$, $\beta \neq -2$, *and pick* $q \in \mathbb{K}^{\mathrm{cl}}$ *such that* $q + q^{-1} = \beta$. *Then* $q^i \neq 1$ *for* $1 \leqslant i \leqslant d$.

(ii) *Suppose* $\beta = 2$ *and* $\mathrm{char}(\mathbb{K}) = p$, $p \geqslant 3$. *Then* $d < p$.

(iii) *Suppose* $\beta = -2$ *and* $\mathrm{char}(\mathbb{K}) = p$, $p \geqslant 3$. *Then* $d < 2p$.

(iv) *Suppose* $\beta = 0$ *and* $\mathrm{char}(\mathbb{K}) = 2$. *Then* $d \leqslant 3$.

Proof. (i) Using (86), we find $q^i = 1$ implies $\theta_i = \theta_0$ for $1 \leqslant i \leqslant d$.

(ii) Suppose $d \geqslant p$. Setting $i = p$ in (87), and recalling p is congruent to 0 modulo p, we find $\theta_p = \theta_0$, a contradiction. Hence, $d < p$.

(iii) Suppose $d \geqslant 2p$. Setting $i = 2p$ in (88), and recalling p is congruent to 0 modulo p, we find $\theta_{2p} = \theta_0$, a contradiction. Hence, $d < 2p$.

(iv) Suppose $d \geqslant 4$. Setting $i = 4$ in (87), we find $\theta_4 = \theta_0$ in view of the comment at the end of Lemma 9.2. This is a contradiction, so $d \leqslant 3$. □

Lemma 9.4. *With reference to Definition* 9.1, *assume* $\theta_0, \theta_1, \ldots, \theta_d$ *are distinct. Pick any integers* i, j, r, s $(0 \leqslant i, j, r, s \leqslant d)$ *and assume* $i + j = r + s$, $r \neq s$. *Then* (i)–(iv) *hold below:*

(i) *Suppose* $\beta \neq 2$, $\beta \neq -2$. *Then*

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \frac{q^i - q^j}{q^r - q^s}, \tag{89}$$

where $q + q^{-1} = \beta$.

(ii) *Suppose* $\beta = 2$ *and* $\mathrm{char}(\mathbb{K}) \neq 2$. *Then*

$$\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \frac{i-j}{r-s}. \tag{90}$$

(iii) *Suppose* $\beta = -2$ *and* $\mathrm{char}(\mathbb{K}) \neq 2$. *Then*

$$\frac{\theta_i-\theta_j}{\theta_r-\theta_s}=\begin{cases}(-1)^{i+r}\dfrac{i-j}{r-s} & \text{if } i+j \text{ is even,}\\ (-1)^{i+r} & \text{if } i+j \text{ is odd.}\end{cases} \tag{91}$$

(iv) *Suppose* $\beta=0$ *and* $\mathrm{char}(\mathbb{K})=2$. *Then*

$$\frac{\theta_i-\theta_j}{\theta_r-\theta_s}=\begin{cases}0 & \text{if } i=j,\\ 1 & \text{if } i\neq j.\end{cases} \tag{92}$$

Proof. To get (i), evaluate the left-hand side in (89) using (86), and simplify the result. Cases (ii)–(iv) are very similar. □

We complete this section with an observation.

Lemma 9.5. *With the notation and assumptions of Lemma* 9.4, *the scalar*

$$\frac{\theta_i-\theta_j}{\theta_r-\theta_s}$$

depends only on i, j, r, s *and* β, *and not on* $\theta_0, \theta_1, \ldots, \theta_d$.

Proof. This is immediate from the data in Lemma 9.4. □

10. A sum

Definition 10.1. Throughout this section d will denote an integer at least 1, and $\theta_0, \theta_1, \ldots, \theta_d$ will denote a sequence of distinct scalars in $\mathbb{K}$. We let β denote any scalar in $\mathbb{K}$.

With reference to Definition 10.1, we now consider the sums

$$\sum_{h=0}^{i-1}\frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}, \tag{93}$$

where $0\leqslant i\leqslant d+1$. Denoting the sum in (93) by ϑ_i, we remark

$$\vartheta_0=0,\quad \vartheta_1=1,\quad \vartheta_d=1,\quad \vartheta_{d+1}=0. \tag{94}$$

Moreover,

$$\vartheta_i=\vartheta_{d-i+1}\quad (0\leqslant i\leqslant d+1) \tag{95}$$

and

$$\vartheta_{i+1}-\vartheta_i=\frac{\theta_i-\theta_{d-i}}{\theta_0-\theta_d}\quad (0\leqslant i\leqslant d). \tag{96}$$

It turns out the sums (93) play an important role a bit later, so we will examine them carefully. We begin by giving explicit formulae for the sums (93) under the

assumption the sequence $\theta_0, \theta_1, \ldots, \theta_d$ is recurrent. To avoid trivialities, we assume $d\geqslant 3$.

Lemma 10.2. *With reference to Definition* 10.1, *assume* $d\geqslant 3$, *and assume* $\theta_0, \theta_1, \ldots, \theta_d$ *is* β*-recurrent. Then for all integers* i $(0\leqslant i\leqslant d+1)$, *we have the following*:

(i) *Suppose* $\beta\neq 2$, $\beta\neq -2$. *Then*

$$\sum_{h=0}^{i-1}\frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}=\frac{(q^i-1)(q^{d-i+1}-1)}{(q-1)(q^d-1)}, \tag{97}$$

where $q+q^{-1}=\beta$.

(ii) *Suppose* $\beta=2$ *and* $\mathrm{char}(\mathbb{K})\neq 2$. *Then*

$$\sum_{h=0}^{i-1}\frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}=\frac{i(d-i+1)}{d}. \tag{98}$$

(iii) *Suppose* $\beta=-2$, $\mathrm{char}(\mathbb{K})\neq 2$, *and* d *odd. Then*

$$\sum_{h=0}^{i-1}\frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}=\begin{cases}0 & \text{if } i \text{ is even,}\\ 1 & \text{if } i \text{ is odd.}\end{cases} \tag{99}$$

(iv) *Suppose* $\beta=-2$, $\mathrm{char}(\mathbb{K})\neq 2$, *and* d *even. Then*

$$\sum_{h=0}^{i-1}\frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}=\begin{cases}i/d & \text{if } i \text{ is even,}\\ (d-i+1)/d & \text{if } i \text{ is odd.}\end{cases} \tag{100}$$

(v) *Suppose* $\beta=0$, $\mathrm{char}(\mathbb{K})=2$, *and* $d=3$. *Then*

$$\sum_{h=0}^{i-1}\frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}=\begin{cases}0 & \text{if } i \text{ is even,}\\ 1 & \text{if } i \text{ is odd.}\end{cases} \tag{101}$$

Proof. The above sums can be computed directly from Lemma 9.4. □

We mention some recursions satisfied by the sums (93).

Lemma 10.3. *With reference to Definition* 10.1, *assume* $\theta_0, \theta_1, \ldots, \theta_d$ *is recurrent, and put*

$$\vartheta_i=\sum_{h=0}^{i-1}\frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}\quad (0\leqslant i\leqslant d+1). \tag{102}$$

Then (i) *and* (ii) *hold below*:

(i) $\vartheta_{i+1}=\vartheta_i\dfrac{\theta_i-\theta_{d-1}}{\theta_{i-1}-\theta_d}+1\quad (1\leqslant i\leqslant d)$,

(ii) $\vartheta_i = \vartheta_{i+1}\dfrac{\theta_i - \theta_1}{\theta_{i+1} - \theta_0} + 1 \quad (0 \leqslant i \leqslant d-1)$.

Proof. (i) These equations are readily verified case by case, using Lemma 10.2.
(ii) Apply (i) above to the sequence $\theta_d, \theta_{d-1}, \ldots, \theta_0$, and use (95). □

Lemma 10.4. *With reference to Definition* 10.1, *assume* $\theta_0, \theta_1, \ldots, \theta_d$ *is recurrent. Let r denote any integer in the range* $1 \leqslant r \leqslant d+1$, *and suppose we are given scalars* $\vartheta_1, \vartheta_2, \ldots, \vartheta_r$ *in* $\mathbb{K}$ *such that*

$$\vartheta_{i+1} = \vartheta_i \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} + \vartheta_1 \quad (1 \leqslant i \leqslant r-1). \tag{103}$$

Then

$$\vartheta_i = \vartheta_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leqslant i \leqslant r). \tag{104}$$

Proof. Define

$$\vartheta_i' = \vartheta_i - \vartheta_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \tag{105}$$

for $1 \leqslant i \leqslant r$, and observe $\vartheta_1' = 0$. Combining Lemma 10.3(i) and (103), we routinely find

$$\vartheta_{i+1}' = \vartheta_i' \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} \quad (1 \leqslant i \leqslant r-1). \tag{106}$$

Apparently $\vartheta_i' = 0$ for $1 \leqslant i \leqslant r$, and the result follows. □

We mention an identity that will be useful later.

Lemma 10.5. *With reference to Definition* 10.1, *assume* $\theta_0, \theta_1, \ldots, \theta_d$ *is recurrent. Then*

$$\frac{\theta_0 - \theta_1 + \theta_{i-1} - \theta_i}{\theta_0 - \theta_i} \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{\theta_0 + \theta_{i-1} - \theta_{d-i+1} - \theta_d}{\theta_0 - \theta_d} \tag{107}$$

for $1 \leqslant i \leqslant d$. (*Caution: the numerator on the far left in* (107) *might be* 0.)

Proof. Add (96) and Lemma 10.3(ii), solve the resulting equation for ϑ_{i+1}, and replace i by $i-1$ in the result. □

Here is another recursion.

Lemma 10.6. *With reference to Definition* 10.1, *assume* $\theta_0, \theta_1, \ldots, \theta_d$ *is* β*-recurrent, and put*

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (0 \leqslant i \leqslant d+1). \tag{108}$$

Then the sequence $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ *is* β*-recurrent.*

Proof. We show

$$\vartheta_{i-2} - (\beta+1)\vartheta_{i-1} + (\beta+1)\vartheta_i - \vartheta_{i+1} \tag{109}$$

is 0 for $2 \leqslant i \leqslant d$. First observe by (84) that

$$\theta_{j-1} - \beta\theta_j + \theta_{j+1} = \theta_{d-j-1} - \beta\theta_{d-j} + \theta_{d-j+1} \quad (1 \leqslant j \leqslant d-1). \tag{110}$$

Eliminating $\vartheta_{i-2}, \vartheta_{i-1}, \vartheta_i, \vartheta_{i+1}$ in (109) using (108), then cancelling terms where possible, and then simplifying the result using (110), we get 0. □

For completness sake, we include a lemma concerning the converse to Lemma 10.6. We do not use the result, so we will not dwell on the proof.

Lemma 10.7. *With reference to Definition* 10.1, *assume* $\theta_0, \theta_1, \ldots, \theta_d$ *is* β*-recurrent. Let* $\vartheta_0, \vartheta_1, \ldots, \vartheta_{d+1}$ *denote a* β*-recurrent sequence of scalars taken from* $\mathbb{K}$, *such that* $\vartheta_0 = 0$, $\vartheta_{d+1} = 0$, *and* $\vartheta_1 = \vartheta_d$. *Then*

$$\vartheta_i = \vartheta_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (0 \leqslant i \leqslant d+1).$$

Proof. Routine calculation using Lemmas 9.2, 9.3, and 10.2. □

11. Some equations involving the split canonical form

In this section, we return to the situation of Definition 4.1, and determine when the products $E_d A^* E_i$ vanish for $0 \leqslant i \leqslant d-2$. We begin with a definition.

Definition 11.1. With reference to Definition 4.1, we define

$$\vartheta_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \quad (1 \leqslant i \leqslant d), \tag{111}$$

and $\vartheta_0 = 0$, $\vartheta_{d+1} = 0$. We observe ϑ_1 equals the scalar ϕ_1 from Definition 7.1.

Our goal in this section is to prove the following theorem.

Theorem 11.2. *With reference to Definition* 4.1, *assume* $d \geqslant 2$. *Then the following are equivalent:*

(i) $E_d A^* E_i = 0 \quad (0 \leqslant i \leqslant d-2)$.

LEM 223 Assume $N \geq 1$

Assume $\{\theta_i\}_{i=0}^{N}$ are mut dist and β-rec

Put

$$\mathcal{g}_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} \qquad 0 \leq i \leq N+1$$

Then $\{\mathcal{g}_i\}_{i=0}^{N+1}$ is β-rec

pf Assume $N \geq 2$ else nothing to prove

$N = 2$:

$$\underset{\overset{\parallel}{0}}{\mathcal{g}_0} - (\beta+1)\underset{\overset{\parallel}{1}}{\mathcal{g}_1} + (\beta+1)\underset{\overset{\parallel}{1}}{\mathcal{g}_2} - \underset{\overset{\parallel}{0}}{\mathcal{g}_3} \overset{?}{=} 0$$

So ok

$N \geq 3$: Consider cases from L 10.2 in Handout

Case $\beta \neq \pm 2$ Pick $0 \neq q \in \overline{\mathbb{F}}$ s.t. $q + q^{-1} = \beta$

For $0 \leq i \leq N+1$

$$\mathcal{g}_i = \frac{(q^i - 1)(q^{N-i+1} - 1)}{(q - 1)(q^N - 1)}$$

$$= R + S q^i + T q^{-i} \qquad \text{some } R, S, T \in \overline{\mathbb{F}}$$

So

$\{\mathcal{g}_i\}_{i=0}^{N+1}$ is β-rec

other cases sim. $\square$

LEM 224 Assume $N \geq 1$

Assume $\{\theta_i\}_{i=0}^{N}$ are mut dist and β-rec.

Given $\{g_i\}_{i=0}^{N+1}$ in $\mathbb{F}$

TFAE

(i) $$g_i = g_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} \qquad 0 \leq i \leq N+1$$

(ii) $\{g_i\}_{i=0}^{N+1}$ is β-rec and

$$g_0 = 0, \qquad g_1 = g_N, \qquad g_{N+1} = 0 \qquad (*)$$

pf (i) $\rightarrow$ (ii)

$\{g_i\}_{i=0}^{N+1}$ is β-rec by L 223

$*$ is clear

(ii) $\rightarrow$ (i) Define

$$\Delta_i = g_i - g_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} \qquad 0 \leq i \leq N+1$$

show

$$\Delta_i = 0 \qquad 0 \leq i \leq N+1$$

By constr

$$\Delta_0 = 0, \qquad \Delta_1 = 0 \qquad \Delta_N = 0 \qquad \Delta_{N+1} = 0$$

By construction and L 223

$\{\Delta_i\}_{i=0}^{N+1}$ is β-rec

Assume $N \geq 3$ else done

Case $\beta \neq \mp 2$ pick $0 \neq q \in \bar{\mathbb{F}}$ $q + q^{-1} = \beta$

$\exists\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ s.t

$$\Delta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i} \qquad 0 \leq i \leq N+1$$

Also

$$q^i \neq 1 \qquad 1 \leq i \leq N$$

[Since $\{\theta_i\}_{i=0}^{N}$ are mut dist and β-rec, invoking L 9.3 in Handout]

Require $0 = \Delta_0,\quad 0 = \Delta_1 \quad 0 = \Delta_N$

Gives

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & q & q^{-1} \\ 1 & q^N & q^{-N} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

show coef matrix is Non sing

$$\det = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & q-1 & q^{-1}-1 \\ 0 & q^N-1 & q^{-N}-1 \end{pmatrix}$$

$$= (q-1)(q^N-1) \det \begin{pmatrix} 1 & -q^{-1} \\ 1 & -q^{-N} \end{pmatrix}$$

$$= (q-1)(q^N-1)(q^{N-1}-1) / q^{-N}$$

Each factor Non 0

$\neq 0$

So $\alpha_i = 0 \quad i = 1,2,3$ ✓

Case $\beta = 2$ char $\mathbb{F} \neq 2$

$\exists$ $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ s.t.

$$\Delta_i = \alpha_1 + \alpha_2 i + \alpha_3 i^2 \qquad 0 \le i \le N+1$$

Since $\{\theta_i\}_{i=0}^{N}$ are mut dist and β-rec

char $\mathbb{F} = 0$ or p, $p \geq N+1$

From $0 = \Delta_0$ $0 = \Delta_1$ $0 = \Delta_N$

get

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & N & N^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Coef matrix has det

$N(N-1)$

$N \neq 0$, $N-1 \neq 0$ in $\mathbb{F}$

so det $\neq 0$

$\alpha_i = 0$ $i = 1, 2, 3$ ✓

Case $\beta = -2$ char $\mathbb{F} \neq 2$

$\exists$ $\alpha_1, \alpha_2, \alpha_3 \in \bar{\mathbb{F}}$ s.t

$$\Delta_i = \alpha_1 + \alpha_2(-1)^i + \alpha_3 i(-1)^i \qquad 0 \le i \le N+1$$

Since $\{\theta_i\}_{i=0}^N$ are mut dist and β-rec

$$\text{char } \mathbb{F} = 0 \quad \text{or} \quad p, \quad 2p > N$$

From
$$0 = \Delta_0, \qquad 0 = \Delta_1 \qquad 0 = \Delta_N$$

get

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & (-1)^N & N(-1)^N \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

show coef mat is non sing

$$\det = \det \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & (-1)^N - 1 & N(-1)^N \end{pmatrix}$$ $*$

For $N = 2n$ even this is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & N \end{pmatrix}$$

$$\det = -2N$$
$$= -2^2 n$$

$2 \neq 0$ in $\mathbb{F}$ $n \neq 0$ in $\mathbb{F}$

det. $\neq 0$ so $\alpha_i = 0$ $i = 1, 2, 3$ ✓

For $N = 2n+1$ odd

$*$ is
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & -2 & -N \end{pmatrix}$$

$\det = -2(1-N)$

$= 2 \cdot 2 \cdot n$

$2 \neq 0$ in $\mathbb{F}$ $\quad n \neq 0$ in $\mathbb{F}$

$\det \neq 0$

$\alpha_i = 0 \quad i = 1, 2, 3$ ✓

Case $\beta = 0$ char $\mathbb{F} = 2$

$N = 3$ only

space of β-rec sequences has basis

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{matrix}$$

$\{\Delta_i\}_{i=0}^{N+1}$ is $\alpha_1(\text{row}1) + \alpha_2(\text{row}2) + \alpha_3(\text{row}3)$

$0 = \Delta_0 = \alpha_1$

$0 = \Delta_1 = \alpha_1 + \alpha_2 \longrightarrow \alpha_2 = 0$

$0 = \Delta_3 = \alpha_1 + \alpha_2 + \alpha_3 \longrightarrow \alpha_3 = 0$

so $\Delta_i = 0 \quad 0 \le i \le 4$ ✓ $\square$

Lec 30 Friday Nov 19

IV the tridiagonal Relations -

$\mathbb{F}$ arb

$N = 0, 1, 2 \ldots$

$V =$ v.s / $\mathbb{F}$ dim $N+1$

Given LS on V

$$\Phi = (A, \{E_i\}_{i=0}^N ; A^*, \{E_i^*\}_{i=0}^N)$$

eigval seq $\{\theta_i\}_{i=0}^N$

dual .. .. $\{\theta_i^*\}_{i=0}^N$

For not. conv

$E_{-1} = 0, \quad E_{N+1} = 0$

$E_{-1}^* = 0 \quad E_{N+1}^* = 0$

Goal is to prove

thm 225 $\exists\ \beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ s.t

$$0 = \left[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*\right], \qquad \text{TD1}$$

$$0 = \left[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A\right] \qquad \text{TD2}$$

Sequence $\beta, \gamma, \gamma^*, \delta, \delta^*$ is unique if $N \geq 3$

LEM 226 Fn $\beta, \gamma, \delta \in \mathbb{F}$ TFAE

(i) β, γ, δ satisfy TD1

(ii) $\{\theta_i\}_{i=0}^N$ is (β, γ, δ)-rec

pf Write

$$P(x, y) = x^2 - \beta x y + y^2 - \gamma(x+y) - \delta$$

Write

$C =$ RHS of TD1

$$C = \sum_{i=0}^N \sum_{j=0}^N E_i C E_j$$

Fn $0 \le i, j \le N$

$$E_i C E_j = E_i A^* E_j \, P(\theta_i, \theta_j)(\theta_i - \theta_j) \qquad *$$

(i) → (ii) $C = 0$

For $1 \le j \le N$

$$\begin{aligned} 0 &= E_{j-1} C E_j \\ &= \underset{\neq 0}{E_{j-1} A^* E_j} \, P(\theta_{j-1}, \theta_j) \underset{\neq 0}{(\theta_{j-1} - \theta_j)} \end{aligned}$$

$P(\theta_{j-1}, \theta_j) = 0$

(ii) → (i) Fn $0 \le i, j \le N$ in RHS(*) at least one factor is 0

so $E_i C E_j = 0$

so $C = 0$

$\square$

LEM 227

The following hold for $0 \le i, j \le N$

(i) $E_i^* A^r E_j^* = 0$ if $0 \le r < |i-j|$

(ii) $E_i^* A^r E_j^* \ne 0$ if $r = |i-j|$

(iii) For $0 \le r, s \le N$

$$E_i^* A^r A^* A^s E_j^* = \begin{cases} \theta_{j+s}^* E_i^* A^{r+s} E_j^* & \text{if } i-j = r+s \\ \theta_{i+r}^* E_i^* A^{r+s} E_j^* & \text{if } j-i = r+s \\ 0 & \text{if } |i-j| > r+s \end{cases}$$

pf Represent Φ by matrices in $Mat_{N+1}(\mathbb{F})$

A^*: $\operatorname{diag}(\theta_i^*)_{i=0}^N$

A: irred tridiag

then E_i^*: $\operatorname{diag}(0, 0 \ldots, \underset{i}{1}, 0, \ldots 0)$ $\quad 0 \le i \le N$

Above facts routinely checked. $\square$

Let $\mathcal{D}$ = subalg of End V gen by A

LEM 22 8 Put

$$L_i = E_0 + E_1 + \cdots + E_i \qquad 0 \le i \le N$$

then

(i) $\{L_i\}_{i=0}^N$ is basis for $\mathcal{D}$

(ii) $$L_i A^* - A^* L_i = E_i A^* E_{i+1} - E_{i+1} A^* E_i \qquad 0 \le i \le N$$

pf (i) Recall $\{E_i\}_{i=0}^N$ is basis for $\mathcal{D}$

(ii) For $0 \le j \le N$

$$\begin{aligned} E_j A^* &= E_j A^* (E_0 + \cdots + E_N) \\ &= E_j A^* E_{j-1} + E_j A^* E_j + E_j A^* E_{j+1} \qquad (*) \end{aligned}$$

Sim

$$A^* E_j = E_{j-1} A^* E_j + E_j A^* E_j + E_{j+1} A^* E_j \qquad (**)$$

Now sum $*$ over $j = 0, 1, \ldots, i$

" $**$ "

take difference and cancel terms. $\square$

LEM 229

$$\mathrm{Span}\{ X A^* Y - Y A^* X \mid X, Y \in \mathcal{D} \} = \{ Z A^* - A^* Z \mid Z \in \mathcal{D} \}$$

pf

$$\begin{aligned} LHS &= \mathrm{Span}\{ E_i A^* E_j - E_j A^* E_i \mid 0 \le i, j \le N \} \\ &= \mathrm{Span}\{ E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \le i \le N \} \\ &= \mathrm{Span}\{ L_i A^* - A^* L_i \mid 0 \le i \le N \} \\ &= RHS \end{aligned}$$

$\square$

pf of th 225

First assume $N \geq 3$

By L 229 (with $X = A^2$, $Y = A$) $\exists\ Z \in \mathcal{D}$ s.t.

$$A^2 A^* A - A A^* A^2 = Z A^* - A^* Z \qquad (*)$$

$\exists$ poly $p \in \mathbb{F}[x]$ of deg $\leq N$ s.t

$$Z = p(A)$$

Let

$$k = \deg p$$

$$c = \text{leading coef of } p$$

show $k = 3$

Suppose $k > 3$, From $*$

$$\underbrace{E_k^*\left(A^2A^*A - AA^*A^2\right)E_0^*}_{\overset{\text{L 227}}{=}\ 0} = \underbrace{E_k^*\left(ZA^* - A^*Z\right)E_0^*}_{\overset{\text{L 227}}{=}\ c\,(\theta_0^* - \theta_k^*)\,E_k^* A^k E_0^*}$$

$$\underset{\neq 0}{c}\ \underbrace{(\theta_0^* - \theta_k^*)}_{\neq 0}\ \underbrace{E_k^* A^k E_0^*}_{\neq 0 \text{ L 227}}$$

cont

11/19/10

7

Suppose $k<3$

$$\underbrace{E_3^*\left(A^2A^*A - AA^*A^2\right)E_0^*}_{\| \ L227} = \underbrace{E_3^*\left(ZA^*-A^*Z\right)E_0^*}_{\| \ L227 \atop 0}$$

$$(\theta_1^* - \theta_2^*)\ \underbrace{E_3^*A^3E_0^*}_{\neq \ L227 \atop 0}$$
$\neq$
0

cont

So $k=3$. Def $\beta = c^{-1}-1$ so $\beta+1 = c^{-1}$

Now in * divide thru by c. Get

$$(\beta+1)\left(A^2A^*A - AA^*A^2\right) =$$

$$A^3A^* - A^*A^3 - \gamma\left(A^2A^* - A^*A^2\right) - \delta\left(AA^* - A^*A\right)$$

for some $\gamma, \delta \in \mathbb{F}$

This gives TD1.

For $2 \le i \le N-1$

$$0 = E_{i-2}^*\left(\text{RHS of TDD}\right)E_{i+1}^*$$

$$= \underbrace{E_{i-2}^*A^3E_{i+1}^*}_{\neq \atop 0}\left(\theta_{i-2}^* - (\beta+1)\theta_{i-1}^* + (\beta+1)\theta_i^* - \theta_{i+1}^*\right) \qquad \text{by } L\ 227$$

So $\{\theta_i^*\}_{i=0}^N$ is β-rec

So $\exists\ \gamma^* \in \mathbb{F}$ s.t

$\{\theta_i^*\}_{i=0}^N$ is (β, γ^*)-rec

So $\exists\ \delta^* \in \mathbb{F}$ s.t

$$\{\theta_i^*\}_{i=0}^N \quad \text{is} \quad (\beta, \gamma^*, \delta^*)\text{-rec}$$

By this and L226 (applied to Φ^*)

$$\beta, \gamma^*, \delta^* \text{ sat TD2}$$

So far we have shown $\exists\ \beta, \gamma, \gamma^*, \delta, \delta^*$ that sat TD1, TD2 (For $N \geq 3$)

show this seq is unique.

Given any $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ that sat TD1, TD2

By L226

$$\{\theta_i\}_{i=0}^N \quad \text{is} \quad (\beta, \gamma, \delta)-\text{rec}$$

$$\quad\quad\quad\quad \text{"} \quad (\beta, \gamma)-\text{rec}$$

$$\quad\quad\quad\quad \text{"} \quad \beta-\text{rec}$$

similarly $\{\theta_i^*\}_{i=0}^N$ is $(\beta, \gamma^*, \delta^*)$-rec

$$\quad\quad\quad\quad \text{"} \quad (\beta, \gamma^*)-\text{rec}$$

$$\quad\quad\quad\quad \text{"} \quad \beta-\text{rec}$$

So

$$\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \qquad 2 \leq i \leq N-1$$

$$= \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

is det-

$$\checkmark \quad \gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1} \qquad 1 \leq i \leq N-1$$

$$\checkmark \quad \gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* \qquad \text{"}$$

$$\checkmark \quad \delta = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \qquad 1 \leq i \leq N$$

$$\checkmark \quad \delta^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \qquad \text{"}$$

Done for $N \geq 3$

Case $N \leq 2$ ex

$\square$

Cor 230 the expressn

$$\frac{\theta_{i-2}-\theta_{i-1}}{\theta_{i-1}-\theta_i} \qquad \frac{\theta^*_{i-2}-\theta^*_{i-1}}{\theta^*_{i-1}-\theta^*_i}$$

are equal and indep of i for $2 \le i \le N-1$

Aside We have shown A, A^* sat TD1, TD2

A, A^* satisfy some relations with lower degree but more parameters called the Askey Wilson rels. (dont need)

thm 231 $\exists\ \beta, \gamma, \gamma^*, \delta, \delta^*, \omega, \eta, \eta^* \in \mathbb{F}$ s.t.

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = \gamma A^2 + \omega A + \eta I \qquad \text{AW1}$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A = \gamma^* A^{*2} + \omega A^* + \eta^* I \qquad \text{AW2}$$

Above Sequence is unique if $N \geq 3$

pf very sim to pf of claim2 in th 180 (which showed AW1, AW2 for case of Krawtchouk type) $\square$

V Conclusion

$\mathbb{F}$ arb

$N = 0, 1, 2, \ldots$

$\{\theta_i\}_{i=0}^N \quad \{\theta_i^*\}_{i=0}^N \quad \{\varphi_i\}_{i=1}^N$ arb. scalars in $\mathbb{F}$

Def $A, A^* \in \mathrm{Mat}_{N+1}(\mathbb{F})$ by

$$A = \begin{pmatrix} \theta_0 & & & \mathbf{0} \\ 1 & \theta_1 & & \\ & 1 & \ddots & \\ \mathbf{0} & & \ddots & \\ & & 1 & \theta_N \end{pmatrix} \qquad A^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & \mathbf{0} \\ & \theta_1^* & \ddots & \\ & & \ddots & \varphi_N \\ \mathbf{0} & & & \theta_N^* \end{pmatrix}$$

LEM 232 Given $\beta, \gamma, \delta \in \mathbb{F}$ s.t.

$\{\theta_i\}_{i=0}^N$ is (β, γ, δ)-rec, $\{\theta_i^*\}_{i=0}^N$ is β-rec.

Consider

$$\left[A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* \right] \qquad (*)$$

then the $(i, i-2)$-entry is

$$g_{i-2} - (\beta+1) g_{i-1} + (\beta+1) g_i - g_{i+1}$$

for $2 \le i \le N$ where

$$g_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_N) \qquad 1 \le i \le N$$

$$g_0 = 0, \qquad g_{N+1} = 0$$

All other entries of $(*)$ are 0.

pf Matrix mult. $\square$

LEM 233 With ref to L 232

$$(*) = 0$$

$\Leftrightarrow$

$$\{g_i\}_{i=0}^{N+1} \text{ is } \beta\text{-rec}$$

pfv $\square$

Pf of th 222

First assume $\exists$ LS

$$\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N) \quad \text{over } \mathbb{F}$$

with PA $(\{\theta_i\}_{i=0}^N, \{\theta_i^*\}_{i=0}^N, \{\varphi_i\}_{i=1}^N, \{\phi_i\}_{i=1}^N)$

show this PA satisfies PA1 — PA5.

PA1: ✓

PA2: ✓

PA5: Cor 230

PA3: Def

$$g_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_N) \qquad 1 \le i \le N$$

show

$$g_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} \qquad 1 \le i \le N$$

Assume $N \ge 1$ else done.
Invoke L224

Pick $\beta \in \mathbb{F}$ st $\{\theta_i\}_{i=0}^N$, $\{\theta_i^*\}_{i=0}^N$ β-rec

Def

$$g_0 = 0, \qquad g_{N+1} = 0$$

Need to show

(i) $g_1 = \phi_1$

(ii) $g_N = g_1$

(iii) $\{g_i\}_{i=0}^{N+1}$ is β-rec (i): By L220(i)

$$\phi_1 = (\theta_1^* - \theta_0^*)(\theta_N - a_0)$$

By L211(i)

$$\varphi_1 = (\theta_1^* - \theta_0^*)(\theta_0 - a_0)$$

So

$$\phi_1 = \varphi_1 - (\theta_1^* - \theta_0^*)(\theta_0 - \theta_N)$$
$$= g_1$$

(ii) Use thm 213. Obs

$$E_N A^* E_i = 0 \qquad 0 \le i \le N-2$$

E_0^* is normalizing by L 193 so

$$E_N E_0^* \ne 0$$

So $\quad g_1 = g_N \quad$ by th 213

(iii) Represent A, A^* by matrices via $\natural$

$$A^\natural = \begin{pmatrix} \theta_0 & \theta_1 & & & 0 \\ 1 & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_N \end{pmatrix} \qquad A^{*\natural} = \begin{pmatrix} \theta_0^* & \varphi_1 & & 0 \\ & \theta_1^* & \ddots & \\ & & \ddots & \varphi_N \\ 0 & & & \theta_N^* \end{pmatrix}$$

Since $\{\theta_i\}_{i=0}^N$ is β-rec $\exists\ \gamma, \delta \in \mathbb{F}$ s.t $\{\theta_i\}_{i=0}^N$ is (β, γ, δ)-rec.

Now β, γ, δ sat TD1 by L226

Now by L233

$$\{ j_{i+1} \}_{i=0}^{N} \quad \text{is } \beta\text{-rec}$$

PA3 proved ✓

PA4 Apply PA3 to Φ^{Ψ}

Done in one direction

Next, given scalars

$$\left(\{\theta_i\}_{i=0}^{N}, \{\theta_i^*\}_{i=0}^{N}, \{\phi_i^*\}_{i=1}^{N} \quad \{\phi_i\}_{i=1}^{N} \right) \qquad *$$

in $\mathbb{F}$ that sat PA1 – PA5 Show $\exists$ LS Φ over $\mathbb{F}$

with PA $*$.

Let V denote a vector space over $\mathbb{F}$ dim $N+1$

Pick basis $\{u_i\}_{i=0}^N$ for V

Define $A, A^* \in \operatorname{End} V$ s.t. rel $\{u_i\}_{i=0}^N$

$$A: \begin{pmatrix} \theta_0 & & & 0 \\ 1 & \theta_1 & & \\ & 1 & \ddots & \\ 0 & & \ddots & \\ & & 1 & \theta_N \end{pmatrix} \qquad A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & 0 \\ & \theta_1^* & \ddots & \\ & & \ddots & \varphi_N \\ 0 & & & \theta_N^* \end{pmatrix}$$

Obs

A is mult free eigvals $\{\theta_i\}_{i=0}^N$

Let E_i = pr idempotent for A and θ_i $(0 \le i \le N)$

A^* is mult free eigvals $\{\theta_i^*\}_{i=0}^N$

Let E_i^* = pr idemp for A^* and θ_i^* $(0 \le i \le N)$

show $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$

is LS on V with PA *

To show Φ is LS on V it suffices to show the following for $0 \le i, j \le N$

$$E_i^* A E_j^* = 0 \qquad \text{if} \quad 1 < i-j \le N \tag{1}$$

$$E_i^* A E_j^* = 0 \qquad \text{if} \quad 1 < j-i \le N \tag{2}$$

$$E_i A^* E_j = 0 \qquad \text{if} \quad 1 \le j-i \le N \tag{3}$$

$$E_i A^* E_j = 0 \qquad \text{if} \quad 1 < i-j < N \tag{4}$$

$$E_i A^* E_j = 0 \qquad \text{if} \quad 1 < i-j = N \tag{5}$$

$$E_i^* A E_j^* \ne 0 \qquad \text{if} \quad i-j = 1 \tag{6}$$

$$E_i^* A E_j^* \ne 0 \qquad \text{if} \quad j-i = 1 \tag{7}$$

$$E_i A^* E_j \ne 0 \qquad \text{if} \quad i-j = 1 \tag{8}$$

$$E_i A^* E_j \ne 0 \qquad \text{if} \quad j-i = 1 \tag{9}$$

— . —

For $0 \le i \le N$ def

$$U_i = \mathbb{F} u_i$$

So

- $\{U_i\}_{i=0}^N$ is dec of V
- $(A - \theta_i I) U_i = U_{i+1} \qquad 0 \le i \le N$
- $(A^* - \theta_i^* I) U_i = U_{i-1} \qquad 0 \le i \le N$

(1), (3), (6) : Prop 202

(9) : LEM 205

Pick $\beta \in \mathbb{F}$ s.t each of $\{\theta_i\}_{i=0}^N$, $\{\theta_i^*\}_{i=0}^N$ is β-rec.

$\exists\ \gamma, \delta \in \mathbb{F}$ s.t. $\{\theta_i\}_{i=0}^N$ is (β, γ, δ)-rec

For $1 \le i \le N$ def

$$g_i = \varphi_i - (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_N)$$

$$= \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N}$$

Put $g_0 = 0$, $g_{N+1} = 0$

Obs $g_1 = \phi_1$, $g_N = \phi_1$

(4) By L 2.24

$$\{g_i\}_{i=0}^{N+1} \text{ is } \beta\text{-rec}$$

So by L 2.33

$$\beta, \gamma, \delta \text{ sats TD1}$$

For $0 \le i, j \le N$

$$0 = E_i \left(\text{RHS of TD1} \right) E_j$$

$$= E_i A^* E_j \, (\theta_i - \theta_j) \, P(\theta_i, \theta_j)$$

$$P(x, y) = x^2 - \beta x y + y^2 - \gamma(x+y) - \delta$$

$\{\theta_h\}_{h=0}^{N}$ is (β, γ, δ)-rec so

$$P(\theta_{h-1}, \theta_h) = 0 \qquad 1 \le h \le N$$

So for $1 \le h \le N-1$

roots of $P(x, \theta_h)$ are $\theta_{h-1}, \theta_{h+1}$

So

$$P(\theta_i, \theta_j) \neq 0 \qquad 1 < j - i < N$$

So

$$E_i A^* E_j = 0 \qquad 1 < j - i < N$$ ✓

(5) Assume $N \geq 2$ else done, Show $E_N A^* E_0 = 0$

Show
$$E_N A^* E_0 U_i = 0 \qquad 0 \leq i \leq N \qquad *$$

By L194
$$U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V \qquad 0 \leq i \leq N$$
$$U_i + \cdots + U_N = E_i V + \cdots + E_N V \qquad 0 \leq i \leq N$$

For $1 \leq i \leq N$ $*$ holds since
$$\begin{aligned} E_N A^* E_0 U_i &\subseteq E_N A^* E_0 (U_i + \cdots + U_N) \\ &= E_N A^* E_0 (E_i V + \cdots + E_N V) \\ &= 0 \end{aligned}$$

Now show $*$ for $i = 0$

Obs
$$U_0 = E_0^* V$$

Show
$$E_N A^* E_0 E_0^* = 0$$

Invoke Th213 (i)
$$\sum_{i=0}^{N-2} E_N A^* E_i E_0^* (\theta_i - \theta_{N-1}) = (\varphi_1 - \varphi_N) E_N E_0^*$$

$$E_N A^* E_i = 0 \qquad 1 \leq i \leq N-2 \qquad \text{by (4)}$$
$$\varphi_1 - \varphi_N = \phi_1 - \phi_1 = 0$$

Get
$$E_N A^* E_0 E_0^* (\underbrace{\theta_0 - \theta_{N-1}}_{\neq 0}) = 0$$

$$E_N A^* E_0 E_0^* = 0 \quad \checkmark$$

$*$ holds for $0 \leq i \leq N$ so

$$E_N A^* E_0 = 0$$

(8) Def

$$U_i^{\Downarrow} = \tau_i(A) E_0^* V \qquad 0 \le i \le N$$

Applying Prop 201 to

$$\Phi^{\Downarrow} = (A, \{E_{N-i}\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$$

we find

- $\{U_i^{\Downarrow}\}_{i=0}^N$ is dec of V
- $(A - \theta_{N-i} I) U_i^{\Downarrow} = U_{i+1}^{\Downarrow} \qquad 0 \le i \le N$
- $(A^* - \theta_i^* I) U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad 0 \le i \le N$

Def

$$v_i = \tau_i(A) u_0 \qquad 0 \le i \le N$$
$$0 \ne u_0 \in E_0^* V$$

So $\quad 0 \ne v_i \in U_i^{\Downarrow} \qquad 0 \le i \le N$

So $\quad \{v_i\}_{i=0}^N$ basis for V

rel this basis

$$A: \begin{pmatrix} \theta_N & & & & 0 \\ 1 & \theta_{N-1} & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_0 \end{pmatrix}$$

$$A^*:\begin{pmatrix} \theta_0^* & z_1 & & & O \\ & \theta_1^* & z_2 & & \\ & & \ddots & \ddots & \\ & O & & \ddots & z_N \\ & & & & \theta_N^* \end{pmatrix}.$$

Some $\quad z_i \in \mathbb{F} \quad\quad 1 \le i \le N$

Trying to show $\quad E_i A^* E_{i-1} \neq 0 \quad\quad 1 \le i \le N$

Applying L 205 to $\Phi^\Downarrow$ suf to show

$$z_i \neq 0 \quad\quad 1 \le i \le N$$

To do this show $\quad z_i = \phi_i \quad\quad 1 \le i \le N$

By PA4

$$\phi_i - (\theta_i^* - \theta_0^*)(\theta_{N-i+1} - \theta_0) = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} \quad\quad 1 \le i \le N$$

Def

$$g_i^\Downarrow = z_i - (\theta_i^* - \theta_0^*)(\theta_{N-i+1} - \theta_0) \quad\quad 1 \le i \le N$$

Show

$$g_i^\Downarrow = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} \quad\quad 1 \le i \le N$$

Put $\quad g_0^\Downarrow = 0, \quad\quad g_{N+1}^\Downarrow = 0$

By L 224 suf to show

(i) $\quad g_1^\Downarrow = \varphi_1$

(ii) $\quad g_1^\Downarrow = g_N^\Downarrow$

(iii) $\quad \{g_i^\Downarrow\}_{i=0}^{N+1}$ is β-rec

(i) Apply 211 (i) to $\Phi^{\Downarrow}$

$$z_1 = (\theta_1^* - \theta_0^*)(\theta_N - a_0)$$

Apply L211 (i) to Φ

$$\varphi_1 = (\theta_1^* - \theta_0^*)(\theta_0 - a_0)$$

So

$$z_1 - \varphi_1 = (\theta_1^* - \theta_0^*)(\theta_N - \theta_0)$$

$$g_1^{\Downarrow} = z_1 - (\theta_1^* - \theta_0^*)(\theta_N - \theta_0)$$
$$= \varphi_1 \quad \checkmark$$

(ii) Apply th 213 to $\Phi^{\Downarrow}$:

$$\sum_{i=0}^{N-2} E_0 A^* E_{N-i} E_0^* (\theta_{N-i} - \theta_1) = (g_1^{\Downarrow} - g_N^{\Downarrow}) E_0 E_0^*$$

LHS $= 0$ since we have shown

$$E_0 A^* E_j = 0 \qquad 2 \le j \le N$$

Recall E_0^* is normalizing so $E_0 E_0^* \neq 0$

So

$$g_1^{\Downarrow} = g_N^{\Downarrow}$$

(iii) Recall $\{\theta_i\}_{i=0}^{N}$ is (β, γ, δ)-rec and $\{\theta_i^*\}_{i=0}^{N}$ is β-rec.

We saw β, γ, δ sat TD1

Apply L 232, 233 to $\Phi^{\Downarrow}$ to get

$$\{\theta_i^{\Downarrow}\}_{i=0}^{N+1} \text{ is } \beta\text{-rec}$$

We have shown (i)–(iii) so

$$z_i = \phi_i \qquad 1 \le i \le N$$

Now $\quad z_i \ne 0 \qquad 1 \le i \le N$

So $\quad E_i A^* E_{i-1} \ne 0 \qquad 1 \le i \le N$ ✓

(2), (7):

Apply "3/4 thm" 18 9

In that thm we showed (i), (iii), (iv).

So (ii) holds.

— , —

We have shown (1) – (9) so

Φ is LS on V

By cnstr

$\{\theta_i\}_{i=0}^N$ is eigval seq of Φ

$\{\theta_i^*\}_{i=0}^N$ is dual eigval seq of Φ

$\{\varphi_i\}_{i=1}^N$ is 1st split seq of Φ

By Note 216

2nd split seq of Φ is $\{z_i\}_{i=1}^N$

We showed

$z_i = \phi_i \qquad 1 \le i \le N$

So $\{\phi_i\}_{i=1}^N$ is 2nd split seq of Φ

So $(\{\theta_i\}_{i=0}^N, \{\theta_i^*\}_{i=0}^N, \{\varphi_i\}_{i=1}^N, \{\phi_i\}_{i=1}^N)$

is PA of Φ

Φ unique up to iso by Cor 219

$\square$

Good

LEM Assume

$E_i^* A E_j^* \neq 0$ if $|i-j|=1$ $\quad 0 \le i,j \le N$

then V is indec as module for A, A^*

pf W submod contain $E_0^*V + \dots + E_N^*V$

Good

Lem Assume

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j|>1 \\ \neq 0 & \text{if } |i-j|=1 \end{cases}$$

$$E_i A^* E_j = 0 \text{ if } |i-j|>1$$

For $|i-j|=1$,

$\to$ $E_i A^* E_j \neq 0$

or $E_j A^* E_i \neq 0$

pf else V is dec

Get φ_i, ϕ_i as before

φ_i, ϕ_i vs a_i as before

$\varphi_i \neq 0$ or $\phi_0 \neq 0$

PA3 $\quad \varphi_i = \phi_1 \sum \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + (\ )(\ )$

PA4 $\quad \phi_i = \varphi_1 \sum \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + (\ )(\ )$

L 226 OK

L 225 (1st half)

$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}$ + ...

PA5 equal indep of i

TD1

$\{\theta_i^*\}$ β-rec

pf def $g_i = \varphi_i - \phi_i(\ )(\ )$

$g_1 = \phi_1 \qquad g_N = \phi_1$

W Lowering maps

$I \quad \psi \quad \psi A,\ A\psi$

ψ

$\theta_i^* = 0$

$\varphi_i = \phi_i = g_i$

$\theta_i^* = q^i \quad \phi_1 = 0$

q^{-i}

0