Lecture 19 Monday Oct 25

We continue to discuss how the Krawtchouk polynomials

$\{K_n(x; p, N)\}_{n=0}^N$ are related to the Lie algebra $L = sl_2(\mathbb{F})$

Next goal: construct certain L-module.

Given commuting indets y, z

Consider $\mathbb{F}$-algebra $\underbrace{\mathbb{F}[y, z]}_{= A}$

A has basis

$$y^r z^s \qquad 0 \le r, s < \infty$$

For $n = 0, 1, 2, \ldots$ define

$$\underbrace{\mathrm{Hom}_n(A)}_{= H_n} = \mathrm{Span}\{y^{n-i} z^i\}_{i=0}^n$$

"nth Homogeneous component"

$$\dim H_n = n+1 \qquad n = 0, 1, 2 \ldots$$

$$A = \sum_{n=0}^{\infty} H_n \qquad \text{(ds sus)}$$

$$H_n H_m = H_{n+m} \qquad 0 \le n, m < \infty$$

$$H_0 = \mathbb{F}1$$

$$H_1 = \mathbb{F}y + \mathbb{F}z.$$

Aside on Lie algebras

Let V = any vector space over $\mathbb{F}$ (pos $\dim V = \infty$)

$\mathrm{End}(V)$ = (assoc) $\mathbb{F}$-algebra of all lin trans $V \to V$

$gl(V)$ = Lie algebra consisting of $\mathbb{F}$-vectorspace $\mathrm{End}(V)$ together with Lie bracket

$$[\varphi, \phi] = \varphi\phi - \phi\varphi$$

Back to $\mathcal{A}$

A <u>derivation</u> of $\mathcal{A}$ is an element $\partial \in gl(\mathcal{A})$ such that

$$\partial(ab) = \partial(a)b + a\partial(b) \qquad \forall\, a, b \in \mathcal{A}$$

Define
$$\mathrm{Der}(\mathcal{A}) = \text{set of all derivations of } \mathcal{A}$$

One checks
$$\mathrm{Der}(\mathcal{A}) = \text{Lie subalgebra of } gl(\mathcal{A})$$

Given $\partial \in \mathrm{Der}(\mathcal{A})$ observe:

$$\partial(1) = 0$$

$$\partial(a^n) = n a^{n-1} \partial(a) \qquad \begin{array}{l} n = 1, 2, \ldots \\ a \in \mathcal{A} \end{array}$$

For $0 \le r, s < \infty$

$$\partial(y^r z^s) = r y^{r-1} z^s \partial(y) + s y^r z^{s-1} \partial(z)$$

So

∂ is determined by $\partial(y)$, $\partial(z)$

So

∂ is determined by its action on $Hom_1(A)$

We emphasise

$\partial = 0 \iff \partial$ vanishes on $Hom_1(A)$ $\quad *$

LEM 14.0 Given an $\mathbb{F}$-linear map

$$\varphi : \mathrm{Hom}_1(\mathcal{A}) \to \mathcal{A}$$

$\exists$ unique $\partial = \partial_\varphi \in \mathrm{Der}(\mathcal{A})$ s.t.

$$\partial\Big|_{\mathrm{Hom}_1(A)} = \varphi$$

" φ has unique extension in $\mathrm{Der}(\mathcal{A})$ "

pf $\exists$ $\partial \in gl(\mathcal{A})$ s.t.

$$\partial(y^r z^s) = r y^{r-1} z^s \varphi(y) + s y^r z^{s-1} \varphi(z)$$

$$0 \le r, s < \infty$$

One checks

$$\partial \in \mathrm{Der}(\mathcal{A})$$

By constr

$$\partial(y) = \varphi(y), \qquad \partial(z) = \varphi(z)$$

So

$$\partial\Big|_{\mathrm{Hom}_1(A)} = \varphi$$

We have shown ∂ exists.

∂ is unique by $\ast$.

$\square$

The Lie algebra $L = \mathfrak{sl}_2(\mathbb{F})$ acts by left mult on vector space $\mathbb{F}^2$ (col vectors)

$\mathrm{Hom}_1(A)$ has basis y, z

Consider vector space iso

$$\mathrm{Hom}_1(A) \to \mathbb{F}^2$$
$$y \to \begin{pmatrix}1\\0\end{pmatrix}$$
$$z \to \begin{pmatrix}0\\1\end{pmatrix}$$

This induces L-module structure on $\mathrm{Hom}_1(A)$ s.t.

$e.y = 0$ $\qquad$ $e.z = y$

$h.y = y$ $\qquad$ $h.z = -z$

$f.y = z$ $\qquad$ $f.z = 0$

LEM 141

(i) The map $L \longrightarrow \mathrm{Der}(A)$

$\varphi \longrightarrow \partial_\varphi$

is a homomorphism of Lie algebras.

(ii) This hom is injective.

pf (i) Given $\varphi, \phi \in L$

$$\partial_{[\varphi, \phi]} \stackrel{?}{=} [\partial_\varphi, \partial_\phi]$$

Both sides in $\mathrm{Der}(A)$

Both sides agree on $\mathrm{Hom}_1(A)$

Both sides are equal by $*$.

(ii) By construction. $\square$

We have now shown the following.

Thm 142. $\mathcal{A}$ has an L-module structure such that each element of L acts on $\mathcal{A}$ as a derivation and

$$e.y = 0 \qquad e.z = y$$
$$h.y = y \qquad h.z = -z$$
$$f.y = z \qquad f.z = 0$$

$\square$

From now on we fix an integer $N \geq 0$ and consider the homogenous component

$$\mathrm{Hom}_N(A) = H_N = V$$

V has a basis

$$\{ y^{N-i} z^i \}_{i=0}^{N}$$

LEM 143 With respect to above basis the matrices rep e, h, f are

$$e: \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & \ddots & \\ & & & \ddots & N \\ & & & & 0 \end{pmatrix}$$

$$h: \begin{pmatrix} N & & & & \\ & N-2 & & & \\ & & N-4 & & \\ & & & \ddots & \\ & & & & -N \end{pmatrix}$$

$$f: \begin{pmatrix} 0 & & & & \\ N & 0 & & & \\ & N-1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

pf For $i = 0, 1, \dots, N$

$$e \cdot y^{N-i} z^{i} = (N-i)\, y^{N-i-1} z^{i} \underbrace{e \cdot y}_{0} + i\, y^{N-i} z^{i-1} \underbrace{e \cdot z}_{y}$$

$$= i\, y^{N-i+1} z^{i-1}.$$

The case of h, f similar. $\square$

By LEM 143 $\quad V$ is L-submodule of $\mathcal{A}$.

One checks L-module V is irred.

For $i = 0, 1, \ldots, N$ define

$$V_i = \mathbb{F} y^{N-i} z^i$$

So

$$\dim V_i = 1$$

V_i is $\underbrace{\text{eigenspace of } h}_{\text{"}h\text{-weight space"}}$ for eigval $N-2i$

$$V = \sum_{i=0}^{N} V_i \qquad \text{(ds of vs)}$$

" h-wt space decomp of V "

find h^*- wt space decomp of V :

Recall y, z is basis for $\mathrm{Hom}_1(A)$

Recall matrix

$$R = \begin{pmatrix} 1-p & 1-p \\ p & p-1 \end{pmatrix}$$

View R as transition matrix from y, z to new basis y^*, z^*:

$$y^* = (1-p)y + pz$$

$$z^* = (1-p)y + (p-1)z$$

Obs

$$y = y^* + \frac{p}{1-p} z^*$$

$$z = y^* - z^*$$

LEM 144. The elements e^*, h^*, f^* act on y^*, z^* as follows

$$e^*. y^* = 0 \qquad e^*. z^* = y^*$$

$$h^*. y^* = y^* \qquad h^*. z^* = -z^*$$

$$f^*. y^* = z^* \qquad f^*. z^* = 0$$

pf

$$e^*. y^* = \left((p-1)e + ph + \frac{p^2}{1-p} f\right)\left((1-p)y + pz\right)$$

$$= (p-1)\left(0 + py\right)$$

$$+ p\left((1-p)y - pz\right)$$

$$+ \frac{p^2}{1-p}\left((1-p)z + 0\right)$$

$$= 0$$

The other cases are similar. $\square$

By cnstr the following is a basis for V

$$\left\{ (y^*)^{N-i} z^{*i} \right\}_{i=0}^{N}$$

LEM 145 With respect to above basis the matrices rep e^*, h^*, f^* are

$$e^*: \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & 2 & & \\ & & 0 & \ddots & \\ & \mathbf{0} & & \ddots & N \\ & & & & 0 \end{pmatrix}$$

$$h^*: \begin{pmatrix} N & & & & \mathbf{0} \\ & N-2 & & & \\ & & N-4 & & \\ & \mathbf{0} & & \ddots & \\ & & & & -N \end{pmatrix}$$

$$f^*: \begin{pmatrix} 0 & & & & \mathbf{0} \\ N & 0 & & & \\ & N-1 & 0 & & \\ & & \ddots & \ddots & \\ & \mathbf{0} & & 1 & 0 \end{pmatrix}$$

pf Since e^* acts on $\mathcal{A}$ as a derivation,

$$e^* . (y^*)^{N-i} (z^*)^i =$$

$$(N-i)(y^*)^{N-i-1}(z^*)^i \underbrace{e^* . y^*}_{0}$$

$$+ \quad i (y^*)^{N-i} (z^*)^{i-1} \underbrace{e^* . z^*}_{y^*}$$

$$= i (y^*)^{N-i+1} (z^*)^{i-1}.$$

The case of h^*, f^* is similar. $\square$

For $i = 0, 1, 2, \ldots N$

define

$$V_i^* = \mathbb{F}(y^*)^{N-i}(z^*)^i$$

So

$$\dim V_i^* = 1$$

V_i^* is h^*-wt space for eigval $N-2i$

$$V = \sum_{i=0}^{N} V_i^* \qquad (\text{ds of vs})$$

" h^*-wt space decomp of V "

DEF 146 For an L-module V we define the lin trans

$$A : V \to V \qquad A^* : V \to V$$

by

$$A = \frac{NI - a}{2} \qquad A^* = \frac{NI - a^*}{2}$$

LEM 147

(i) With respect to the basis $\{y^{N-i} z^i\}_{i=0}^N$ the matrices rep A and A^* are

A : B^t

A^* : D

(ii) With respect to the basis $\{(y^*)^{N-i} (z^*)^i\}_{i=0}^N$ the matrices rep A, A^* are

A : D

A^* : B^t

pf (i) Use L 143 and

$$a = 2(1-p)e + (1-2p)h + 2pf$$
$$a^* = h$$

(ii) Use L 145 and

$$a = h^*$$
$$a^* = 2(1-p)e^* + (1-2p)h^* + 2pf^*$$

$\square$

Lecture 20 Wednesday Oct 27

We continue to discuss the L-submodule

$V = \mathrm{Hom}_N(A)$ of $A = \mathbb{F}[y,z]$

We now define a symmetric bilinear form

$$\langle\ ,\ \rangle : V \times V \to \mathbb{F}$$

As we will see, both

$$\langle V_i, V_j \rangle = 0 \quad \text{if } i \neq j \qquad 0 \le i,j \le N \qquad *$$

$$\langle V_i^*, V_j^* \rangle = 0 \quad \text{if } i \neq j \qquad 0 \le i,j \le N \qquad **$$

DEF 148 We define $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ such that

$$\left\langle y^{N-i} z^i, \; y^{N-j} z^j \right\rangle = \delta_{ij} \frac{(N-i)!\, i!}{N!\, (1-p)^{N-i} p^i} = \frac{1}{k_i (1-p)^N}$$

$(0 \le i,j \le N)$

Obs $\langle\ ,\ \rangle$ is sym, nondeg, and satisfies $(*)$.

LEM 149 We have

$$\langle \varphi . u, v \rangle = \langle u, \varphi^{+} . v \rangle \qquad \forall \varphi \in L \quad \forall u, v \in V$$

where $+$ is the antiaut of L from L135.

pf WLOG $\varphi \in \{ e, h, f \}$

WLOG u, v in the basis $\{ y^{N-i} z^{i} \}_{i=0}^{N}$

Case $\varphi = e$

$$\underbrace{\langle \underbrace{e . y^{N-i} z^{i}}_{= \, i y^{N-i+1} z^{i-1}}, y^{N-j} z^{j} \rangle}_{= \, i \, \delta_{i-1,j} \frac{1}{k_j (1-p)^N}} \overset{?}{=} \underbrace{\langle y^{N-i} z^{i}, \underbrace{e^{+}}_{= \, \frac{p}{1-p} f} . y^{N-j} z^{j} \rangle}_{= \, \frac{p}{1-p} (N-j) y^{N-j-1} z^{j+1}}$$

$$\frac{p}{1-p} (N-j) \, \delta_{i,j+1} \frac{1}{k_i (1-p)^N}$$

$$\left[\begin{array}{ll} \delta_{i-1,j} = \delta_{i,j+1} & \\ \text{For } i-j=1 & k_i = k_j \frac{p}{1-p} \frac{N-j}{i} \end{array} \right]$$

$$LHS = RHS \checkmark$$

Cases $\varphi = h$, $\varphi = f$ similar

$\square$

LEM 150 $(**)$ holds.

pf

Assume $i \neq j$

Given $u \in V_i^*$ $v \in V_j^*$

$h^* u = (N-2i)u$ $h^* v = (N-2j)v$

$$(N-2i)\langle u, v\rangle = \langle h^* u, v\rangle$$
$$= \langle u, h^{*t} v\rangle \qquad h^{*t} = h^*$$
$$= \langle u, h^* v\rangle$$
$$= \langle u, v\rangle (N-2j)$$

So $\langle u, v\rangle = 0$ $\square$

LEM 151 With respect to $\langle\, , \rangle$ the basis for V dual to $\{y^{N-i}z^i\}_{i=0}^N$ is

$$y^{N-i} z^i \underbrace{\frac{N! \, (1-p)^{N-i} p^i}{(N-i)! \, i!}}_{k_i (1-p)^N} \qquad 0 \le i \le N$$

pf For $0 \le i, j \le N$

$$\langle y^{N-i} z^i, \; y^{N-j} z^j k_j (1-p)^N \rangle = \frac{\delta_{ij}}{k_i (1-p)^N} k_j (1-p)^N$$

$$= \delta_{ij} \qquad \square$$

LEM 152 For the dual bases in L 151

The sum of the basis vectors is $(y^*)^N$

pf Check

$$(y^*)^N \stackrel{?}{=} \sum_{i=0}^{N} y^{N-i} z^i k_i (1-p)^N$$

$$(y^*)^N = \left((1-p)y + pz\right)^N$$

binom thm

$$= \sum_{i=0}^{N} \underbrace{\binom{N}{i} (1-p)^{N-i} p^i}_{k_i (1-p)^N} y^{N-i} z^i$$

$\square$

LEM 153 For $0 \le i, j \le N$

$$\left\langle (y^*)^{N-i}(z^*)^i,\ (y^*)^{N-j}(z^*)^j \right\rangle = \delta_{ij} \underbrace{\frac{(N-i)!\, i!}{N!} \frac{(1-p)^i}{p^i}}_{k_i^{-1}}$$

pf Assume $i=j$ else done by L150

Ind on i

Case $i=0$: show

$$\| (y^*)^N \|^2 = 1$$

$$\|(y^*)^N\|^2 = \left\| \sum_{i=0}^{N} y^{N-i} z^i k_i (1-p)^N \right\|^2 \quad \text{by Lem 152}$$

$$= \sum_{i=0}^{N} \| y^{N-i} z^i k_i (1-p)^N \|^2 \quad \text{by L150}$$

$$= \sum_{i=0}^{N} k_i (1-p)^N \quad \text{by def of } \langle , \rangle$$

$$= \sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i$$

$$= (1-p+p)^N$$

$$= 1$$

Case $i \ge 1$:

By LEM 149

$$\left\langle e^*_\circ (y^*)^{N-i}(z^*)^i , (y^*)^{N-i+1}(z^*)^{i-1} \right\rangle$$

$$= \left\langle (y^*)^{N-i}(z^*)^i , \underbrace{(e^*)^+}_{= \frac{p}{1-p} f^*} (y^*)^{N-i+1}(z^*)^{i-1} \right\rangle$$

$$LHS = \| (y^*)^{N-i+1} (z^*)^{i-1} \|^2 \, i \qquad \text{by } L145$$

$$= \frac{i}{k_{i-1}} \qquad \text{by ind}$$

$$RHS = \| (y^*)^{N-i} (z^*)^i \|^2 (N-i+1) \frac{p}{1-p} \qquad \text{by } L145$$

So

$$\| (y^*)^{N-i} (z^*)^i \|^2 = \frac{i}{N-i+1} \frac{1}{k_{i-1}} \frac{1-p}{p}$$

$$= \frac{1}{k_i} \quad \checkmark$$

$\square$

LEM 154 With respect to $\langle\, ,\rangle$ the basis for V dual to $\{(y^*)^{N-i}(z^*)^i\}_{i=0}^N$ is

$$(y^*)^{N-i}(z^*)^i \underbrace{\frac{N! \qquad p^i}{(N-i)!\; i!\; (1-p)^i}}_{k_i} \qquad 0 \le i \le N$$

pf By L153, for $0 \le i, j \le N$

$$\left\langle (y^*)^{N-i}(z^*)^i,\; (y^*)^{N-j}(z^*)^j k_j \right\rangle = \frac{\delta_{ij}}{k_i} k_j$$
$$= \delta_{ij}$$

$\square$

LEM 155 For the dual basis in L 154 the sum of the basis vectors is y^N

pf (sim to Lem 152)

$$y^N \stackrel{?}{=} \sum_{i=0}^{N} (y^*)^{N-i} (z^*)^i k_i$$

$$y^N = \left(y^* + \frac{p}{1-p} z^* \right)^N$$

$$= \sum_{i=0}^{N} (y^*)^{N-i} (z^*)^i \underbrace{\binom{N}{i} \left(\frac{p}{1-p}\right)^i}_{k_i}$$

$\square$

We now find the transition matrices between the bases $\{y^{N-i} z^i\}_{i=0}^N$ and $\{(y^*)^{N-i}(z^*)^i\}_{i=0}^N$

LEM 156 For $j = 0, 1, 2, \ldots, N$

(i) $$(y^*)^{N-j}(z^*)^j = \sum_{i=0}^N y^{N-i} z^i \binom{N}{i} (1-p)^{N-i} p^i \, {}_2F_1\left(\begin{matrix} -i, -j \\ -N \end{matrix} \Big| \frac{1}{p}\right)$$

(ii) $$y^{N-j} z^j = \sum_{i=0}^N (y^*)^{N-i} (z^*)^i \binom{N}{i} \left(\frac{p}{1-p}\right)^i \, {}_2F_1\left(\begin{matrix} -i, -j \\ -N \end{matrix} \Big| \frac{1}{p}\right)$$

pf : this is our gen function th 106 in disguise

By th 106 for $j = 0, 1, \dots N$

$$\left(1 - \frac{1-p}{p} t\right)^{j} (1+t)^{N-j} = \sum_{i=0}^{N} \binom{N}{i} {}_2F_1\left({-i, -j \atop -N} \,\Big|\, \frac{1}{p} \right) t^i$$

$t =$ indet

Write $t = \frac{v}{u}$ u, v indets
Get

$$\left(u - \frac{1-p}{p} v\right)^{j} (u+v)^{N-j} = \sum_{i=0}^{N} \binom{N}{i} {}_2F_1\left({-i, -j \atop -N} \,\Big|\, \frac{1}{p} \right) u^{N-i} v^i$$

To get (i) take

$u = (1-p) y,$ $v = p z$

obs $u + v = y^*$

$u - \frac{1-p}{p} v = z^*$

To get (ii) take

$u = y^*$ $v = \frac{p}{1-p} z^*$

obs

$u + v = y$

$u - \frac{1-p}{p} v = z$

$\square$

We now find the inner products between the bases $\{y^{N-i}z^i\}_{i=0}^N$ and $\{(y^*)^{N-j}(z^*)^j\}_{j=0}^N$

LEM 157 For $0 \le i, j \le N$

$$\left\langle y^{N-i}z^i, (y^*)^{N-j}(z^*)^j \right\rangle = {}_2F_1\left(\begin{matrix}-i & -j\\ & -N\end{matrix}\Big/ \frac{1}{p}\right)$$

pf

$$\left\langle y^{N-i}z^i, (y^*)^{N-j}(z^*)^j \right\rangle = \sum_{k=0}^N \left\langle y^{N-i}z^i,\ y^{N-k}z^k \binom{N}{k}(1-p)^{N-k}p^k\, {}_2F_1\left(\begin{matrix}-k & -j\\ & -N\end{matrix}\Big/ \frac{1}{p}\right)\right\rangle$$

by L 156

$$= {}_2F_1\left(\begin{matrix}-i & -j\\ & -N\end{matrix}\Big/ \frac{1}{p}\right)$$

by Def 148

$\square$

In the following pages we summarize our results

Recall $A: V \to V$, $A^*: V \to V$

from Def 146.

Recall matrices U, B, D, K from Def 115

Matrices that represent A and A^*

basis	A	A^*
$\{y^{N-i} z^i\}_{i=0}^{N}$	B^t	D
$\{y^{N-i} z^i k_i (1-p)^N\}_{i=0}^{N}$	B	D
$\{(y^*)^{N-i} (z^*)^i k_i\}_{i=0}^{N}$	D	B
$\{(y^*)^{N-i} (z^*)^i\}_{i=0}^{N}$	D	B^t

Transition matrices

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\{y^{N-i}z^i\}_{i=0}^{N} \arrow[r, shift left, "K(1-p)^N"] \arrow[d, shift right, "KU(1-p)^N"'] \arrow[dr, shift left, "KUK(1-p)^N"] & \{y^{N-i}z^i k_i (1-p)^N\}_{i=0}^{N} \arrow[l, shift left, "K^{-1}(1-p)^{-N}"'] \arrow[d, shift left, "UK"] \arrow[dl, shift left, "U"] \\
\{(y^*)^{N-i}(z^*)^i\}_{i=0}^{N} \arrow[r, shift right, "K"'] \arrow[u, shift right, "KU"'] \arrow[ur, shift left, "KUK(1-p)^N"] & \{(y^*)^{N-i}(z^*)^i k_i\}_{i=0}^{N} \arrow[l, shift right, "K^{-1}"] \arrow[u, shift left, "UK(1-p)^N"] \arrow[ul, shift left, "U"]
\end{tikzcd}
$$

key:

$$k_i = \binom{N}{i}\left(\frac{p}{1-p}\right)^i \qquad i=0,1,\ldots N$$

$$\{u_i\}_{i=0}^{N} \xrightarrow{M} \{v_i\}_{i=0}^{N}$$

means $\quad v_j = \sum_{i=0}^{N} M_{ij} u_i \qquad j=0,1,\ldots,N$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\{y^{N-i} z^i\}_{i=0}^{N} \arrow[loop above, "K^{-1}(1-p)^{-N}"] \arrow[r, dash, "I"] \arrow[d, dash, "U"'] \arrow[dr, "UK" near start] & \{y^{N-i} z^i k_i (1-p)^N\}_{i=0}^{N} \arrow[loop above, "K(1-p)^N"] \arrow[d, dash, "KUK(1-p)^N"] \arrow[dl, "KU(1-p)^N"' near start] \\
\{(y^*)^{N-i}(z^*)^i\}_{i=0}^{N} \arrow[loop below, "K^{-1}"] \arrow[r, dash, "I"'] \arrow[ur, "UK(1-p)^N"' near start] & \{(y^*)^{N-i}(z^*)^i k_i\}_{i=0}^{N} \arrow[loop below, "K"] \arrow[ul, "KU"' near start]
\end{tikzcd}
$$

key: $\{u_i\}_{i=0}^{N} \xrightarrow{M} \{v_i\}_{i=0}^{N}$

means $M_{ij} = \langle u_i, v_j \rangle \quad (0 \le i,j \le N)$

Note Above diagrams match the ones drawn earlier with $\mu_0 = (1-p)^{-N}$

$\square$

Lecture 21 Friday Oct 21

Continue to discuss the L-module

$V = \mathrm{Hom}_N(A)$ of $A = \mathbb{F}[y, z]$

thm 158 For $j = 0, 1, \ldots, N$

(i) $$K_j(A; p, N)\, y^N = y^{N-j} z^j$$

(ii) $$K_j(A^*; p, N)\, (y^*)^N = (y^*)^{N-j} (z^*)^j$$

pf (i)

eigenvector for A^* eigval i

$$K_j(A; p, N)\, y^N = K_j(A; p, N) \sum_{i=0}^{N} (y^*)^{N-i} (z^*)^i k_i$$

by L154

$$= \sum_{i=0}^{N} (y^*)^{N-i} (z^*)^i k_i \; K_j(i; p, N)$$

$$= \sum_{i=0}^{N} (y^*)^{N-i} (z^*)^i k_i \; {}_2F_1\left(\begin{matrix} -i, -j \\ -N \end{matrix} \,\middle|\, \frac{1}{p} \right)$$

$$= y^{N-j} z^j$$

by L156 (ii)

(ii) Sim.

$\square$

Recall our situation...

$A = \mathbb{F}[y, z]$ is gen by y, z and also by y^*, z^*

$$y^* = (1-p)y + pz \qquad\qquad y = y^* + \frac{p}{1-p} z^*$$

$$z^* = (1-p)y + (p-1)z \qquad\qquad z = y^* - z^*$$

$$a y^* = y^* \qquad a z^* = -z^* \qquad\qquad 1-2p = \langle a, a^* \rangle \qquad p \neq 0, 1$$

$$a^* y = y \qquad a^* z = -z$$

For $0 \le i, j \le N$,

$$\langle y^{N-i} z^i, (y^*)^{N-j} (z^*)^j \rangle = {}_2F_1\left({-i, -j \atop -N} \Big| \frac{1}{p} \right)$$

Suppose we replace a by $-a$ and leave a^* alone

Consider new system $\qquad \tilde{y}, \tilde{z}, \tilde{y^*}, \tilde{z^*}$

$$\tilde{a} = -a \qquad \tilde{a^*} = a^*$$

$$\langle \tilde{a}, \tilde{a^*} \rangle = 1 - 2\tilde{p}$$

$$\| $$

$$-\langle a, a^* \rangle = 2p - 1$$

$$\tilde{p} = 1 - p$$

Define

$$\tilde{y} = \frac{1-p}{p} y \qquad\qquad \tilde{z} = -z$$

$$\tilde{y^*} = z^* \qquad\qquad \tilde{z^*} = y^*$$

Then

$$\tilde{y^*} = (1-\tilde{p})\tilde{y} + \tilde{p}\tilde{z} \qquad\qquad \tilde{y} = \tilde{y^*} + \frac{\tilde{p}}{1-\tilde{p}} \tilde{z^*}$$

$$\tilde{z^*} = (1-\tilde{p})\tilde{y} + (\tilde{p}-1)\tilde{z} \qquad\qquad \tilde{z} = \tilde{y^*} - \tilde{z^*}$$

$$\tilde{a}\tilde{y^*} = \tilde{y^*} \qquad \tilde{a}\tilde{z^*} = -\tilde{z^*}$$

$$\tilde{a^*}\tilde{y} = \tilde{y} \qquad \tilde{a^*}\tilde{z} = -\tilde{z}$$

antiaut $\dagger : L \to L$ assoc with $\tilde{a}, \tilde{a}^*$ is same as $\widetilde{\text{orig } \dagger}$

Bil form $\widetilde{\langle , \rangle}$ on $V = \mathrm{Hom}_N(A)$ agrees with orig $\langle , \rangle$

up to scalar ε $\qquad \widetilde{\langle \ \rangle} = \varepsilon \langle \ \rangle$

Find ε

By def 148

$$\| y^N \|^2 = (1-p)^{-N}$$

$$\widetilde{\| \tilde{y}^N \|^2} = (1-\tilde{p})^{-N}$$

$$\| \tilde{y}^N \|^2 = \widetilde{\left\| \tfrac{1-p}{p} y \right\|}^2 = \left(\frac{1-p}{p}\right)^{2N} \widetilde{\| y^N \|}^2 = \varepsilon \| y^N \|^2, \qquad (1-\tilde{p})^{-N} = p^{-N}$$

$$\varepsilon = \left(\frac{p}{1-p}\right)^N$$

Applying L157 to $\sim$ system

$$\left(\frac{p}{1-p}\right)^N \left\langle \tilde{y}^{N-i} \tilde{z}^i, (\tilde{y}^*)^{N-j} (\tilde{z}^*)^j \right\rangle = {}_2F_1\left({-i \ \ -j \atop -N} \Big/ \frac{1}{1-p} \right) \qquad 0 \le i, j \le N$$

$$= \left(\frac{p}{p-1}\right)^i \left\langle y^{N-i} z^i, (y^*)^j (z^*)^{N-j} \right\rangle$$

$$= {}_2F_1\left({-i \ \ j-N \atop -N} \Big/ \frac{1}{p} \right)$$

We have shown:

thm 159 For $p \in F$ $p \neq 0, p \neq 1$ and $N = 0,1,2\ldots$

$$ {}_2F_1\left(\begin{matrix}-i & -j\\ & -N\end{matrix}\middle/ \frac{1}{1-p}\right) = \left(\frac{p}{p-1}\right)^i {}_2F_1\left(\begin{matrix}-i & j-N\\ & -N\end{matrix}\middle/ \frac{1}{p}\right) $$

$0 \le i, j \le N$

Direct pf using gen. functions

By th 106

$$ \left(1 - \frac{1-p}{p}t\right)^j (1+t) = \sum_{n=0}^{N} \binom{N}{n} {}_2F_1\left(\begin{matrix}-n & -j\\ & -N\end{matrix}\middle/ \frac{1}{p}\right) t^n $$

Suf to show

$$ \sum_{n=0}^{N} \binom{N}{n} {}_2F_1\left(\begin{matrix}-n & -j\\ & -N\end{matrix}\middle/ \frac{1}{1-p}\right) t^n \overset{?}{=} \sum_{n=0}^{N} \binom{N}{n}\left(\frac{p}{p-1}\right)^n {}_2F_1\left(\begin{matrix}-n & j-N\\ & -N\end{matrix}\middle/ \frac{1}{p}\right) t^n $$

// th 106 $p \to 1-p$ // th 106 $j \to N-j$, $t \to \frac{p}{p-1}t$

$$ \left(1 - \frac{p}{1-p}t\right)^j (1+t)^{N-j} \qquad \left(1 - \frac{1-p}{p}\frac{p}{p-1}t\right)^{N-j}\left(1 + \frac{p}{p-1}t\right)^j $$

$$ = \qquad (1+t)^{N-j} \qquad \left(1 - \frac{p}{1-p}t\right)^j $$

$\checkmark$

$\square$

Cor 160 Fix $p \in \mathbb{F}$ $p \neq 0, p \neq 1$ and N ...

$$ {}_2F_1\left({-i \;\; -j \atop -N} \;\middle|\; \frac{1}{p} \right) = \left(\frac{p}{p-1}\right)^{N-i-j} {}_2F_1\left({i-N \;\; j-N \atop -N} \;\middle|\; \frac{1}{p} \right) $$

$0 \le i, j \le N$

pf Apply Thm 159 twice, once to i and once to j $\square$

th 159 is special case of

Th 161 (Pfaff transform) For $a, b \in \mathbb{F}$ and $n = 0, 1, 2, \ldots$

$${}_2F_1\left(\begin{matrix}-n & a\\ & b\end{matrix}\Big/ x\right) = (1-x)^n \, {}_2F_1\left(\begin{matrix}-n & b-a\\ & b\end{matrix}\Big/ \frac{x}{x-1}\right)$$

pf $\quad RHS = (1-x)^n \displaystyle\sum_{r=0}^{n} \frac{(-n)_r\,(b-a)_r}{b_r \quad r!} \left(\frac{x}{x-1}\right)^r$

$$= \sum_{r=0}^{n} \frac{(-n)_r\,(b-a)_r}{b_r \quad r!} \; x^r \underbrace{(1-x)^{n-r}}_{\sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} x^i} (-1)^r$$

[change var $i \to j = r+i$]

$$= \sum_{r=0}^{n} \frac{(-n)_r\,(b-a)_r}{b_r \quad r!} \sum_{j=r}^{n} x^j (-1)^j \binom{n-r}{j-r}$$

$$= \sum_{j=0}^{n} x^j (-1)^j \sum_{r=0}^{j} \frac{(-n)_r\,(b-a)_r}{b_r \quad r!} \binom{n-r}{j-r}$$

For $0 \le j \le n$ compare coeff of x^j in LHS, RHS

show
$$\frac{(-n)_j \, a_j}{b_j \quad j!} \overset{?}{=} (-1)^j \sum_{r=0}^{j} \frac{(-n)_r\,(b-a)_r}{b_r \quad r!} \binom{n-r}{j-r}$$

$$RHS = (-1)^j \sum_{r=0}^{j} \frac{(-j)_r\,(b-a)_r}{b_r \quad r!} \underbrace{\frac{(-n)_r}{(-j)_r} \binom{n-r}{j-r}}_{\binom{n}{j}}$$

$$= (-1)^j \binom{n}{j} \sum_{r=0}^{j} \frac{(-j)_r\,(b-a)_r}{b_r \quad r!}$$

$$= \underbrace{(-1)^j \binom{n}{j}}_{\frac{(-n)_j}{j!}} \; \underbrace{{}_2F_1\left(\begin{matrix}-j & b-a\\ & b\end{matrix}\Big/ 1\right)}_{\overset{\text{chu vand}}{=}\ \frac{a_j}{b_j}}$$

$= LHS$ $\square$

Back to $L - \ldots$

Next goal: factor $U = (\text{Lower tr})(\text{upper tr})$

Recall for $0 \le i, j \le N$

$$U_{ij} = {}_2F_1\left(\begin{matrix} -i & -j \\ & -N \end{matrix} \,\middle|\, \frac{1}{p}\right)$$

$$= \sum_{r=0}^{N} \frac{(-i)_r (-j)_r}{(-N)_r \, r!} \frac{1}{p^r}$$

$$= \binom{N}{i}^{-1} \sum_{r=0}^{\min(i,j)} \binom{N-r}{N-i} \binom{j}{r} \left(\frac{-1}{p}\right)^r$$

View sum as expressing matrix product

$$U = \mathrm{diag}\left(\binom{N}{i}^{-1}\right)_{i=0}^{N} \begin{pmatrix} \ddots & & 1 & & 0 \\ \cdots & & 3 & 1 & \\ \cdots & & 3 & 2 & 1 \\ \cdots & 1 & 1 & 1 & 1 \end{pmatrix} \mathrm{diag}\left(\left(\frac{-1}{p}\right)^i\right)_{i=0}^{N} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 3 & \\ & & 1 & 3 & \ddots \\ & & & 1 & \\ 0 & & & & \ddots \end{pmatrix}$$

$$= \left(\quad \text{Lower triang} \quad \right) \left(\quad \text{upper triang} \quad \right)$$

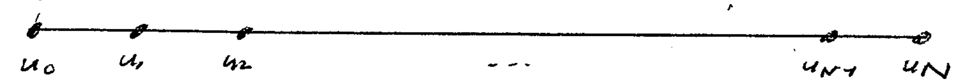
We now interpret this. Given our setup, more natural to work with

$$kU = \mathrm{diag}\left(\left(\frac{p}{1-p}\right)^i\right)_{i=0}^{N} \begin{pmatrix} \cdot & & & 0 \\ \cdots & & 1 & \\ & & 2 & 1 \\ \cdots & & 1 & 1 & 1 \end{pmatrix} \mathrm{diag}\left(\left(\frac{-1}{p}\right)^i\right)_{i=0}^{N} \begin{pmatrix} 1 & 1 & 1 & \cdots \\ & 1 & 2 & \cdots \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix}$$

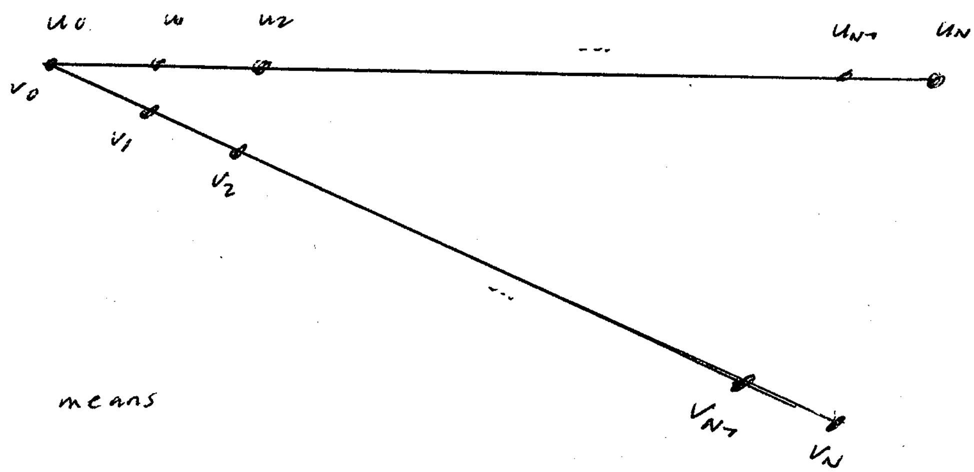
A diagram

Consider any basis $\{u_i\}_{i=0}^{N}$ for V

Represent this basis by a line segment.



Consider 2nd basis $\{v_i\}_{i=0}^{N}$ for V. The diagram



means

$$\mathrm{Span}(u_0, u_1, \ldots, u_i) = \mathrm{Span}(v_0, v_1, \ldots, v_i)$$

$0 \le i \le N$

Recall two - /

$\{y^{N-i} z^i\}_{i=0}^N$ $\qquad\qquad$ $\{(y^*)^{N-i} (z^*)^i\}_{i=0}^N$

$y^* = (1-p)y + pz$ $\qquad\qquad$ $y = y^* + \frac{p}{1-p} z^*$

$z^* = (1-p)y + (p-1)z$ $\qquad\qquad$ $z = y^* - z^*$

$ay^* = y^*$ $\qquad$ $az^* = -z^*$ $\qquad\qquad$ $1-2p = \langle a, a^* \rangle$ $\qquad$ $p \neq 0, 1$

$a^* y = y$ $\qquad$ $a^* z = -z$

For dist $u, v \in \{y, z, y^*, z^*\}$

u, v gen A so

$\{u^{N-i} v^i\}_{i=0}^N$ is basis for V

this gives 6 bases for V that are related

as follows.

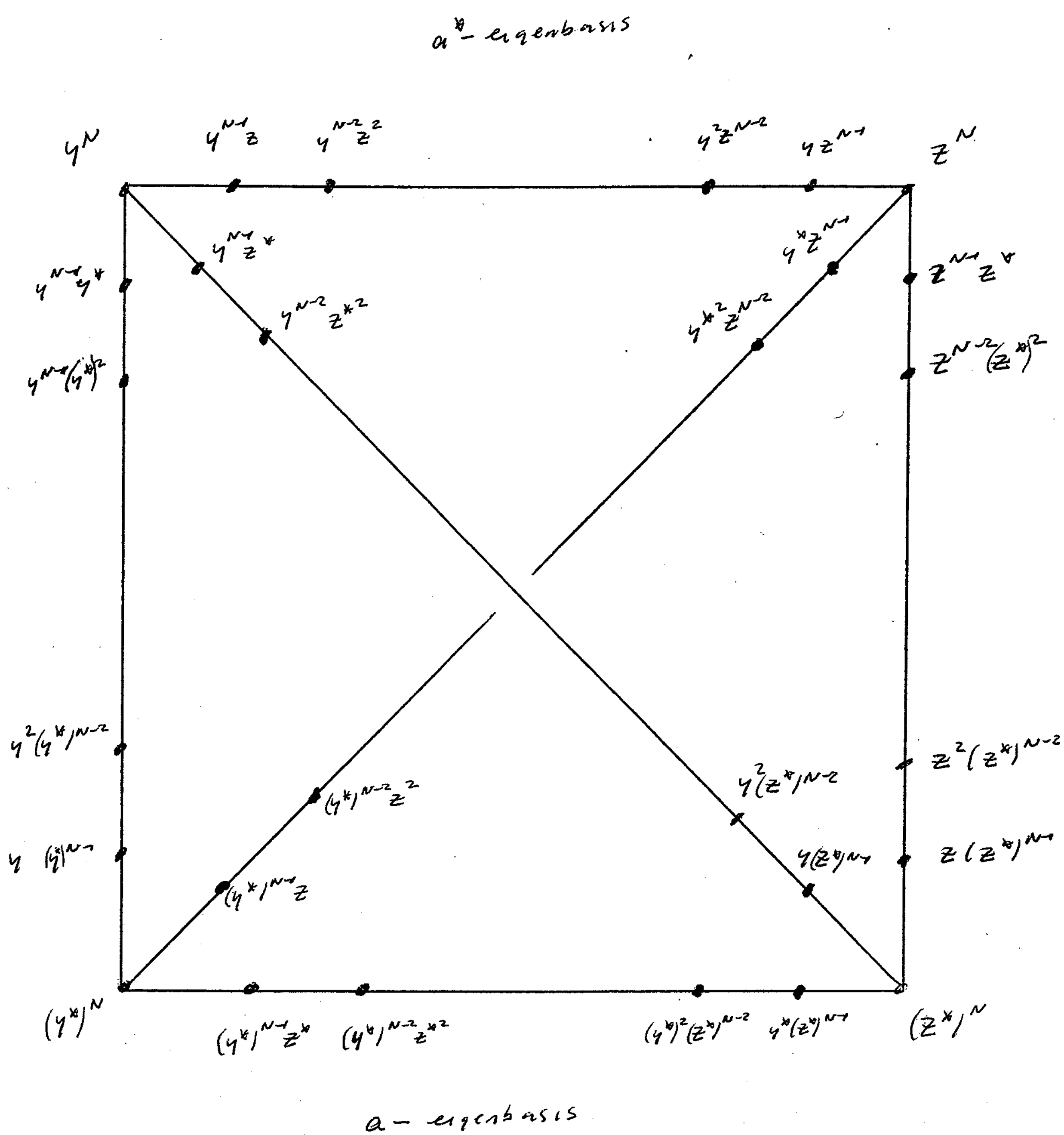
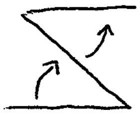


Diagram displays

a-eigenbasis
a^*-eigenbasis
four "split" bases for a, a^*

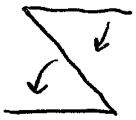
(i)



$$\{(y^*)^{N-i}(z^*)^i\}_{i=0}^N \longrightarrow \{y^{N-i}(z^*)^i\}_{i=0}^N \longrightarrow \{y^{N-i}z^i\}_{i=0}^N$$

$$kU = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix} \quad \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$$

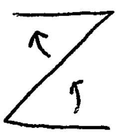
(ii)



$$\{y^{N-i}z^i\}_{i=0}^N \longrightarrow \{y^{N-i}(z^*)^i\}_{i=0}^N \longrightarrow \{(y^*)^{N-i}(z^*)^i\}_{i=0}^N$$

$$kU(1-p)^N = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} \quad \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$$

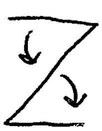
(iii)



$$\{(y^*)^{N-i}(z^*)^i\}_{i=0}^N \longrightarrow \{(y^*)^{N-i}z^i\}_{i=0}^N \longrightarrow \{y^{N-i}z^i\}_{i=0}^N$$

$$kU = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} \quad \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix}$$

(iv)



$$\{y^{N-i}z^i\}_{i=0}^N \longrightarrow \{(y^*)^{N-i}z^i\}_{i=0}^N \longrightarrow \{(y^*)^{N-i}(z^*)^i\}_{i=0}^N$$

$$kU(1-p)^N = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix} \quad \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$$

trans matrix KU
is product of :

$$\left\{ (y^*)^{N-i} (z^*)^{i} \right\}_{i=0}^{N}$$

$$\Big\downarrow \qquad \mathrm{diag}\left(\left(\frac{p}{1-p}\right)^{i} \right)_{i=0}^{N}$$

$$\left\{ (y^*)^{N-i} \left(\frac{p z^*}{1-p}\right)^{i} \right\}_{i=0}^{N}$$

$$\Big\downarrow \qquad \begin{pmatrix} \ddots & & & & 0 \\ \ddots & 1 & & & \\ & 3 & 1 & & \\ & 3 & 2 & 1 & \\ \cdots & 1 & 1 & 1 & 1 \end{pmatrix} \qquad y = y^* + \frac{p z^*}{1-p}$$

$$\left\{ y^{N-i} \left(\frac{p z^*}{1-p}\right)^{i} \right\}_{i=0}^{N}$$

$$\Big\downarrow \qquad \mathrm{diag}\left(\left(\frac{-1}{p}\right)^{i} \right)_{i=0}^{N}$$

$$\left\{ y^{N-i} \left(\frac{z^*}{p-1}\right)^{i} \right\}_{i=0}^{N}$$

$$\Big\downarrow \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 3 & \cdots \\ & & 1 & 3 & \\ & & & 1 & \\ & 0 & & & \ddots \end{pmatrix} \qquad z = y + \frac{z^*}{p-1}$$

$$\left\{ y^{N-i} z^{i} \right\}_{i=0}^{N}$$

this gives our initial factorization of KU

(ii) This expresses "Double [illegible] ...

(iii) Duplicate of (ii)

(iv) Duplicate of (i)

Next goal: Find action of a, a^* on split bases

LEM 162 The following occurs in $\mathrm{Hom}_{\mathbb{1}}(A)$

basis	matrix rep a	matrix rep a^*
y, y^*	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2(1-p) \\ 0 & -1 \end{pmatrix}$
y, z^*	$\begin{pmatrix} 1 & 0 \\ \frac{2p}{p-1} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2(1-p) \\ 0 & -1 \end{pmatrix}$
z, y^*	$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -2p \\ 0 & 1 \end{pmatrix}$
z, z^*	$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 2(1-p) \\ 0 & 1 \end{pmatrix}$

Thm 16.3 Rel each split basis the matrix rep a (resp a^*) is lower bidiagonal (resp. upper bidiagonal). The entries are given below.

basis	a: (i,i)-entry	a: $(i,i-1)$-entry	a^*: (i,i)-entry	a^*: $(i-1,i)$-entry
$\{y^{N-i} y^{*i}\}_{i=0}^{N}$	$2i-N$	$2(N-i+1)$	$N-2i$	$2(1-p)i$
$\{y^{N-i}(z^*)^i\}_{i=0}^{N}$	$N-2i$	$\frac{2p}{p-1}(N-i+1)$	$N-2i$	$2(1-p)i$
$\{z^{N-i}(y^*)^i\}_{i=0}^{N}$	$2i-N$	$2(N-i+1)$	$2i-N$	$-2pi$
$\{z^{N-i}(z^*)^i\}_{i=0}^{N}$	$N-2i$	$2(N-i+1)$	$2i-N$	$2(1-p)i$

pf By L162 and since a, a^* act as derivations —

For instance

$$a\, y^{N-i}(y^*)^i = (N-i)\, y^{N-i-1}(y^*)^i \overbrace{a(y)}^{-y+2y^*} + i\, y^{N-i}(y^*)^{i-1} \underbrace{a(y^*)}_{y^*}$$

$$= (2i-N)\, y^{N-i}(y^*)^i + 2(N-i)\, y^{N-i-1}(y^*)^{i+1}$$

$\square$

Note Given any basis $\{v_i\}_{i=0}^N$ the inverted basis is $\{v_{N-i}\}_{i=0}^N$. In Th 163, with respect to each inverted basis the matrices rep a and a^* are upper bidiag and lower bidiag, resp.

topics on sl_2 / Krawtchouk

- use the diagram below thm 161 to motivate the tetrahedron algebra ⊠

- Using ⊠ define Ψ, Δ as Sarah did. show $\Delta = \exp(\Psi)$

- how are 4 relations of Δ related
  Ψ related

- . . .

- transition matrices as exponentials

- go thru Marked problems and solve them for Krawtchouk

Given $p \in \mathbb{F}$ $p \neq 0, p \neq 1$ Given integer $N \geq 0$

consider the tridiagonal matrix $b \in Mat_{N+1}(\mathbb{F})$

with entries

$(i, i-1)$	(i, i)	$(i-1, i)$
$2(1-p)i$	$(N-2i)(1-2p)$	$2p(N-i+1)$

So b has constant row sum N

Obs. $b = NI - 2B$

B from Def 115

b = matrix rep a rel basis $\{y^{N-i} z^i k_i (1-p)^N\}_{i=0}^N$ for V.

action of a on V has eigvals $\{N-2i\}_{i=0}^N$

so b has eigval $\{N-2i\}_{i=0}^N$.

Problem.

Show from scratch that b has eigvals $\{N-2i\}_{i=0}^N$

Sol 1 Define $U \in Mat_{N+1}(\mathbb{F})$ s.t.

$$U_{ij} = {}_2F_1\left({-i, -j \atop -N} \Big/ \frac{1}{p} \right) \qquad 0 \le i, j \le N$$

Define

$$d = diag(N-2i)_{i=0}^{N}$$

By matrix mult

$$b\,U = U\,d \qquad (*)$$

So for $i = 0, 1, 2, \ldots, N$

col i of U is eigvector for b with eigval $N-2i$

So b has eigvals

$$\{N-2i\}_{i=0}^{N}$$

Difficulty with Sol 1: Hard to verify $(*)$

since entries of U are sums.

Inspired by th 163, instead of making b diagonal

we just make it ~~lower bidiagonal~~ upper bidiag.

Sol 2 Define lower triangular matrix

$T \in Mat_{N+1}(\mathbb{F})$ s.t.

$$T_{ij} = \binom{i}{j} \qquad 0 \le j \le i \le N$$

So

$$T = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

T^{-1} exists.

Define upper bidiagonal $C \in Mat_{N+1}(\mathbb{F})$ by

(i,i)-entry	$(i-1,i)$-entry
$N-2i$	$2p(N-i+1)$

Obs C has eigenvalues $\{N-2i\}_{i=0}^{N}$

By matrix mult

$$bT = TC \qquad (**)$$

So

$$T^{-1} b T = C$$

So b, C have same eigvals

So b has eigvals $\{N-2i\}_{i=0}^{N}$

Note $**$ much easier to verify than $*$

Ex $N=3$

$$
b: \begin{pmatrix} 3(1-2p) & 6p & 0 & 0 \\ 2(1-p) & 1-2p & 4p & 0 \\ 0 & 4(1-p) & 2p-1 & 2p \\ 0 & 0 & 6(1-p) & 3(2p-1) \end{pmatrix}
$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 6p & 0 & 0 \\ 0 & 1 & 4p & 0 \\ 0 & 0 & -1 & 2p \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Finding the eigenvalues of a tridiag matrix, cont

Given an irred tridiag matrix $B \in Mat_{N+1}(\mathbb{F})$:

$$B = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & c_N & a_N \end{pmatrix}$$

Let $\{\theta_i\}_{i=0}^N$ denote the roots of the char poly for B

(poss θ_i not distinct)

Goal: find $\{\theta_i\}_{i=0}^N$

Recall

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad i = 0, 1, \ldots, N$$

Recall

$$B^t = K B K^{-1}$$

where

$$K = \operatorname{diag}(k_i)_{i=0}^N$$

Define polynomials $\{p_i\}_{i=0}^N$ in $\mathbb{F}[x]$ by

$$x p_i = c_i p_{i-1} + a_i p_i + b_i p_{i+1} \qquad 0 \le i \le N-1$$

$$p_0 = 1, \qquad p_{-1} = 0$$

So for $0 \le i \le N$

$$\deg p_i = i$$

$$\text{coeff of } x^i \text{ in } p_i \text{ is } \frac{1}{b_0 b_1 \cdots b_{i-1}}$$

For $0 \le i \le N$ define $\tau_i \in \mathbb{F}[x]$ by

$$\tau_i = (x-\theta_0)(x-\theta_1) \cdots (x-\theta_{i-1})$$

τ_i monic deg i

For $0 \le i \le N$ pick

$$0 \ne z_i \in \mathbb{F}$$

Write $\{p_i\}_{i=0}^{N}$ in terms of

$$\left\{ \frac{\tau_i}{z_1 z_2 \cdots z_i} \right\}_{i=0}^{N}$$

$\exists$ upper triangular $S \in \mathrm{Mat}_{N+1}(\mathbb{F})$ s.t.

$$p_j = \frac{1}{k_j} \sum_{i=0}^{j} S_{ij} \frac{\tau_i}{z_1 z_2 \cdots z_i} \qquad j = 0, 1, \ldots, N$$

Obs S^{-1} exists.

LEM 164 With above notation

$$S B S^{-1} = \begin{pmatrix} \theta_0 & & & & 0 \\ z_1 & \theta_1 & & & \\ & z_2 & \theta_2 & & \\ & & \ddots & \ddots & \\ 0 & & & z_N & \theta_N \end{pmatrix}$$

pf Let V denote a vector space over $\mathbb{F}$ with dim $N+1$. Pick a basis for V:

$$\{v_i\}_{i=0}^{N} \qquad (*)$$

Let B represent a lin trans $A : V \to V$ rel $\{v_i\}_{i=0}^{N}$:

$$A v_j = \sum_{i=0}^{N} B_{ij} v_i \qquad j = 0, 1, \dots, N$$

One checks

$$v_j = k_j\, p_j(A)\, v_0 \qquad j = 0, 1, \dots, N$$

Obs $\{A^i v_0\}_{i=0}^{N}$ are lin indep, hence basis for V.

So vectors

$$\left\{ \frac{\tau_i(A) v_0}{z_1 z_2 \cdots z_i} \right\}_{i=0}^{N} \qquad (**)$$

form a basis for V.

S is the transition matrix from $(**)$ to $(*)$.

The matrix rep A rel $(**)$ is $\begin{pmatrix} \theta_0 & & & 0 \\ z_1 & \theta_1 & & \\ & z_2 & \theta_2 & \\ 0 & & \ddots & \ddots \\ & & z_N & \theta_N \end{pmatrix}$

Result follows. $\square$

Let B, K, S as above

Define
$$T = K^{-1} S^t$$

Obs T is lower triangular

Obs for $0 \le j \le N$
$$p_j = \sum_{i=0}^{j} T_{ji} \frac{\tau_i}{z_1 z_2 \cdots z_i}$$

LEM 165 With above notation
$$T^{-1} B T = \begin{pmatrix} \theta_0 & z_1 & & & 0 \\ & \theta_1 & z_2 & & \\ & & \theta_2 & \ddots & \\ & 0 & & \ddots & z_N \\ & & & & \theta_N \end{pmatrix}$$

pf In LEM 164 Take transpose of each side and use
$$B^t = K B K^{-1}$$ $\square$

Note 166 To find $\{\theta_i\}_{i=0}^{N}$ it is often a good strategy to hunt for S or T from L164, 165

These matrices can be very nice with entries in factored form. See for example the matrix T from Sol 2 of the previous problems.

We now display a general class of tridiag matrices B for which the matrix T from L165 is nice.

Th 167 Referring to the data

$$B, \{\theta_i\}_{i=0}^N, \{p_i\}_{i=0}^N$$

from above L164.

Assume $\exists$ scalars $\{\theta_i^*\}_{i=0}^N$, $\{\varphi_i\}_{i=1}^N$ in $\mathbb{F}$ s.t.

(i) $\theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \le i, j \le N)$

(ii) $\varphi_i \neq 0$ $1 \le i \le N$

(iii) For $0 \le j \le N$

$$p_j = \sum_{i=0}^{j} \frac{(x-\theta_0)(x-\theta_1)\cdots(x-\theta_{i-1})(\theta_j^*-\theta_0^*)(\theta_j^*-\theta_1^*)\cdots(\theta_j^*-\theta_{i-1}^*)}{\varphi_1\varphi_2\cdots\varphi_i}$$

Define a Lower triangular $T \in \mathrm{Mat}_{N+1}(\mathbb{F})$ by

$$T_{ij} = (\theta_i^*-\theta_0^*)(\theta_i^*-\theta_1^*)\cdots(\theta_i^*-\theta_{j-1}^*)$$

$$0 \le i, j \le N$$

Then

$$T^{-1} B T = \begin{pmatrix} \theta_0 & \varphi_1 & & & 0 \\ & \theta_1 & \varphi_2 & & \\ & & \theta_2 & \ddots & \\ & & & \ddots & \varphi_N \\ 0 & & & & \theta_N \end{pmatrix}$$

pf Apply L165 with

$$z_i = \varphi_i \qquad 1 \le i \le N$$

$\square$

Ex 168 The Krawtchouk polys $\{K_n(x; p, N)\}_{n=0}$

have form of Th 167 (iii) with

$\theta_i = i \qquad \theta^*_i = i \qquad \varphi_i = -pi(N-i+1)$

<u>Problem</u> Given a poly sequence

$\{p_i\}_{i=0}^N$ in $\mathbb{F}[x]$ $\quad$ ($N = \infty$ or $N < \infty$)

Find all sequences of scalars

$$\left(\{\theta_i\}_{i=0}^N ,\ \{\theta_i^*\}_{i=0}^N ,\ \{\varphi_i\}_{i=1}^N \right)$$

such that

$$p_j \in \mathbb{F} \sum_{i=0}^{j} \frac{(x-\theta_0)(x-\theta_1)\cdots(x-\theta_{i-1})(\theta_j^*-\theta_0^*)(\theta_j^*-\theta_1^*)\cdots(\theta_j^*-\theta_{i-1}^*)}{\varphi_1\varphi_2\cdots\varphi_i}$$

Do this for each polynomial sequence in the Askey scheme.

Expect:

family	# sols
Askey Wilson	4
q-Racah	2
Hermite	0

Show how each solution corresponds to a basis that makes A^* upper bidiag and A lower bidiag. where A is "mult by x" and A^* is the q-difference operator

Lecture 23 Wednesday Nov 3

A characterization of the Krawtchouk polys using the notion of a Leonard pair

<u>Motivation</u> Recall setup

char $\mathbb{F} = 0$, $L = sl_2(\mathbb{F})$

$a, a^* \in L$ (normalized, semi simple, generate L)

$N = 0, 1, 2, \ldots$

L-module $V = Hom_N(A)$ $A = \mathbb{F}[y, z]$

basis for V	matrix rep a	matrix rep a^*
$\{y^{N-i} z^i\}_{i=0}^N$	irred tridiag	diag
$\{(y^*)^{N-i} (z^*)^i\}_{i=0}^N$	diag	irred tridiag

Motivated by this we now define a Leonard pair.

Until further notice:

$\mathbb{F}$ arb

Fix integer $N \geq 0$

Fix V a vector space over $\mathbb{F}$ with dimension $N+1$

DEF 169 A <u>Leonard pair</u> on V is an ordered pair of Linear trans

$A: V \to V$, $A^*: V \to V$

such that

(i) $\exists$ basis for V with respect to which the matrix rep A is irred tridiag and the matrix rep A^* is diag

(ii) $\exists$ basis for V with respect to which the matrix rep A is diag and the matrix rep A^* is irred. tridiag.

Note Given LP A, A^* on V

then A^*, A is LP on V

Also for $\alpha, \alpha^*, \beta, \beta^* \in \mathbb{F}$ $\alpha \neq 0$, $\alpha^* \neq 0$

$\alpha A + \beta I$, $\alpha^* A^* + \beta^* I$ is LP on V.

Notation An element $A \in \operatorname{End} V$ is called <u>multiplicity-free</u> whenever the roots of its char poly are mutually distinct and contained in $\mathbb{F}$

So

A multfree $\iff$ A is diagonalizable and all eigenspaces have dim 1.

LEM 170 Let A, A^* denote LP on V.

Then each of A, A^* is mult free.

pf Concerning A.

By Def 169 (ii) $\exists$ basis for V consisting of eigenvectors for A.

So all eigenvalues of A are in $\mathbb{F}$

+ min poly of A has no repeated roots

By L47

min poly of A = char poly of A

So char poly of A has no repeated roots.

So A mult free.

Similarly A^* is mult free $\square$

Notation

Given MF $\qquad A \in \operatorname{End} V$

Given ordering $\{\theta_i\}_{i=0}^N$ of eigvals of A

For $0 \le i \le N$ let $V_i \subseteq V$ be the eigenspace of A for θ_i

So
$$V = \sum_{i=0}^N V_i \qquad (ds)$$

For $0 \le i \le N$ define $E_i \in \operatorname{End} V$ s.t.
$$(E_i - I)V_i = 0$$
$$E_i E_j V = 0 \quad \forall\, j \ne i \quad (0 \le j \le N)$$

Call E_i = __primitive__ __idempotent__ of A for θ_i

Obs
$$E_i E_j = \delta_{ij} E_i \qquad (0 \le i,j \le N)$$
$$I = \sum_{i=0}^N E_i$$
$$A = \sum_{i=0}^N \theta_i E_i$$
$$V_i = E_i V \qquad 0 \le i \le N$$
$$E_i = \prod_{\substack{0 \le j \le N \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad 0 \le i \le N$$

Let $\mathcal{D}$ = subalg of $\operatorname{End} V$ gen by A.

$\{A^i\}_{i=0}^N$ is basis for $\mathcal{D}$ and $\prod_{i=0}^N (A - \theta_i I) = 0$

$\{E_i\}_{i=0}^N$ is basis for $\mathcal{D}$.

$\operatorname{tr} E_i = 1 \qquad 0 \le i \le N$

rank $E_i = 1$

DEF 171 By a Leonard system on V

we mean a sequence

$$\Phi = \left(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N \right)$$

s.t.

(i) Each of A, A^* is mult free el of $\operatorname{End} V$

(ii) $\{E_i\}_{i=0}^N$ is an ordering of the pr idempotents of A

(iii) $\{E_i^*\}_{i=0}^N$ ... A^*

(iv) $$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j|>1 \\ \neq 0 & \text{if } |i-j|=1 \end{cases} \qquad 0 \le i,j \le N$$

(v) $$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j|>1 \\ \neq 0 & \text{if } |i-j|=1 \end{cases} \qquad 0 \le i,j \le N$$

+

Note Given LS Φ on V as in Def 171

Given $\alpha, \alpha^*, \beta, \beta^* \in \mathbb{F}$ $\alpha \neq 0, \alpha^* \neq 0$.

then

$$\left(\alpha A + \beta I, \ \{E_i\}_{i=0}^N, \ \alpha^* A^* + \beta^* I, \ \{E_i^*\}_{i=0}^N \right)$$

is LS on V

Also each of the following is LS on V

$$\Phi^* := \left(A^*, \ \{E_i^*\}_{i=0}^N, \ A, \ \{E_i\}_{i=0}^N \right)$$

$$\Phi^{\Downarrow} := \left(A, \ \{E_{N-i}\}_{i=0}^N, \ A^*, \ \{E_i^*\}_{i=0}^N \right)$$

$$\Phi^{\downarrow} := \left(A, \ \{E_i\}_{i=0}^N, \ A^*, \ \{E_{N-i}^*\}_{i=0}^N \right)$$

LP vs LS:

Given LP A, A^* on V

let $\{v_i\}_{i=0}^N$ denote a basis for V from Def 169 (ii)

let $\{v_i^*\}_{i=0}^N$ " " 169 (i)

Each v_i is eigvector for A, let E_i denote corresp prim idl

Each v_i^* ... A^*, E_i^* ...

then $(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ is LS on V

Given LS on V

$$(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$$

For $0 \leq i \leq N$ pick

$$0 \neq v_i \in E_i V \qquad 0 \neq v_i^* \in E_i^* V$$

then

$\{v_i\}_{i=0}^N$ is basis for V that sats Def 16.9 (ii)

$\{v_i^*\}_{i=0}^N$ - - - - 16.9 (i)

So A, A^* is LP on V

Given LS $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ on V

Then A, A^* is LP on V

We say A, A^* and Φ are <u>associated</u>.

Obs A, A^* is assoc to $\Phi, \Phi^\downarrow, \Phi^\Downarrow, \Phi^{\downarrow\Downarrow}$

and no other LS.

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DEF 172 Given LS $\Phi = (A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$

For $0 \le i \le N$ let

θ_i = eigval of A for E_i

θ_i^* = eigval of A^* for E_i^*

Call $\{\theta_i\}_{i=0}^N$ the <u>eigenvalue</u> <u>sequence</u> of Φ

.. $\{\theta_i^*\}_{i=0}^N$. <u>dual</u> <u>eigenvalue</u> <u>sequence</u> of Φ

Def 173

Given LP A, A^* on V

By an eigenvalue sequence for A, A^* we mean the eigval seq for an assoc LS.

By a dual eigenvalue sequence for A, A^* we mean the dual eigval seq for an assoc LS

So if $\{\theta_i\}_{i=0}^N$ is an eigval seq for A, A^*
so is $\{\theta_{N-i}\}_{i=0}^N$ and no other eigval seq.

Sim if $\{\theta_i^*\}_{i=0}^N$ .. dual eigval seq ..
so is $\{\theta_{N-i}^*\}_{i=0}^N$ - - - dual eigval seq.

Prop 174 Given LS $(A, \{E_i\}_{i=0}^N, A^*, \{E_i^*\}_{i=0}^N)$ on V. Then the following is a basis for End V:

$$A^i E_0^* A^j \qquad 0 \le i, j \le N \qquad *$$

pf Pick

$$0 \ne v_i \in E_i^* V \qquad 0 \le i \le N$$

So

$$\{v_i\}_{i=0}^N \text{ is basis for } V$$

WLOG identify each $Y \in \text{End } V$ with the matrix that represents Y with respect to $\{v_i\}_{i=0}^N$.

So

$$E_0^* = \text{diag}(1, 0, 0, \ldots, 0)$$

$$A = \text{irred. tridiag}$$

One checks $*$ are lin indep and hence a basis for End V $\square$

COR 175 Ref to Prop 174, each of the following is a generating set for End$V$

(i) A, E_0^*

(ii) A^*, E_0

(iii) A, A^*

pf (i) By Prop 174

(ii) Apply (i) to A^*, A

(iii) By (i) and since E_0^* is a poly in A^* $\square$

COR 17.6 Let A, A^* denote LP on V

then V is irred. as a module for A, A^*

$\left[\begin{array}{l} \text{ie} \quad \nexists \text{ subspace } W \subseteq V \text{ s.t.} \\ \quad AW \subseteq W, \quad A^*W \subseteq W, \quad W \neq 0, \quad W \neq V \end{array}\right]$

pf ✓ $\square$

DEF 177 Assume char $\mathbb{F}=0$

Given LP A, A^* on V

This LP has <u>Krawtchouk type</u> whenever both

$\{N-2i\}_{i=0}^N$ is an eigenvalue sequence for A, A^*

.. dual eigenvalue .. ...

LEM 178 Assume char $\mathbb{F}=0$ Given LP A, A^* on V

Given eigenvalue sequence $\{\theta_i\}_{i=0}^N$ for A, A^*

... dual eigval .. $\{\theta_i^*\}_{i=0}^N$ ..

TFAE

(i) Each of $\{\theta_i\}_{i=0}^N$, $\{\theta_i^*\}_{i=0}^N$ are in arithmetic progression

(ii) $\exists\ \alpha, \alpha^*, \beta, \beta^* \in \mathbb{F}$ $\alpha \neq 0, \alpha^* \neq 0$ s.t. LP

$\alpha A + \beta I,\ \alpha^* A^* + \beta^* I$ has Krawtchouk type.

pf clear. $\square$

In the following two theorems we characterize the LP of Krawtchouk type using $L = sl_2(\mathbb{F})$

Thm 179 Assume char $\mathbb{F} = 0$
Let a, a^* denote normalized semi simple elements that generate L.
Let V denote a finite dim'l irred. L-module.
Then a, a^* act on V as a LP of Krawtchouk type.

pf Let $N = \dim V - 1$

Up to iso $\exists$ unique irred L-module dim $N+1$

WLOG $V = Hom_N(A)$ $\quad A = \mathbb{F}[y, z]$

as discussed earlier.

By discussion prior Def 169

- a, a^* act on V as LP
- $\{N-2i\}_{i=0}^{N}$ is an eigenvalue sequence for A, A^*
- $\{N-2i\}_{i=0}^{N}$ is a dual eigenvalue sequence for A, A^*

Result follows. $\square$

thm 180 Assume char $\mathbb{F}=0$

Given $N=0,1,2..$

Given a vector space V over $\mathbb{F}$ dim $N+1$

Given LP A, A^* on V of Krawtchouk type.

Then $\exists$ L module structure on V and a pair of normalized s.s. generators for L that act on V as A, A^*. This L-mod str is uniqued.

pf Assume $N \geq 2$ else trivial.

By assumption A, A^* has an eigenvalue sequence $\{\theta_i\}_{i=0}^N$ and a dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^N$ such that

$$\theta_i = N-2i \qquad \theta_i^* = N-2i \qquad 0 \leq i \leq N$$

For $0 \leq i \leq N$ let

E_i = prim idempotent of A for θ_i

E_i^* = ... A^* .. θ_i^*

<u>claim 1</u> A, A^* satisfy

$$[A,[A,[A,A^*]]] = 4[A,A^*] \qquad DG1$$

$$[A^*,[A^*,[A^*,A]]] = 4[A^*,A] \qquad DG2$$

"Dolan Grady relations"

pf cl 1 let

$$C = \text{LHS of DG1} - \text{RHS of DG1}$$

show $C = 0$

Since $I = \sum_{i=0}^{N} E_i$ suf to show

$$E_i C E_j = 0 \qquad 0 \le i, j \le N$$

Using $A E_i = \theta_i E_i$, $E_j A = \theta_j E_j$ we have

$$E_i C E_j = E_i A^* E_j \left(\theta_i^3 - 3\theta_i^2\theta_j + 3\theta_i\theta_j^2 - \theta_j^3 - 4(\theta_i - \theta_j) \right)$$

$$= \underbrace{E_i A^* E_j}_{\substack{\shortparallel \\ 0 \text{ if } |i-j|>1}} \left(\underbrace{\theta_i - \theta_j}_{\substack{\shortparallel \\ 0 \text{ if } i=j}} \right) \underbrace{\left(\theta_i - \theta_j - 2 \right)\left(\theta_i - \theta_j + 2 \right)}_{\substack{\shortparallel \\ 0 \text{ if } |i-j|=1}}$$

$$= 0$$

We have shown DG1.

DG2 is sim.

claim 1 proved ✓

<u>claim 2</u> $\exists\ \rho \in \mathbb{F}$ such that

$$[A,[A,A^*]] = 4(2\rho-1)A + 4A^* \qquad \text{AW1}$$

$$[A^*,[A^*,A]] = 4A + 4(2\rho-1)A^* \qquad \text{AW2}$$

pf By DG1 A commutes with

$$[A,[A,A^*]] - 4A^*$$

Let $\mathcal{D}$ = subalg of End V gen by A

$= \{ Y \in \text{End}V \mid [Y,A]=0 \}$ (since A MF)

So $[A,[A,A^*]] - 4A^* \in \mathcal{D}$

So $[A,[A,A^*]] - 4A^* = \sum_{i=0}^{N} \alpha_i A^i \qquad \alpha_i \in \mathbb{F}$

show $\alpha_i = 0 \qquad 3 \le i \le N$

suppose not, let

$k = \max\{ i \mid 3 \le i \le N,\ \alpha_i \ne 0 \}$

Consider

$$\underbrace{E_k^*\left([A,[A,A^*]] - 4A^*\right)E_0^*}_{\overset{\shortparallel}{0}} = \underbrace{E_k^*\left(\sum_{i=0}^{k} \alpha_i A^i\right)E_0^*}_{\overset{\shortparallel}{\underset{\ne 0}{\alpha_k}\ \underbrace{E_k^* A^k E_0^*}_{\ne 0}}}$$

cont.

show $\alpha_2 = 0$:

$$\underbrace{E_2^*\left([A,[A,A^*]] - 4A^*\right)E_0^*}_{=\, E_2^*A^2E_0^*\left(\underbrace{\theta_0^* - 2\theta_1^* + \theta_2^*}_{=0}\right)} = \underbrace{E_2^*\left(\alpha_0 I + \alpha_1 A + \alpha_2 A^2\right)E_0^*}_{=\, \underbrace{E_2^*A^2E_0^*}_{\neq 0}\,\alpha_2}$$

So $\alpha_2 = 0$ ✓

show $\alpha_0 = 0$:

In
$$[A,[A,A^*]] - 4A^* = \alpha_0 I + \alpha_1 A$$

take trace of each side

$$\operatorname{tr} A = \operatorname{tr} A^* = \sum_{i=0}^{N}(N-2i)$$
$$= 0$$

$$\operatorname{tr} I = \dim V$$
$$= N+1$$

$$\operatorname{tr}\left([A,[A,A^*]]\right) = 0 \quad \text{since}$$

$$\operatorname{tr} uv = \operatorname{tr} vu \quad \forall u,v$$
$$\operatorname{tr}[u,v] = 0$$

So
$$0 = \alpha_0(N+1)$$

So
$$\alpha_0 = 0 \checkmark$$

So far

$$[A,[A,A^*]] - 4A^* = \alpha_1 A \qquad (*)$$

Similarly $\exists\ \alpha_1^* \in \mathbb{F}$ s.t.

$$[A^*,[A^*,A]] - 4A = \alpha_1^* A^* \qquad (**)$$

show $\alpha_1 = \alpha_1^*$:

In $(*)$ take commutator of each term with A^*

$$\Big[[A,[A,A^*]], A^*\Big] = \alpha_1 [A,A^*]$$

Similarly using $**$

$$\Big[A, [A^*,[A^*,A]]\Big] = \alpha_1^* [A,A^*]$$

In above two equations the HS coincide by the Jacobi identity so

$$\alpha_1 [A,A^*] = \alpha_1^* [A^*,A]$$

But $[A,A^*] \neq 0$ since V is irred as (A,A^*)-module

So $\alpha_1 = \alpha_1^*$.

Now define $p \in \mathbb{F}$ s.t

$$4(2p-1) = \alpha_1$$

then $*$, $**$ become AW1, AW2 $\checkmark$ claim 2 proved.

claim 3 $\quad \rho \neq 0, \ \rho \neq 1$

pf Suppose $\rho = 0$. Then AW1, AW2 become

$$[A,[A,A^*]] = -4A + 4A^* \qquad (1)$$

$$[A^*,[A^*,A]] = 4A - 4A^* \qquad (2)$$

With respect to some basis for V

$$A: \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & c_N & a_N \end{pmatrix} \qquad A^* = \begin{pmatrix} N & & & 0 \\ & N-2 & & \\ & & \ddots & \\ 0 & & & -N \end{pmatrix}$$

$c_i b_{i-1} \neq 0 \quad 1 \le i \le N$

Using (2) and matrix mult

$$a_i = N - 2i \qquad (0 \le i \le N)$$

Now using (1) and matrix mult $\qquad$ cont.

$$b_0 c_1 = 0$$

So $\rho \neq 0$

Similarly one shows $\rho \neq 1$. $\qquad$ claim 3 proved.

By Claims 2, 3 A, A^* satisfy the defining relations for L in the presentation involving two normalized s.s generators (Th 1.27).

Therefore $\exists$ L-module structure on V and a pair of normalized s.s. gens for L that act on V as A, A^*.

This L-mod str is irred since V is irred as a module for A, A^*. $\qquad \square$

For the combinatorialists in the audience

we now relate the Krawtchouk polys to the Hamming

graphs $H(N,r)$

Given integers $N \geq 1$ $r \geq 2$

Given a set S $|S| = r$

Let Γ denote an undirected graph with vertex set

$$X = \underbrace{S \times S \times \cdots \times S}_{N}$$

vertices $x,y \in X$ are adjacent in Γ whenever they differ in exactly one coordinate.

Γ called the <u>Hamming graph</u> $H(N,r)$

Γ is connected

Let ∂ = path-length distance in Γ

For $x,y \in X$ and $0 \leq i \leq N$

$$\partial(x,y) = i \iff x,y \text{ differ in exactly } i \text{ coordinates}$$

So

$$N = \max\{\partial(x,y) \mid x,y \in X\}$$

"diameter of Γ"

For $x \in X$ and $i \in \mathbb{Z}$ define

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}$$

Abbr $\Gamma(x) = \Gamma_1(x)$

For $0 \le i \le N$ and $x, y \in X$ at $\partial(x,y) = i$

$$\underbrace{|\Gamma(x) \cap \Gamma_{i-1}(y)|}_{c_i} = i$$

$$\underbrace{|\Gamma(x) \cap \Gamma_i(y)|}_{a_i} = i(r-2)$$

$$\underbrace{|\Gamma(x) \cap \Gamma_{i+1}(y)|}_{b_i} = (N-i)(r-1)$$

" Γ is distance-regular with intersection numbers c_i, a_i, b_i "

For $0 \le i \le N$ def

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$$

$$= \binom{N}{i} (r-1)^i$$

then

$$|\Gamma_i(x)| = k_i \qquad x \in X$$

For $0 \le i \le N$ define $A_i \in \mathrm{Mat}_X(\mathbb{C})$ by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \ne i \end{cases} \qquad x, y \in X$$

" ith distance matrix "

thm 181 With above notation

$$\frac{A_i}{k_i} = K_i(A; p, N) \qquad 0 \le i \le N$$

where

$$p = \frac{r-1}{r}$$

$$A = \frac{(r-1)N I - A_1}{r}$$

pf Define

$$B_i = \frac{A_i}{k_i} \qquad 0 \le i \le N$$

By combinatorial counting

$$A_1 B_i = c_i B_{i-1} + a_i B_i + b_i B_{i+1} \qquad 0 \le i \le N$$

$$B_0 = I, \qquad B_{-1} = 0, \quad B_{N+1} = 0$$

Now

$$A B_i = c_i' B_{i-1} + a_i' B_i + b_i' B_{i+1} \qquad * \qquad 0 \le i \le N$$

where

$$c_i' = i(p-1)$$

$$a_i' = (1-p)i + p(N-i)$$

$$b_i' = (i-N)p$$

Comparing $*$ with th 99 we find

$$B_i = K_i(A; p, N) \qquad 0 \le i \le N$$

$\square$