The Krawtchouk polynomials

Until further notice $\mathbb{F}$ arb

$$\text{char}\,\mathbb{F} = 0$$

Fix integer $N \geq 0$

Fix $p \in \mathbb{F}$ $p \neq 0$, $p \neq 1$

Def 94 For $n = 0, 1, 2, \ldots, N$ define

$$\underbrace{K_n(x; p, N)}_{K_n(x)} = {}_2F_1\left(\begin{matrix} -n & -x \\ & -N \end{matrix} \,\middle|\, \frac{1}{p}\right)$$

"nth Krawtchouk polynomial in variable x"

n	K_n
0	1
1	$1 - \frac{x}{N}\frac{1}{p}$
2	$1 - 2\frac{x}{N}\frac{1}{p} + \frac{x(x-1)}{N(N-1)}\frac{1}{p^2}$
3	$1 - 3\frac{x}{N}\frac{1}{p} + 3\frac{x(x-1)}{N(N-1)}\frac{1}{p^2} - \frac{x(x-1)(x-2)}{N(N-1)(N-2)}\frac{1}{p^3}$
$\vdots$	$\vdots$

For $n = 0, 1, \ldots N$

K_n has degree n

coeff of x^n is $\frac{1}{(-N)_n \, p^n}$

In the study of the Krawtchouk polys, instead of working with the basis $\{x^n\}_{n\geq 0}^{\infty}$ for $\mathbb{F}[x]$, more natural to use basis

$$(-x)_n \qquad n = 0, 1, \ldots$$

LEM 95 $\quad$ For $n = 0, 1, 2 \ldots$

$$x\,(-x)_n = n(-x)_n - (-x)_{n+1}$$

pf $\quad (n-x)(-x)_n = (-x)_{n+1}$ $\quad \square$

Next goal: 3-term rec for K_n

Aside: A few facts about $(-x)_n$

- Gen function

$$\sum_{n=0}^{\infty} \frac{(-x)_n t^n}{n!} = {}_1F_0(-x \mid t)$$

$$= (1-t)^x$$

- For $n = 0, 1, 2, \ldots$

$$(-x-y)_n = \sum_{i=0}^{n} \binom{n}{i} (-x)_i (-y)_{n-i}$$

x, y indets

[this is Chu-Vandermonde in Disguise]

- For $n = 0, 1, 2 \ldots$

$$\frac{(-x)_{n+1} - (-y)_{n+1}}{y-x} = \sum_{\substack{0 \le i \\ 0 \le j \\ i+j \le n}} \binom{n-i}{j} \binom{n-j}{i} (-x)_i (-y)_j$$

Since the sequence $\{k_n\}_{n=0}^{N}$ is finite

we must handle the "end" somehow.

View $\mathbb{F}[x]$ as $\mathbb{F}$-algebra of all polynomial functions $\mathbb{F} \to \mathbb{F}$

View $\{0, 1, \ldots, N\}$ as subset of $\mathbb{F}$

Let V = $\mathbb{F}$-algebra of all functions $\{0, 1, \ldots, N\} \to \mathbb{F}$

$\dim V = N+1$

Have surjective $\mathbb{F}$-algebra hom

$$\sigma \quad \begin{array}{ccc} \mathbb{F}[x] & \to & V \\ f & \to & f\big|_{\{0,1,\ldots,N\}} \end{array}$$

kernel of σ = ideal of $\mathbb{F}[x]$ gen by $(-x)_{N+1}$

$$= \mathbb{F}[x]\,(-x)_{N+1}$$

$$\mathbb{F}[x] = \operatorname{Span}\{x^i\}_{i=0}^{N} + \mathbb{F}[x]\,(-x)_{N+1} \quad \text{(ds of vs)}$$

σ induces $\mathbb{F}$-alg iso

$$\underbrace{\mathbb{F}[x] \big/ \mathbb{F}[x]\,(-x)_{N+1}}_{\text{call this } \mathbb{F}[x]_N} \longrightarrow V$$

Identify

$\mathbb{F}[x]_N$ with V

via σ.

Abusing notation, we often use the same notation for an element of $\mathbb{F}[x]$ and its image in $\mathbb{F}[x]_N = V$

From this viewpt

$$\{x^i\}_{i=0}^N \quad \text{is basis for } \mathbb{F}[x]_N$$

$\forall f \in \mathbb{F}[x]$ TFAE

(i) $f(x) = 0 \quad \forall x = 0, 1, \ldots, N$

(ii) $f = 0$ in $\mathbb{F}[x]_N$

Sometimes view $\{K_n(x; p, N)\}_{n=0}^N$ as basis for $\mathbb{F}[x]_N$

Def 96 For $i = 0, 1, \ldots, N$ define

$$f_i = \frac{(-x)_i}{(-N)_i} \frac{(-1)^i}{p^i}$$

obs

$\{f_i\}_{i=0}^N$ is basis for $\mathbb{F}[x]_N$

For notational conv set $f_{N+1} = 0$

LEM 97 For $n = 0, 1, \ldots, N$

$$K_n = \sum_{i=0}^{n} \binom{n}{i} f_i \qquad (\text{in } \mathbb{F}[x] \text{ or } \mathbb{F}[x]_N)$$

pf

$$K_n = \sum_{i=0}^{n} \frac{(-n)_i (-x)_i}{(-N)_i \, i!} \frac{1}{p^i}$$

$$= \sum_{i=0}^{n} \underbrace{\frac{(-n)_i (-1)^i}{i!}}_{\binom{n}{i}} \underbrace{\frac{(-x)_i (-1)^i}{(-N)_i \, p^i}}_{f_i}$$

$\square$

In next result we give 2 versions

LEM 98

(i) For $i = 0, 1, \ldots, N$

$$x f_i = i f_i + p(i-N) f_{i+1} \quad \text{in } \mathbb{F}[x]_N$$

(ii) For $i, x = 0, 1, \ldots, N$

$$x f_i(x) = i f_i(x) + p(i-N) f_{i+1}(x) \quad \text{in } \mathbb{F}[x]$$

pf (i) Use L 95

(ii) By (i) □

As we proceed with more results each time there are two versions as in (i), (ii) above. Usually we just state one version.

Thm 99. For $n = 0, 1, \ldots, N$ the following holds in $\mathbb{F}[x]_N$:

$$-x K_n =$$

K_{n-1}	$n(1-p)$
K_n	$n(p-1) + (n-N)p$
K_{n+1}	$(N-n)p$

pf

$$-x K_n = -\sum_{i=0}^{n} \binom{n}{i} x f_i$$

$$= -\sum_{i=0}^{n} \binom{n}{i} \left(i f_i + p(i-N) f_{i+1} \right)$$

$$= -\sum_{i=0}^{n} f_i \left(i \binom{n}{i} + p(i-1-N) \binom{n}{i-1} \right)$$

$$= -\sum_{i=0}^{n} f_i \left(i \binom{n}{i} + p(i-1-n)\binom{n}{i-1} + p(n-N)\binom{n}{i-1} \right)$$

$$i \binom{n}{i} = \frac{i\, n!}{i!\,(n-i)!} = \frac{n!}{(i-1)!\,(n-i)!} = n \binom{n-1}{i-1}$$

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

$$(i-1-n)\binom{n}{i-1} = \frac{-(n-i+1)\, n!}{(i-1)!\,(n-i+1)!} = \frac{-n!}{(i-1)!\,(n-i)!}$$

$$= -n \binom{n-1}{i-1}$$

… the above sum using this to get result □

Next goal: Foreward/backward shift operators

for Krawtchouk polys

Def 100 We define $\mathbb{F}$-linear trans

$\nabla, \Delta : \mathbb{F}[x] \to \mathbb{F}[x]$

such that $\forall f \in \mathbb{F}[x]$

$$(\Delta f)(x) = f(x+1) - f(x)$$

$$(\nabla f)(x) = f(x) - f(x-1)$$

f	Δf	∇f
1	0	0
x	1	1
x^2	$2x+1$	$2x-1$
x^3	$3x^2+3x+1$	$3x^2-3x+1$
$\vdots$	$\vdots$	$\vdots$

Caution: Ideal $\mathbb{F}[x](-x)_{N+1}$ not closed under Δ or ∇.

So ∇, Δ do not exist at level of $\mathbb{F}[x]_N$

LEM 101

(i) $\Delta\,(-x)_n = -n\,(-x)_{n-1}$ $\quad n=1,2\ldots$

(ii) $\nabla\,(-x)_n = -n\,(1-x)_{n-1}$ $\quad n=1,2\ldots$

(iii) $x\nabla\,(-x)_n = n\,(-x)_n$ $\quad n=0,1\ldots$

pf. routine □

Recall

$$A : \begin{array}{ccc} \mathbb{F}[x] & \to & \mathbb{F}[x] \\ f & \to & xf \end{array}$$

LEM 102

(i) Δ, ∇ commute

(ii) $\Delta A - A\Delta = \Delta + I$

(iii) $\nabla A - A\nabla = I - \nabla$

(iv) $(1-\nabla)(1+\Delta) = I$

pf (i) $\forall f \in \mathbb{F}[x]$

$$\Delta \nabla f = f(x+1) - 2f(x) + f(x-1)$$

$$= \nabla \Delta f$$

(ii) $\forall f \in \mathbb{F}[x]$

$$\Delta A f = \Delta\, xf$$

$$= (x+1) f(x+1) - x f(x)$$

$$A\Delta f = A\left(f(x+1) - f(x)\right)$$

$$= x f(x+1) - x f(x)$$

$$(\Delta + I) f = f(x+1) - f(x) + f(x)$$

$$= f(x+1)$$

Result follows

(iii) Sim to (ii)

□

LEM 103 (Forward shift) Fn $N \geq 1$

$$\Delta\, k_n(x;p,N) = \frac{-n}{Np} k_{n-1}(x;p,N-1)$$

$n = 0,1,\ldots,N$

pf

$$\Delta k_n = \Delta \sum_{i=0}^{n} \binom{n}{i} f_i \qquad f_i = \frac{(-x)_i}{(-N)_i}\frac{(-1)^i}{p^i}$$

$$= \sum_{i=1}^{n} \binom{n}{i} \frac{-i(-x)_{i-1}}{(-N)_i}\frac{(-1)^i}{p^i}$$

$$\left[\ \binom{n}{i} i = n\binom{n-1}{i-1} \qquad (-N)_i = -N(1-N)_{i-1}\ \right]$$

$$= \frac{-n}{Np}\sum_{i=1}^{n}\binom{n-1}{i-1}\frac{(-x)_{i-1}}{(1-N)_{i-1}}\frac{(-1)^{i-1}}{p^{i-1}}$$

$[\,i-1 \to j\,]$

$$= \frac{-n}{Np}\sum_{j=0}^{n-1}\binom{n-1}{j}\underbrace{\frac{(-x)_j}{(1-N)_j}\frac{(-1)^j}{p^j}}_{f_j}$$

$$= \frac{-n}{Np} k_{n-1}(x;p,N-1)$$

$\square$

Continue to discuss Krawtchouk polys $K_n(x; p, N)$

LEM 104 (Backward shift) For $n = 0, 1, \dots, N$

$$(*) \quad (N+1-x)\, K_n(x, p, N) - x\,\frac{1-p}{p}\, K_n(x-1, p, N)$$

in $\mathbb{F}[x]$

$$= (N+1)\, K_{n+1}(x, p, N+1)$$

pf Recall

$$K_n(x; p, N) = {}_2F_1\left(\begin{matrix} -n \; -x \\ -N \end{matrix} \,\Big|\, \frac{1}{p}\right)$$

$$= \sum_{\ell=0}^{N \to N+1} \frac{(-n)_\ell (-x)_\ell}{(-N)_\ell\, \ell!} \frac{1}{p^\ell}$$

$[(-n)_\ell = 0 \text{ if } \ell > n]$

View each term in $(*)$ as power series in $\frac{1}{p}$

For $0 \le \ell \le N+1$ compare coeffs of $\frac{1}{p^\ell}$

$\ell = 0$:

$$N+1-x + x = N+1 \quad \checkmark$$

$\ell \ge 1$:

$$(N+1-x)\frac{(-x)_\ell (-n)_\ell}{(-N)_\ell\, \ell!} + x\,\frac{(1-x)_\ell (-n)_\ell}{(-N)_\ell\, \ell!} - x\,\frac{(1-x)_{\ell-1} (-n)_{\ell-1}}{(-N)_{\ell-1}\, (\ell-1)!}$$

$$\stackrel{?}{=} (N+1)\frac{(-x)_\ell (-n-1)_\ell}{(-N-1)_\ell\, \ell!}$$

this is routinely checked

$\square$

Note: Backward shift has following interp using ∇

$$\nabla f(x) = f(x) - f(x-1)$$ for all functions f on integer $x = 0, 1, \ldots$

(*) is asserting

$$\nabla\left(\binom{N}{x} \left(\frac{p}{1-p}\right)^x K_n(x; p, N) \right) = \binom{N+1}{x} \left(\frac{p}{1-p}\right)^x K_{n+1}(x; p, N+1)$$

check

$$\binom{N}{x}\left(\frac{p}{1-p}\right)^x K_n(x; p, N) - \binom{N}{x-1}\left(\frac{p}{1-p}\right)^{x-1} K_n(x-1; p, N)$$ ← 0 if $x=0$

$$\stackrel{?}{=} \binom{N+1}{x}\left(\frac{p}{1-p}\right)^x K_{n+1}(x; p, N+1)$$

$x=0$: $1 - 0 = 1$ ✓

$x \geq 1$

$$\frac{N!}{x!(N-x)!} K_n(x; p, N) - \frac{N!}{(x-1)!(N-x+1)!} \frac{1-p}{p} K_n(x-1; p, N)$$

$$\stackrel{?}{=} \frac{(N+1)!}{x(N-x+1)!} K_{n+1}(x; p, N+1)$$

$$(N-x+1) K_n(x; p, N) - x \frac{1-p}{p} K_n(x-1; p, N)$$

$$\stackrel{?}{=} (N+1) K_{n+1}(x; p, N+1)$$

✓ □

LEM 105 (Rodrigues-type formula)

For $n, x = 0, 1, \dots N$

$$\binom{N}{x}\left(\frac{p}{1-p}\right)^{x} K_n(x; p, N) = \nabla^n\left(\binom{N-n}{x}\left(\frac{p}{1-p}\right)^{x}\right)$$

pf Apply above note repeatedly $\square$

Thm 106 For $x = 0, 1, \ldots, N$

$$\left(1 - \frac{1-p}{p} t\right)^x (1+t)^{N-x} = \sum_{n=0}^{N} \binom{N}{n} K_n(x; p, N) t^n \quad (*)$$

t = indet

pf View each side as power series in $\frac{1}{p}$

LHS :

$$\left(1 - \frac{1-p}{p} t\right)^x = \left(1+t - \frac{t}{p}\right)^x$$

$$= \sum_{\ell=0}^{x} \frac{1}{p^\ell} (-1)^\ell t^\ell (1+t)^{x-\ell} \binom{x}{\ell}$$

$$\binom{x}{\ell} = \frac{(-x)_\ell (-1)^\ell}{\ell!}$$

$(-x)_\ell = 0$ if $\ell > x$

$$= \sum_{\ell=0}^{N} \frac{1}{p^\ell} t^\ell (1+t)^{x-\ell} \frac{(-x)_\ell}{\ell!}$$

For $0 \le \ell \le N$ in LHS of $(*)$ the coeff of $\frac{1}{p^\ell}$ is

$$\frac{t^\ell (1+t)^{x-\ell} (-x)_\ell}{\ell!} (1+t)^{N-x}$$

$$= t^\ell \frac{(1+t)^{N-\ell} (-x)_\ell}{\ell!}$$

RHS : Recall

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-x & -n\\ & -N\end{matrix} \middle| \frac{1}{p}\right)$$

$$= \sum_{\ell=0}^{N} \frac{(-x)_\ell (-n)_\ell}{(-N)_\ell\, \ell!} \frac{1}{p^\ell}$$

For $0 \le \ell \le N$ in RHS of (*) the coeff of $\frac{1}{p^\ell}$ is

$$\frac{(-x)_\ell}{\ell!} \frac{1}{(-N)_\ell} \sum_{n=0}^{N} \binom{N}{n} t^n (-n)_\ell$$

Show: for $0 \le \ell \le N$

$$t^\ell (1+t)^{N-\ell} \overset{?}{=} \frac{1}{(-N)_\ell} \sum_{n=0}^{N} \binom{N}{n} t^n (-n)_\ell \qquad **$$

$(-n)_\ell = 0$ if $\ell > n$

Sum is $\sum_{n=\ell}^{N}$ ch vars $r = n - \ell$

$$\text{RHS of } ** = \frac{1}{(-N)_\ell} \sum_{r=0}^{N-\ell} \binom{N}{r+\ell} t^{r+\ell} (-r-\ell)_\ell$$

$$\frac{1}{(-N)_\ell} = \frac{(-1)^\ell (N-\ell)!}{N!}, \qquad \binom{N}{r+\ell} = \frac{N!}{(r+\ell)!(N-r-\ell)!}, \qquad (-r-\ell)_\ell = (-1)^\ell \frac{(r+\ell)!}{r!}$$

$$= t^\ell \sum_{r=0}^{N-\ell} \binom{N-\ell}{r} t^r$$

$$= t^\ell (1+t)^{N-\ell} \checkmark$$

□

Another generating function

<u>Notation</u> Given power series

$$f(t) = \sum_{i=0}^{\infty} a_i t^i$$

$$[f(t)]_N \text{ means } \sum_{i=0}^{N} a_i t^i$$

Thm 107 For $x = 0, 1, \ldots, N$

$$\left[e^t \, {}_1F_1\left(\begin{matrix} -x \\ -N \end{matrix} \middle| \frac{-t}{p} \right) \right]_N = \sum_{n=0}^{N} K_n(x; p, N) \frac{t^n}{n!} \quad *$$

pf For $0 \le \ell \le N$ compare coeff of $\frac{1}{p^\ell}$ on LHS, RHS of $*$

LHS: Recall

$$e^t = \sum_{i=0}^{\infty} \frac{t^i}{i!}$$

Coeff of $\frac{1}{p^\ell}$ in LHS of $*$ is

$$\left[\frac{(-x)_\ell}{(-N)_\ell} \frac{(-1)^\ell t^\ell}{\ell!} \sum_{i=0}^{\infty} \frac{t^i}{i!} \right]_N$$

$$= \frac{(-x)_\ell}{(-N)_\ell} \frac{(-1)^\ell t^\ell}{\ell!} \sum_{i=0}^{N-\ell} \frac{t^i}{i!}$$

RHS : coeff of $\frac{1}{p^l}$ is

$$\sum_{n=0}^{N} \frac{(-x)_l (-n)_l}{(-N)_l \, l!} \frac{t^n}{n!}$$

$$= \frac{(-x)_l}{(-N)_l \, l!} \sum_{n=0}^{N} \frac{(-n)_l \, t^n}{n!}$$

$(-n)_l = 0$ if $l > n$

$$= \frac{(-x)_l}{(-N)_l \, l!} \sum_{n=l}^{N} \frac{(-n)_l \, t^n}{n!}$$

$(-n)_l = \frac{n! \, (-1)^l}{(n-l)!}$

$$= \frac{(-x)_l (-1)^l}{(-N)_l \, l!} \sum_{n=l}^{N} \frac{t^n}{(n-l)!}$$

$i = n-l$

$$= \frac{(-x)_l (-1)^l}{(-N)_l \, l!} t^l \sum_{i=0}^{N-l} \frac{t^i}{i!}$$

✓

□

Next goal: difference equation for $K_n(x;p,N)$

Obs that for $n,x = 0,1,\dots,N$

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n\ -x\\ -N\end{matrix}\middle|\ \frac{1}{p}\right) \qquad *$$

$$= K_x(n;p,N)$$

In 3-term rec thm 99 interchange $n \leftrightarrow x$ and use $*$ to get

thm 108 For $n,x = 0,1,\dots,N$

$$-n K_n(x;p,N) = x(1-p)\Big(K_n(x-1;p,N) - K_n(x;p,N)\Big)$$
$$+ (N-x)p\Big(K_n(x+1;p,N) - K_n(x;p,N)\Big)$$

In other words $y = K_n(x;p,N)$ is a solution to

$$\Big((p-1)A\nabla + p(NI-A)\Delta + nI\Big)y = 0$$

$$[A : f \to xf]$$

Define $\mathbb{F}$-linear map

$$A^*: \mathbb{F}[x] \to \mathbb{F}[x]$$

by

$$A^* = (1-p)\, A\nabla + p(A - NI)\,\Delta$$

So for $n = 0, 1, \ldots, N$

$K_n(x; p, N)$ is eigenvector for A^* with eigenval n

LEM 109 The maps A, A^* satisfy

(i) $[A, [A, A^*]] = (2p-1)A + A^* - pNI$

(ii) $[A^*, [A^*, A]] = (2p-1)A^* + A - pNI$

pf. Use Lem 102 and the def of A^*. $\square$

LEM 110 In $\mathbb{F}[x]$ the ideal

$$\mathbb{F}[x]\,(-x)_{N+1}$$

is closed under each of A, A^*

pf

A: any ideal closed under mult by x

A^*: Recall above ideal has basis

$$(-x)_n \qquad N+1 \le n < \infty$$

For $n \ge N+1$ Apply A^* to $(-x)_n$

$$A^*(-x)_n = (1-p)\underbrace{x\,\nabla(-x)_n}_{n(-x)_n} + p(x-N)\underbrace{\Delta(-x)_n}_{-n(-x)_{n-1}}$$

$(-x)_{N+1}$ divides $(-x)_n$ ✓

... $(x-N)(-x)_{n-1}$ ✓

So $A^*(-x)_n$ is in the ideal

□

By Lem 110 the action of A, A^* on $\mathbb{F}[x]$ induces an action of A, A^* on

$$\mathbb{F}[x] / \mathbb{F}[x](-x)_{N+1} = \mathbb{F}[x]_N = V$$

By const each of A, A^* is diagonalizable on V with eigvalues $0, 1, 2, \ldots, N$

For convenience define a, a^* on $\mathbb{R}[x]$ by

$$a = NI - 2A$$

$$a^* = NI - 2A^*$$

Obs each of a, a^* is diagonalizable on V with eigvalues

$$N, N-2, N-4, \ldots, -N$$

LEM 111 On $\mathbb{F}[x]$ or V

$$[a, [a, a^*]] = 4(2p-1)a + 4a^*$$

$$[a^*, [a^*, a]] = 4a + 4(2p-1)a^*$$

pf In L109 eliminate A, A^* using

$$A = \frac{a - NI}{2}, \qquad A^* = \frac{a^* - NI}{2}$$ □

We will show the relations in L 111 give a presentation of the Lie algebra $sl_2(\mathbb{F})$.

Next goal: orthog relations for $\{K_n(x;p,N)\}_{n=0}^{N}$

We apply th 69, suitably interpreted.

th 69 was about an infinite sequence $\{p_n\}_{n=0}^{\infty}$

We fixed $d \geq 0$ and only considered $\{p_n\}_{n=0}^{d+1}$.

Remaining polys play no role. So we apply th 69 with:

$$d = N$$

$$p_i = K_i(x;p,N) \qquad 0 \leq i \leq N$$

$$p_{N+1} = (-x)_{N+1}$$

$$\left[\text{so } x p_N \in \text{Span}(p_{N-1}, p_N, p_{N+1}) \right]$$

Recall $\{\theta_i\}_{i=0}^{N}$ are the roots of p_{N+1}. Take

$$\theta_i = i \qquad 0 \leq i \leq N$$

By th 99

$$c_i = i(p-1) \qquad 1 \leq i \leq N$$

$$b_i = (i-N)p \qquad 0 \leq i \leq N-1$$

$$c_i + a_i + b_i = 0 \qquad 0 \leq i \leq N$$

$$c_0 = 0, \quad b_N = 0$$

For $0 \leq i \leq N$

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$$

$$= \binom{N}{i} \left(\frac{p}{1-p}\right)^i$$

LEM 112 For the Krawtchouk polynomials $K_n(x;p,N)$ the Christoffel numbers are

$$m_i = \mu_0 \binom{N}{i} p^i (1-p)^{N-i} \qquad i = 0,1,\dots,N$$

Pf Use Cor 71

$$\frac{\mu_0}{m_i} = \frac{K_N(\theta_i;p,N)}{c_1 c_2 \cdots c_N} \prod_{\substack{0 \le j \le N \\ j \ne i}} (\theta_i - \theta_j) \qquad *$$

Obs

$$\begin{aligned} K_N(\theta_i;p,N) &= K_N(i;p,N) \\ &= {}_2F_1\left(\begin{matrix} -i, -N \\ -N \end{matrix} \,\middle|\, \frac{1}{p}\right) \\ &= {}_1F_0\left(-i \,\middle|\, \frac{1}{p}\right) \\ &= \sum_{j=0}^{i} \frac{(-i)_j}{j!} \frac{1}{p^j} \\ &= \sum_{j=0}^{i} \binom{i}{j} (-1)^j \frac{1}{p^j} \\ &= \left(1 - \frac{1}{p}\right)^i \end{aligned}$$

Eval $*$ using this and data above the lemma statement. Result follows.

□

Note 113   Ref to LEM 112, for $0 \le i \le N$

$$m_i = \mu_0 (1-p)^N \binom{N}{i} \left(\frac{p}{1-p}\right)^i$$

$$= \mu_0 (1-p)^N k_i$$

the parameter μ_0 is free so WLOG

$$\mu_0 = (1-p)^{-N}$$

In this case

$$m_i = k_i$$

Thm 114 The Krawtchouk polynomials satisfy

(i) $$\sum_{i=0}^{N} K_n(i;p,N)\, K_m(i;p,N) \binom{N}{i} p^i (1-p)^{N-i}$$

$$= \delta_{n,m} \binom{N}{n}^{-1} \left(\frac{1-p}{p}\right)^n \qquad 0 \le n,m \le N$$

(ii) $$\sum_{n=0}^{N} K_n(i;p,N)\, K_n(j;p,N) \binom{N}{n} p^n (1-p)^{N-n}$$

$$= \delta_{ij} \binom{N}{i}^{-1} \left(\frac{1-p}{p}\right)^i \qquad 0 \le i,j \le N$$

pf: Apply Th 69 using LEM 112 and data above it. □

Next goal: express our results on the Krawtchouk polynomials in terms of matrices.

DEF 115 For the Krawtchouk polynomials $\{K_n(x; p, N)\}_{n=0}^{N}$ we define:

Matrix Name	matrix entries
U	$\left({}_2F_1\left(\begin{matrix} -i & -j \\ -N & \end{matrix} \middle/ \frac{1}{p} \right) \right)_{0 \le i,j \le N}$
B	$\begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & c_N & a_N \end{pmatrix}$ $c_i = i(p-1) \quad 1 \le i \le N$ $b_i = (i-N)p \quad 0 \le i \le N-1$ $a_i = -c_i - b_i \quad 0 \le i \le N \quad c_0 = 0, \; b_N = 0$
D	$\operatorname{diag}(0, 1, 2, \ldots, N)$
K	$\operatorname{diag}\left(\binom{N}{i} \left(\frac{p}{1-p}\right)^i \right)_{0 \le i \le N}$

thm 116 With ref to Def 115

Result	meaning	ref
$U^t = U$	clear	
$B^t = KBK^{-1}$	B is symmetrizable	L 50
$UD = BU$	3-term rec	th 99
$DU = UB^t$	difference equation	th 108
$(1-p)^N UKUK = I$	orthogonality	th 114

pf Each ref write the result in matrix form. □

With ref to Def 115 and thm 116

Sometimes useful to work with

$$P = UK$$ $\star$

instead of U.

thm 117 With ref to Def 115 and ($\star$)

$$P^t = KPK^{-1}$$

$$B^t = KBK^{-1}$$

$$PD = BP$$

$$PB = DP$$

$$P^2 = (1-p)^{-N} I$$

pf In thm 116 elim U using ($\star$) $\square$

Next goal: express our results on the Krawtchouk polynomials in terms of $A, A^*, V, \langle , \rangle$

Recall our convention

$$\mathbb{F}[x] / \mathbb{F}[x](-x)_{N+1} = \mathbb{F}[x]_N = V = \mathbb{F}\text{-algebra of all functions } \{0,1,\ldots,N\} \to \mathbb{F}$$

Two linear trans on V:

$$A : V \to V \qquad\qquad A^* : V \to V$$
$$f \to xf \qquad\qquad \text{from Lem 110}$$

Basis for V:

$$\{ k_i(x; p, N) \}_{i=0}^N$$

Significance of A^* is that

$$A^* k_i(x; p, N) = i\, k_i(x; p, N) \qquad i = 0, 1, \ldots, N$$

Another basis $\{e_i\}_{i=0}^N$ for V comes from Gauss quadrature

$$e_i(j) = \delta_{ij} \qquad 0 \le i, j \le N$$

So

$$e_i e_j = \delta_{ij} e_i \qquad 0 \le i, j \le N$$

$$1 = \sum_{i=0}^N e_i$$

$$x e_i = i e_i \qquad i = 0, 1, \ldots, N$$

$$e_i = \prod_{\substack{0 \le j \le N \\ j \ne i}} \frac{x-j}{i-j} \qquad i = 0, 1, \ldots, N$$

Note that

$$A e_i = i e_i \qquad 0 \le i \le N$$

Recall bilinear form from Gauss quadrature

$$\langle , \rangle : \quad V \times V \longrightarrow \mathbb{F}$$
$$f \quad g \longrightarrow \sum_{i=0}^{N} f(i)\, g(i)\, m_i$$

So

$$\langle e_i, e_j \rangle = \delta_{ij}\, m_i \qquad 0 \le i, j \le N$$

By thm 114

$$\langle k_i(x; p, N),\ k_j(x; p, N) \rangle = \delta_{ij} \frac{\mu_0}{k_i}$$
$$0 \le i, j \le N$$

By construction

$$\langle Af, g \rangle = \langle f, Ag \rangle \qquad \forall f, g \in V$$
$$\langle A^* f, g \rangle = \langle f, A^* g \rangle$$

Relative $\langle , \rangle$ the basis for V dual to $\{ k_i(x; p, N) \}_{i=0}^{N}$

is

$$\left\{ k_i(x; p, N) \frac{k_i}{\mu_0} \right\}_{i=0}^{N}$$

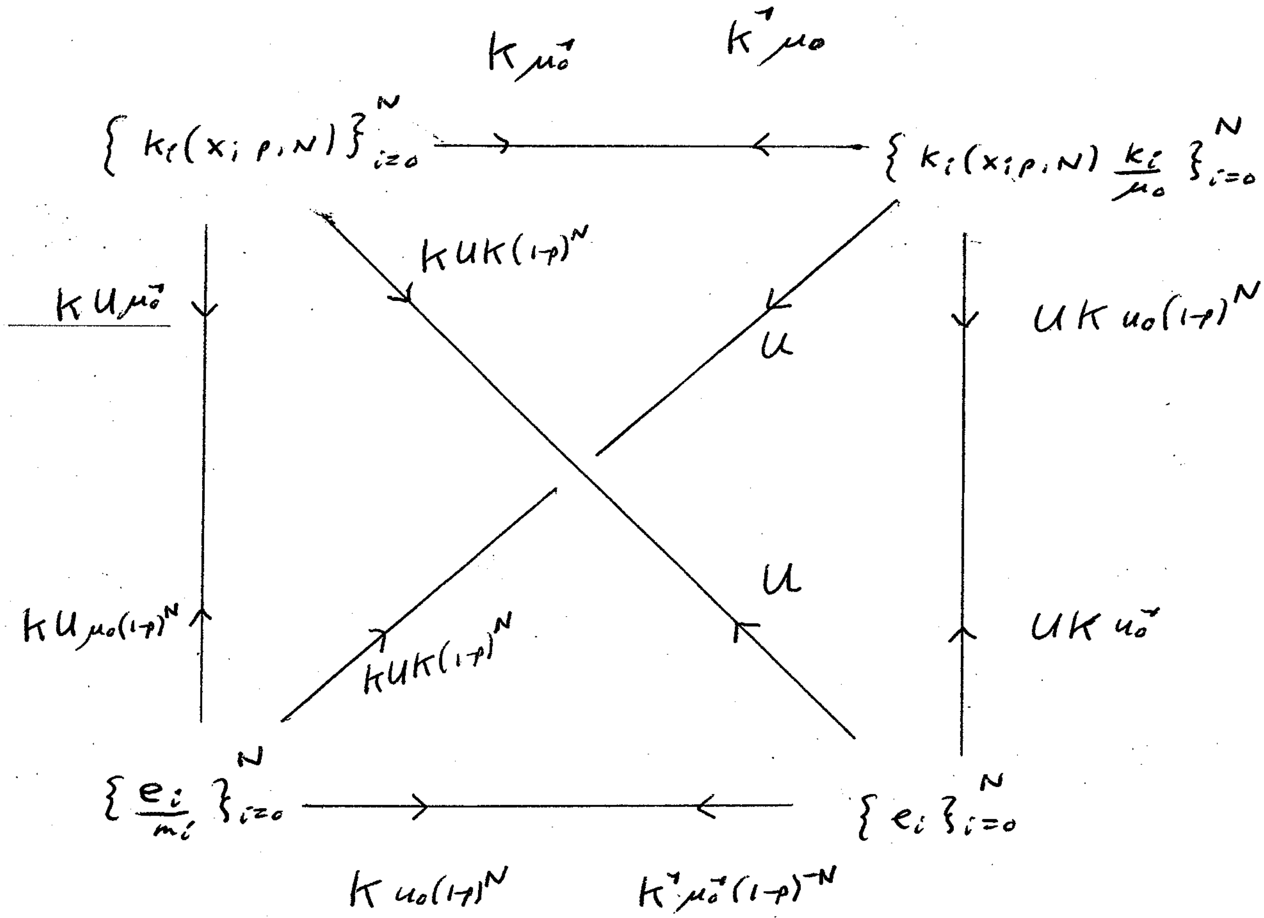
Relative $\langle , \rangle$ the basis for V dual to $\{ e_i \}_{i=0}^{N}$ is

$$\{ e_i / m_i \}_{i=0}^{N}$$

Matrices that rep . . . . . .

basis	matrix rep A	matrix rep A^*
$\{k_i(x;p,N)\}_{i=0}^{N}$	B^t	D
$\{k_i(x;p,N)\,k_i/u_0\}_{i=0}^{N}$	B	D
$\{e_i\}_{i=0}^{N}$	D	B
$\{e_i/m_i\}_{i=0}^{N}$	D	B^t

transition matrices

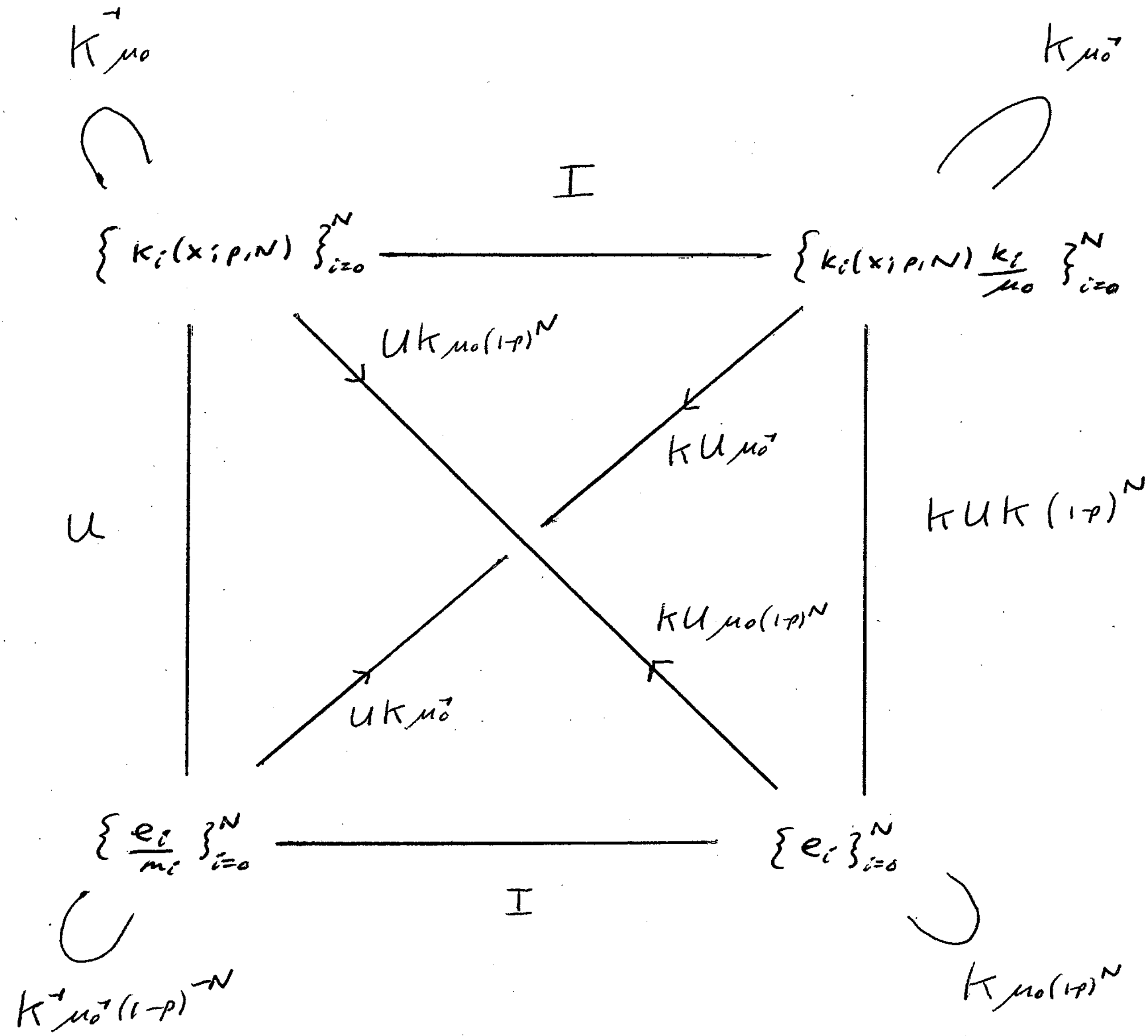


key:

$$\{u_i\}_{i=0}^N \xrightarrow{M} \{v_i\}_{i=0}^N$$

means $\quad v_j = \sum_{i=0}^N M_{ij} u_i \quad (0 \le j \le N)$

<u>Matrices that represent ⟨ , ⟩</u>



Key: $\{u_i\}_{i=0}^N \xrightarrow{M} \{v_i\}_{i=0}^N$

means $M_{ij} = \langle u_i, v_j \rangle \quad (0 \le i,j \le N)$

LEM 118 For $i=0,1,\ldots,N$

(i) $K_i(A;p,N)\,v_0 = v_i$

where $v_i = K_i(x;p,N)$

(ii) $K_i(A^*;p,N)\,v_0^* = v_i^*$

where $v_i^* = e_i/m_i$

pf (i) A is mult by x

(ii) Matrix rep A rel $\{v_i\}_{i=0}^N = B^t$

$=$ matrix rep A^* rel $\{v_i^*\}_{i=0}^N$

Result follows. □

LEM 119 With ref to Def 115

$$[B, [B, D]] = (2p-1) B + D - pN I,$$

$$[D, [D, B]] = (2p-1) D + B - pN I.$$

pf In Lem 109 we saw A, A^* satisfy the above relations.

With respect to the basis $\{ k_i(x; p, N) \frac{k_i}{\mu_0} \}_{i=0}^{N}$

A, A^* are rep by B, D resp.

Result follows. □

Recall the Lie algebra

$$\mathfrak{sl}_2(\mathbb{F}) = \{ y \in \mathrm{Mat}_2(\mathbb{F}) \mid \mathrm{tr}(y) = 0 \}$$

$\parallel$

L

Lie bracket $[y, z] = yz - zy$

L has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h$$

Given $y \in L$ we have

$$\mathrm{ad}_y : \begin{array}{ccc} L & \longrightarrow & L \\ z & \rightarrow & [y, z] \end{array} \qquad \text{" adjoint map "}$$

Recall Killing form

$$(\, , \,) \quad \begin{array}{ccc} L \times L & \longrightarrow & \mathbb{F} \\ y \quad z & \rightarrow & \mathrm{tr}(\mathrm{ad}_y \, \mathrm{ad}_z) \end{array}$$

$$\parallel \; 4\,\mathrm{tr}(yz)$$

	e	h	f
e	0	0	4
h	0	8	0
f	4	0	0

For notational convenience def

$$\langle\, ,\rangle : L \times L \to \mathbb{F}$$
$$y, z \to (y, z)/8 .$$

So

$\langle , \rangle$	e	h	f
e	0	0	$\frac{1}{2}$
h	0	1	0
f	$\frac{1}{2}$	0	0

Given $y \in L$ write

$$y = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

Obs

$$\underset{\|y\|^2}{\overset{\shortparallel}{\langle y, y\rangle}} = \begin{pmatrix} \beta & \alpha & \gamma \end{pmatrix} \begin{pmatrix} & & \frac{1}{2} \\ & 1 & \\ \frac{1}{2} & & \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}$$

$$= \beta\gamma + \alpha^2$$

$$= -\det(y)$$

Let r, s denote the eigvals of y (in alg closure of $\mathbb{F}$)

$$r + s = \operatorname{tr}(y) = 0$$

$$rs = \det(y)$$

Case $r = s$:

$r = 0, \; s = 0, \qquad y^2 = 0 \qquad \det(y) = 0, \qquad \|y\|^2 = 0$

Case $r \neq s$:

$r \neq 0, \; s \neq 0 \qquad y$ is semisimple $\qquad \det(y) \neq 0 \qquad \|y\|^2 \neq 0$

LEM 120 $\quad \forall y \in L \qquad$ TFAE

(i) $\quad \|y\|^2 = 1$

(ii) $\quad \det(y) = -1$

(iii) $\quad y$ is s.s. with eigvalues $1, -1$

(iv) $\quad \exists$ invertible $\Delta \in Mat_2(\mathbb{F})$ s.t

$$y = \Delta h \Delta^{-1} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

pf clear.

$\square$

DEF 121 A s.s element of L is <u>normalized</u>

whenever it has eigenvalues 1, -1.

Def 122 Given two normalized s.s elements

$$a, a^* \in L$$

Define $\rho \in \mathbb{F}$ s.t

$$\langle a, a^* \rangle = 1 - 2\rho$$

Ex 123 With above notation take

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \qquad \alpha^2 + \beta\gamma = 1$$

$$a^b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$\underset{1-2p}{\underset{\shortparallel}{\langle a, a^b \rangle}} = (\beta \quad \alpha \quad \gamma) \begin{pmatrix} & & \frac{1}{2} \\ & 1 & \\ \frac{1}{2} & & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \alpha$$

So

$$\alpha = 1 - 2p \qquad p = \frac{1-\alpha}{2}$$

$$\beta\gamma = 1 - \alpha^2$$
$$= 4p(1-p)$$

Obs

$$[a, a^b] = [\beta e + \alpha h + \gamma f, h]$$
$$= -2\beta e + 2\gamma f$$

LEM 124 Given normalized s.s. elements

$$a,\ a^* \in L$$

with

$$\langle a, a^* \rangle = 1-2p$$

then

$\langle , \rangle$	a	a^*	$[a,a^*]$
a	1	$1-2p$	0
a^*	$1-2p$	1	0
$[a,a^*]$	0	0	$-16p(1-p)$

the above matrix has det $-64p^2(1-p)^2$

pf obs

$$\langle a, [a,a^*] \rangle = \langle \underset{0}{\underset{\parallel}{[a,a]}}, a^* \rangle = 0$$

Sim $\langle a^*, [a,a^*] \rangle = 0$

To find $\|[a,a^*]\|^2$ WLOG a, a^* as in EX 123

$$\|[a,a^*]\|^2 = \|-2\beta e + 2\gamma f\|^2$$

$$= \begin{pmatrix} -2\beta & 0 & 2\gamma \end{pmatrix} \begin{pmatrix} & & \frac{1}{2} \\ & 1 & \\ \frac{1}{2} & & \end{pmatrix} \begin{pmatrix} -2\beta \\ 0 \\ 2\gamma \end{pmatrix}$$

$$= -4\beta\gamma$$

$$= -16p(1-p)$$

□

LEM 125 With ref to LEM 124 TFAE

(i) $\rho \neq 0$ and $\rho \neq 1$

(ii) $a, a^*, [a, a^*]$ is basis for L

(iii) a, a^* generate L

pf (i) → (ii) In Lem 124 the matrix of inner products is non deg so

$a, a^*, [a, a^*]$ lin indep

hence a basis for L.

(ii) → (i) $\langle , \rangle$ is non deg on L so matrix in LEM 124 has nonzero det.

(ii) ⟺ (iii) Since $\dim L = 3$. □

LEM 126 Given normalized semi simple

$$a, a^* \in L$$

write

$$\langle a, a^* \rangle = 1-2\rho$$

Then

(i) $$[a, [a, a^*]] = 4(2\rho-1)a + 4a^*$$

(ii) $$[a^*, [a^*, a]] = 4a + 4(2\rho-1)a^*$$

pf (ii) WLOG represent a, a^* by the matrices in EX 124

So

$$[a, a^*] = -2\beta e + 2\gamma f$$

So

$$[a^*, a] = 2\beta e - 2\gamma f$$

Now

$$[a^*, [a^*, a]] = [h, 2\beta e - 2\gamma f]$$

$$= 4\beta e + 4\gamma f$$

Also

$$4a + 4(2\rho-1)a^* = 4(\beta e + \alpha h + \gamma f) + 4\underbrace{(2\rho-1)}_{=-\alpha}h$$

$$= 4\beta e + 4\gamma f \quad \checkmark$$

(i) Sim.

□

thm 12.7 Given $p \in \mathbb{F}$ with $p \neq 0$, $p \neq 1$.

Let $\mathcal{L}$ denote the Lie algebra over $\mathbb{F}$ defined by generators u, v and relations

$$[u,[u,v]] = 4(2p-1)u + 4v,$$

$$[v,[v,u]] = 4u + 4(2p-1)v$$

Then $\mathcal{L}$ is isomorphic to $sl_2(\mathbb{F})$. An iso is

$$u \to 2(1-p)e + (1-2p)h + 2pf$$

$$v \to h$$

The inverse of this iso sends

$$h \to v$$

$$e \to \frac{2u - 2(1-2p)v - [u,v]}{8(1-p)}$$

$$f \to \frac{2u - 2(1-2p)v + [u,v]}{8p}$$

pf One checks each map above is a Lie algebra homomorphism.

□

Given normalized semi simple elements a, a^* that generate $L = sl_2(\mathbb{F})$. Write

$$\langle a, a^* \rangle = 1 - 2p .$$

So $p \neq 0$, $p \neq 1$ by LEM 125

So the following is a basis for L:

$$a, \quad a^*, \quad [a, a^*].$$

In view of LEM 126 and THM 127, from now on we identify

$$a = 2(1-p)e + (1-2p)h + 2pf$$

$$= \begin{pmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{pmatrix}$$

$$a^* = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Obs

$$[a, a^*] = 4(p-1)e + 4pf$$

$$= \begin{pmatrix} 0 & 4(p-1) \\ 4p & 0 \end{pmatrix}$$

By an <u>automorphism</u> of L we mean a Lie algebra isomorphism $L \to L$

LEM 128 $\exists$ unique automorphism of L that sends

$$a \to a^*, \qquad a^* \to a.$$

Denoting this aut by $*$ we have

$$(y^*)^* = y \qquad \forall y \in L$$

pf In THM 127 the relations are invariant under the swap $u \leftrightarrow v$. So the aut exists. It is unique since a, a^* generate L. The last assertion is clear. $\square$

Another view of map *

Define

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1-\frac{1}{p} \end{pmatrix}, \qquad W = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}$$

One checks

$$W\,U\,W\,U = (1-p)\,I$$

Define

$$R = WU$$

So

$$R^2 = (1-p)\,I$$

We have

$$R = \begin{pmatrix} 1-p & 1-p \\ p & p-1 \end{pmatrix}$$

$$R^{-1} = (1-p)^{-1} R$$

$$= \begin{pmatrix} 1 & 1 \\ \frac{p}{1-p} & -1 \end{pmatrix}$$

LEM 129 With above notation

$$y^* = R y R^{-1} \qquad \forall y \in L$$

pf Obs the map

$$L \longrightarrow L$$
$$y \longrightarrow R y R^{-1}$$

is an aut of L.

One checks

$$R a R^{-1} = a^* \qquad R a^* R^{-1} = a.$$

The result follows by LEM 128 □

LEM 130 $\forall y, z \in L$

$$\langle y^*, z^* \rangle = \langle y, z \rangle$$

pf

$$LHS = \left\langle RyR^{-1}, RzR^{-1} \right\rangle$$

$$= \frac{1}{2} tr\left(RyR^{-1} RzR^{-1} \right)$$

$$= \frac{1}{2} tr\left(RyzR^{-1} \right)$$

$$= \frac{1}{2} tr(yz)$$

$$= \langle y, z \rangle$$

□

LEM 131 The following diag...

$$
\begin{array}{ccc}
L & \xrightarrow{\ *\ } & L \\
{\scriptstyle \mathrm{ad}\, y}\downarrow & & \downarrow {\scriptstyle \mathrm{ad}(y^*)} \\
L & \xrightarrow[\ *\ ]{} & L
\end{array}
$$

pf $\forall z \in L$ chase z around diag

$$
\begin{array}{ccc}
z & \longrightarrow & z^* \\
\downarrow & & \downarrow \\
 & & [y^*, z^*] \\
 & & " \\
[y, z] & \longrightarrow & [y, z]^*
\end{array}
$$

$\square$

$\square$

Applying the map $*$ to the basis e, h, f for L

we get another basis for L:

$$e^*, \quad h^*, \quad f^*.$$

By LEM 129

$$e^* = ReR^{-1} \qquad h^* = RhR^{-1} \qquad f^* = RfR^{-1}$$

This gives

$$e^* = \begin{pmatrix} p & p-1 \\ \frac{p^2}{1-p} & -p \end{pmatrix}$$

$$= (p-1)e + ph + \frac{p^2}{1-p} f$$

$$h^* = \begin{pmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{pmatrix}$$

$$= 2(1-p)e + (1-2p)h + 2pf$$

$$= a$$

$$f^* = \begin{pmatrix} 1-p & 1-p \\ p-1 & p-1 \end{pmatrix}$$

$$= (1-p)e + (1-p)h + (p-1)f$$

In summary we have the following bases for L:

$$e \quad h \quad f \qquad (I)$$

$$a \quad a^* \quad [a, a^*] \qquad (II)$$

$$e^* \quad h^* \quad f^* \qquad (III)$$

LEM 132 For the bases (I) – (III) the transition matrices are given below.

$$
\begin{array}{c|ccc|}
 & a & a^* & [a, a^*] \\
e & 2(1-\rho) & 0 & 4(\rho-1) \\
h & 1-2\rho & 1 & 0 \\
\hline
f & 2\rho & 0 & 4\rho
\end{array}
$$

$$
\begin{array}{c|ccc}
 & e & h & f \\
\hline
a & \dfrac{1}{4(1-p)} & 0 & \dfrac{1}{4p} \\[2ex]
a^* & \dfrac{2p-1}{4(1-p)} & 1 & \dfrac{2p-1}{4p} \\[2ex]
\hline
[a, a^*] & \dfrac{1}{8(p-1)} & 0 & \dfrac{1}{8p}
\end{array}
$$

$$
\begin{array}{c} \\ e^* \\ h^* \\ f^* \end{array}
\begin{array}{c}
\begin{array}{ccc} a & a^* & [a, a^*] \end{array} \\
\left(\begin{array}{ccc} 0 & 2(1-p) & 4(1-p) \\ 1 & 1-2p & 0 \\ 0 & 2p & -4p \end{array} \right)
\end{array}
$$

	e^*	h^*	f^*
a	$\frac{2p-1}{4(1-p)}$	1	$\frac{2p-1}{4p}$
a^*	$\frac{1}{4(1-p)}$	0	$\frac{1}{4p}$
$[a, a^*]$	$\frac{1}{8(1-p)}$	0	$\frac{-1}{8p}$

trans

	e^*	h^*	f^*
e	$p\urcorner$	$2(1-p)$	$1-p$
h	p	$1-2p$	$1-p$
f	$\frac{p^2}{1-p}$	$2p$	$p\urcorner$

trans

$$
\begin{array}{c|ccc}
 & e & h & f \\
\hline
e^* & p-1 & 2(1-p) & 1-p \\
h^* & p & 1-2p & 1-p \\
f^* & \frac{p^2}{1-p} & 2p & p-1
\end{array}
$$

LEM 133 For each pair of bases among (I) – (III) the matrix rep $\langle , \rangle$ is given below

$\langle , \rangle$	e	h	f
e	0	0	$\frac{1}{2}$
h	0	1	0
f	$\frac{1}{2}$	0	0

$\langle , \rangle$	a	a^*	$[a, a^*]$
a	1	$1-2p$	0
a^*	$1-2p$	1	0
$[a, a^*]$	0	0	$-16p(1-p)$

$\langle , \rangle$	e^*	h^*	f^*
e^*	0	0	$\frac{1}{2}$
h^*	0	1	0
f^*	$\frac{1}{2}$	0	0

$\langle , \rangle$	e^*	h^*	f^*
e	$\frac{p^2}{2(1-p)}$	p	$\frac{p-1}{2}$
h	p	$1-2p$	$1-p$
f	$\frac{p-1}{2}$	$1-p$	$\frac{1-p}{2}$

$\langle \cdot, \cdot \rangle$	a	a^*	$[a, a^*]$
e	p	0	$2p$
h	$1-2p$	1	0
f	$1-p$	0	$2(p-1)$

$\langle \cdot, \cdot \rangle$	a	a^*	$[a, a^*]$
e^*	0	p	$-2p$
h^*	1	$1-2p$	0
f^*	0	$1-p$	$2(1-p)$

LEM 134 Relative each basis (I)–(III) the matrices representing $\operatorname{ad} a$, $\operatorname{ad} a^*$ are given below

Relative the basis $a, a^*, [a, a^*]$:

$$\operatorname{ad} a: \begin{pmatrix} 0 & 0 & 4(2p-1) \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\operatorname{ad} a^*: \begin{pmatrix} 0 & 0 & -4 \\ 0 & 0 & 4(1-2p) \\ -1 & 0 & 0 \end{pmatrix}$$

Relative the basis $e \quad h \quad f$

$$\mathrm{ad}\,a : \begin{pmatrix} 2(1-2p) & 4(p-1) & 0 \\ -2p & 0 & 2(1-p) \\ 0 & 4p & 2(2p-1) \end{pmatrix}$$

$$\mathrm{ad}\,a^{*} : \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Relative the basis e^*, h^*, f^*

$$\text{ad } a: \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\text{ad } a^*: \quad \begin{pmatrix} 2(1-2p) & 4(p-1) & 0 \\ -2p & 0 & 2(1-p) \\ 0 & 4p & 2(2p-1) \end{pmatrix}$$

By an <u>antiautomorphism</u> of L we mean an iso of $\mathbb{F}$-vector spaces $\sigma : L \to L$ such that

$$[y, z]^\sigma = [z^\sigma, y^\sigma] \qquad \forall y, z \in L$$

EX the map

$$\begin{aligned} L &\longrightarrow L \\ y &\to -y \end{aligned}$$

is an antiaut of L.

LEM 135 $\exists$ unique antiaut of L that fixes each of a, a^*. Denoting this map by $\dagger$ we have

$$(y^\dagger)^\dagger = y \qquad \forall y \in L$$

pf Routine using LEM 126 and Thm 127 $\square$

LEM 136 We have

$$y^\dagger = W y^t W^{-1} \qquad \forall y \in L$$

(transpose)

where

$$W = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}.$$

pf Obs the map

$$L \longrightarrow L$$

$$y \longrightarrow W y^t W^{-1}$$

is an antiaut of L.

One checks

$$a = W a^t W^{-1}, \qquad a^* = W a^{*t} W^{-1}$$

where

$$a = \begin{pmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{pmatrix} \qquad a^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Result follows. □

LEM 137 The maps $*$, $\dagger$ commute

pf For $y = a$ and $y = a^*$ we have

$$(y^*)^\dagger = (y^\dagger)^* \qquad \square$$

LEM 138 $\forall y, z \in L$

$$\langle y^t, z \rangle = \langle y, z^t \rangle$$

pf check

$$\mathrm{tr}(W y^t W^{-1} z) \overset{?}{=} \mathrm{tr}(y W z^t W^{-1})$$

|| trans || $\mathrm{tr}(rs) = \mathrm{tr}(sr)$

$$\mathrm{tr}(z^t W^{-1} y W) \quad \checkmark \quad \mathrm{tr}(z^t W^{-1} y W)$$

□

LEM139 We have

y	e	h	f
y^t	$\frac{\rho}{1-\rho} f$	h	$\frac{1-\rho}{\rho} e$

y	e^*	h^*	f^*
y^t	$\frac{\rho}{1-\rho} f^*$	h^*	$\frac{1-\rho}{\rho} e^*$

pf

$$e^t = w e^t w^-$$

$$= \begin{pmatrix} 1-\rho & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^t \begin{pmatrix} \frac{1}{1-\rho} & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{\rho}{1-\rho} & 0 \end{pmatrix}$$

$$= \frac{\rho}{1-\rho} f$$

Other cases similar. □

□