General orthog polynomials

We assume $\mathbb{F}=\mathbb{R}$

We need a fact about pos def matrices.

Fix integer $n>0$

Let $M=$ real symmetric $n\times n$ matrix

Recall the eigenvalues of M are in $\mathbb{R}$

M called <u>pos def</u> whenever these eigvals >0

In this case $\det M>0$

Let V denote a vector space over $\mathbb{R}$ with dim n

Fix a basis $\{v_i\}_{i=1}^n$ for V

Define a bilinear form $\langle\, ,\rangle : V\times V\to\mathbb{R}$ s.t

$$\langle v_i, v_j\rangle = M_{ij} \qquad \text{for } 1\le i,j\le n$$

$\langle\, ,\rangle$ sym.

Recall $\langle\, ,\rangle$ is pos def whenever

$$\langle u,u\rangle > 0 \qquad \forall\, 0\ne u\in V$$

TFAE (ex)

(i) M is pos def

(ii) $\langle\, ,\rangle$ is pos def

(iii) $\exists$ $N\in \mathrm{Mat}_n(\mathbb{R})$ s.t. $M=N^tN$

(iv) $\exists$ basis $\{u_i\}_{i=1}^n$ for V s.t $\langle u_i,u_j\rangle=\delta_{ij}$ for $1\le i,j\le n$. "orthonormal basis" wrt $\langle\, ,\rangle$

thm 34 With above notation TFAE

(i) M is pos def

(ii) For $1 \le t \le n$

$$\det\left((M_{ij})_{1 \le i,j \le t} \right) > 0$$

"principal minor"

pf (i) → (ii)

$\langle , \rangle$ is pos def on V

So for $1 \le t \le n$ the restriction of $\langle , \rangle$ to $\mathrm{Span}(v_i)_{i=1}^{t}$

is pos def.

So $(M_{ij})_{1 \le i,j \le t}$ is pos def

So $\det((M_{ij})_{1 \le i,j \le t}) > 0$

(ii) → (i) Ind on n

$n=1$ clear

$n \ge 2$: By ind

$(M_{ij})_{1 \le i,j \le n-1}$ is pos def

Define $W = \mathrm{Span}(v_i)_{i=1}^{n-1}$

Restr of $\langle , \rangle$ to W is pos def

So $\exists$ basis $\{u_i\}_{i=1}^{n-1}$ for W s.t.

$\langle u_i, u_j \rangle = \delta_{ij}$ $1 \le i,j \le n-1$

$\det M \neq 0$ so $\langle , \rangle$ nondeg on V

Define $W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}$

$\dim W^{\perp} = 1$ since $\langle , \rangle$ nondeg

$W \cap W^{\perp} = 0$ since $\langle , \rangle$ is pos def on W

Now $V = W + W^{\perp}$ (orthog dir sum)

Pick $0 \neq v \in W^{\perp}$

Put $a = \langle v, v \rangle$

show $a > 0$

Obs $u_1, u_2, \ldots, u_{n-1}, v$ is basis for V

Matrix of inner products is

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & a \end{array} \right)$$

Let S denote transition matrix from orig basis $\{v_i\}_{i=1}^{n}$ to above basis. Then

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & a \end{array} \right) = S^t M S$$

Apply det

$$a = \underbrace{(\det S)^2}_{\vee\, 0} \underbrace{\det M}_{\vee\, 0}$$

> 0

$\exists\ b \in \mathbb{R} \quad b^2 = a$

Define

$$u_n = \frac{v}{b}$$

So

$$\langle u_n, u_n \rangle = \frac{\langle v, v \rangle}{a} = 1$$

Now $\{u_i\}_{i=1}^{n}$ is an orthonormal basis for V rel $\langle , \rangle$

So M is pos def. $\square$

For general orthogonal polynomials there are 3 natural starting points:

(i) 3-term recurrence

(ii) the bilinear form

(iii) moments

(i) $\to$ (ii) Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2, \ldots$$

$$p_0 = 1 \qquad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \qquad n = 1, 2, \ldots$$

Obs for $n = 0, 1, 2, \ldots$

p_n has degree n

coeff of x^n is $\dfrac{1}{b_0 b_1 \cdots b_{n-1}}$

Obs

$$\{p_n\}_{n=0}^{\infty} \quad \text{is a basis for } \mathbb{R}[x]$$

Fix $u_0 \in \mathbb{R}$ $\quad u_0 > 0$

Define a bilinear form

$$\langle\, , \rangle : \quad \mathbb{R}[x] \times \mathbb{R}[x] \longrightarrow \mathbb{R}$$

such that

$$\langle p_n, p_m \rangle = \delta_{nm}\, u_0 \frac{c_1 c_2 \cdots c_n}{b_0 b_1 \cdots b_{n-1}} \qquad 0 \le n, m < \infty$$

By constr $\langle\, , \rangle$ is symmetric and positive def.

Basis $\{p_n\}_{n=0}^{\infty}$ is orthogonal rel $\langle\, , \rangle$ and

$$\|p_n\|^2 = \frac{c_n}{b_{n-1}} \|p_{n-1}\|^2 \qquad n = 1, 2, \ldots$$

$$\|p_0\|^2 = u_0$$

LEM 35 The above form $\langle\, , \rangle$ is A-invariant.

In other words

$$\langle xf, g \rangle = \langle f, xg \rangle \qquad \forall f, g \in \mathbb{R}[x]$$

pf WLOG

$$f = p_n \qquad g = p_m$$

$$\langle x p_n, p_m \rangle = \langle c_n p_{n-1} + a_n p_n + b_n p_{n+1}, p_m \rangle \qquad *$$

? ((

$$\langle p_n, x p_m \rangle = \langle p_n, c_m p_{m-1} + a_m p_m + b_m p_{m+1} \rangle \qquad **$$

Case $|n-m| \geq 2$: $*$, $**$ both 0

Case $n=m$: $* = **$ ✓

Case $|n-m| = 1$: WLOG $m = n-1$

$$* = c_n \|p_{n-1}\|^2$$

$$** = b_{n-1} \|p_n\|^2$$

So $* = **$ $\square$

(ii) → (iii)

Given symmetric, pos definite, A-invariant bilinear form

$$\langle\, ,\, \rangle : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$$

For $n = 0, 1, 2, \ldots$ define

$$u_n = \langle x^n, 1 \rangle$$

Since $\langle\, ,\, \rangle$ is A-inv

$$\langle x^i, x^j \rangle = u_{i+j} \qquad 0 \le i, j < \infty$$

So for $n = 0, 1, 2 \ldots$ the Hankel matrix

$$\begin{pmatrix} u_0 & u_1 & u_2 & \cdots & u_n \\ u_1 & u_2 & & & \vdots \\ u_2 & & & & \\ \vdots & & & & \\ u_n & & \cdots & & u_{2n} \end{pmatrix} \qquad \ast$$

is pos def.

Define D_n to be the det of $\ast$. Then

$$D_n > 0 \qquad n = 0, 1, 2 \ldots$$

(iii) $\to$ (ii)

Given a sequence of real numbers $\{u_n\}_{n=0}^{\infty}$

For $n=0,1,2..$ we define the nth Hankel matrix as in $*$ and let D_n denote its det.

Assume $$D_n > 0 \qquad n=0,1,2..$$

Define an $\mathbb{R}$-linear map

$$\phi : \mathbb{R}[x] \to \mathbb{R}$$

s.t

$$\phi(x^n) = u_n \qquad \text{for } n=0,1,2..$$

Define

$$\begin{array}{cccccc} \langle , \rangle : & \mathbb{R}[x] & \times & \mathbb{R}[x] & \to & \mathbb{R} \\ & f & & g & \to & \phi(fg) \end{array}$$

Then $\langle , \rangle$ is a bilinear form which is symmetric and A-inv

LEM 36 Above bil form $\langle , \rangle$ is pos def.

pf Given $0 \neq f \in \mathbb{R}[x]$ show $\|f\|^2 > 0$

Write

$$f = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$$\|f\|^2 = (\alpha_0, \ldots, \alpha_n) \begin{pmatrix} & & \\ & \text{Hankel}_n & \\ & & \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix}$$

show Hankel_n is pos def

Its principle minors are

$$D_0, D_1, \ldots, D_n$$

and hence all positive.

Now Hankel_n is pos def by thm 34 $\square$

(ii) → (i) Given a symmetric, pos def, A-inv bilinear form

$$\langle\, , \rangle : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$$

Pick any sequence of nonzero real numbers $\{b_n\}_{n=0}^{\infty}$, $(b_0 = 1)$

Define

$$x_n = \frac{x}{b_n} \qquad n = 0, 1, 2 \ldots$$

Apply Gram-Schmidt to the basis

$$\{x_0 x_1 x_2 \cdots x_{n-1}\}_{n=0}^{\infty}$$

to get a sequence of polys $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$

By construction, for $n = 0, 1, 2, \ldots$

p_n has degree exactly n

coeff of x^n in p_n is $\dfrac{1}{b_0 b_1 \cdots b_{n-1}}$

Also

$$\langle p_n, p_m \rangle = 0 \quad \text{if } n \neq m \qquad 0 \leq n, m < \infty$$

$$\|p_n\|^2 > 0 \qquad n = 0, 1, 2 \ldots$$

LEM 37 The above polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a 3-term recurrence

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0,1,2,\dots \qquad (*)$$

$$(p_{-1} = 0)$$

where

$$c_n b_{n-1} > 0 \qquad n = 1,2,\dots \qquad (**)$$

pf Write $x p_n$ as lin combin of $p_0, p_1, p_2 \dots$

$$x p_n = \sum_{i=0}^{n+1} \alpha_i p_i$$

In LHS coef of x^{n+1} is $\dfrac{1}{b_0 b_1 \cdots b_{n-1}}$

" RHS " " is $\dfrac{\alpha_{n+1}}{b_0 b_1 \cdots b_n}$

So $\alpha_{n+1} = b_n$

Show $\alpha_i = 0 \quad (0 \le i < n-1)$

use $i < n-1$

$$\underbrace{\langle p_i, x p_n \rangle}_{\alpha_i \underbrace{\|p_i\|^2}_{>0}} = \underbrace{\langle \overbrace{x p_i}^{\in \mathrm{Span}(p_0, p_1, \dots, p_{i+1})}, p_n \rangle}_{0}$$

So $\alpha_i = 0$

We now have $(*)$.

Show $(**)$:

$$\langle x p_{n-1}, p_n \rangle = \langle p_{n-1}, x p_n \rangle$$

Expand each side using $(*)$. Get

$$\underbrace{b_{n-1}}_{\neq 0} \underbrace{\|p_n\|^2}_{>0} = c_n \underbrace{\|p_{n-1}\|^2}_{>0}$$

c_n, b_{n-1} have same sign

$\square$

Lecture 6 Friday Sept 17

Continue to discuss general orthog polynomials

thm 38 Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ that satisfy a 3-term recurrence

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2 \ldots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \qquad n = 1, 2, \ldots$$

Let $\{u_n\}_{n=0}^{\infty}$ and $\{D_n\}_{n=0}^{\infty}$ denote the corresp moments and Hankel determinants. Then

(i) $$p_n = \frac{1}{b_0 b_1 \cdots b_{n-1}} \frac{1}{D_{n-1}} \det \begin{pmatrix} u_0 & u_1 & \cdots & u_n \\ u_1 & & \cdots & \vdots \\ \vdots & & & \\ u_{n-1} & & \cdots & u_{2n-1} \\ 1 & x & x^2 \cdots & x^n \end{pmatrix}$$

$n = 1, 2 \ldots$

(ii) $$D_n / D_{n-1} = u_0 \, c_1 c_2 \cdots c_n \, b_0 b_1 \cdots b_{n-1} \qquad n = 1, 2, \ldots$$

(iii) $$D_n = u_0^{n+1} (c_1 b_0)(c_2 b_1) \cdots (c_{n-1} b_{n-2})^2 (c_n b_{n-1}) \qquad n = 0, 1, 2 \ldots$$

pf (i) Similar to pf of thm 33

(ii) Assume $n \geq 1$ else triv.

By (i)

$$\langle p_n, x^n \rangle = \frac{1}{b_0 b_1 \cdots b_{n-1}} \frac{D_n}{D_{n-1}}$$

By our earlier discussion

$$u_0 \frac{c_1 c_2 \cdots c_n}{b_0 b_1 \cdots b_{n-1}} = \|p_n\|^2$$

$$= \langle p_n, p_n \rangle$$

$$= \left\langle p_n, \frac{x^n}{b_0 b_1 \cdots b_{n-1}} + L \text{ terms} \right\rangle$$

$$= \frac{\langle p_n, x^n \rangle}{b_0 b_1 \cdots b_{n-1}}$$

Therefore

$$\frac{D_n}{D_{n-1}} = u_0 \, c_1 c_2 \cdots c_n \, b_0 b_1 \cdots b_{n-1}$$

(iii) By (ii) $\square$

More on pos def matrices

Fix integer $n \geq 1$

Fix $M \in Mat_n(\mathbb{R})$

Consider condition

(SDD) For $1 \leq i \leq n$

$$M_{ii} > \sum_{\substack{1 \leq j \leq n \\ i \neq j}} |M_{ij}|$$ "strict diagonal dominance"

LEM 39 With above notation assume M is SDD

Let $\theta \in \mathbb{C}$ denote an eigenval of M (ie root of char polynomial of M).

Then the real part

$$Re(\theta) > 0$$

[$\theta = a + bi$ $a, b \in \mathbb{R}$ $i^2 = -1$

$Re(\theta) = a$, $Im(\theta) = b$]

pf Pick $0 \neq v \in \mathbb{C}^n$ s.t $Mv = \theta v$

Write $v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

Pick i ($1 \leq i \leq n$) s.t

$$|\alpha_i| \geq |\alpha_j| \quad (1 \leq j \leq n)$$

Obs $\quad |\alpha_i| > 0 \quad$ since $\quad v \neq 0$

In $\quad Mv = \theta v \quad$ consider row i

$$\sum_{j=1}^{n} M_{ij} \alpha_j = \theta \alpha_i$$

Rewrite as

$$(\theta - M_{ii}) \alpha_i = \sum_{\substack{1 \le j \le n \\ i \neq j}} M_{ij} \alpha_j$$

Now

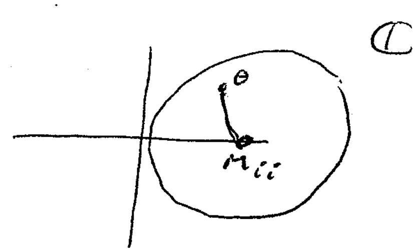
$$
\begin{aligned}
|\theta - M_{ii}|\,|\alpha_i| &= |(\theta - M_{ii}) \alpha_i| \\
&= \Big| \sum_{\substack{1 \le j \le n \\ i \neq j}} M_{ij} \alpha_j \Big| \\
&\le \sum_{\substack{1 \le j \le n \\ i \neq j}} |M_{ij}|\,|\alpha_j| \\
&\le \sum_{\substack{1 \le j \le n \\ i \neq j}} |M_{ij}|\,|\alpha_i| \\
&< M_{ii}\,|\alpha_i|
\end{aligned}
$$

So

$$|\theta - \quad| < M_{ii}$$

So

$$\mathrm{Re}(\theta) > 0$$



$\square$

thm 40 Given $M \in Mat_n(\mathbb{R})$

Assume M is symmetric and SDD

Then M is pos def.

pf For each eigvalue θ of M

$\theta \in \mathbb{R}$ since M is symmetric

so $\theta = Re(\theta) > 0$ by LEM 39.

Now M is pos def. $\square$

[wont use thm 40 right away]

Next goal: Christoffel - Darboux identity

Given polynomial sequence $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2 ..$$

$$p_0 = 1, \qquad p_{-1} = 0$$

$$c_n b_{n-1} \neq 0 \qquad n = 1, 2 ..$$

$\mathbb{F}$ arb

Write
$$z_n = \frac{c_1 c_2 \cdots c_n}{b_0 b_1 \cdots b_{n-1}} \qquad n = 0, 1, 2 \ldots$$

For $n = 0, 1, 2 ..$ find

$$\sum_{i=0}^{n} \frac{p_i(x)\, p_i(y)}{z_i} \qquad \left[x, y \text{ commuting indets}\right]$$

View
$$\sum_{i=0}^{n} \frac{p_i(x)\, p_i(y)}{z_i} = \frac{1}{x-y} \sum_{i=0}^{n} \frac{(x-y)\, p_i(x)\, p_i(y)}{z_i}$$

$$= \frac{1}{x-y} \sum_{i=0}^{n} \left(\left(c_i p_{i-1}(x) + a_i p_i(x) + b_i p_{i+1}(x) \right) \frac{p_i(y)}{z_i} - \frac{p_i(x)}{z_i} \left(c_i p_{i-1}(y) + a_i p_i(y) + b_i p_{i+1}(y) \right) \right)$$

$$= \frac{1}{x-y}\Bigg(\sum_{i=1}^{n} p_{i-1}(x)\,p_i(y) \Big(\underbrace{\frac{c_i}{z_i} - \frac{b_{i-1}}{z_{i-1}}}_{u_0} \Big)$$

$$+ \sum_{i=1}^{n} p_i(x)\,p_{i-1}(y) \Big(\underbrace{\frac{b_{i-1}}{z_{i-1}} - \frac{c_i}{z_i}}_{u_0} \Big)$$

$$- \; p_n(x)\,p_{n+1}(y)\,\frac{b_n}{z_n}$$

$$+ \; p_{n+1}(x)\,p_n(y)\,\frac{b_n}{z_n} \Bigg)$$

$$= \frac{p_{n+1}(x)\,p_n(y) - p_n(x)\,p_{n+1}(y)}{x-y} \quad \frac{b_n}{z_n}$$

$$\left[\begin{array}{l} x p_0 = a_0 p_0 + b_0 p_1 \qquad p_0 = 1 \\ p_1 = \dfrac{x-a_0}{b_0} \\ p_1(x) - p_1(y) = \dfrac{x-y}{b_0} \\ \\ \dfrac{b_n}{(x-y) z_n} = \dfrac{1}{p_1(x)-p_1(y)} \dfrac{b_n}{b_0} \dfrac{1}{z_n} \\ \qquad = \dfrac{1}{p_1(x)-p_1(y)} \quad \dfrac{b_1 b_2 \cdots b_n}{c_1 c_2 \cdots c_n} \end{array}\right]$$

$$= \frac{p_{n+1}(x)\,p_n(y) - p_n(x)\,p_{n+1}(y)}{p_1(x) - p_1(y)} \quad \frac{b_1 b_2 \cdots b_n}{c_1 c_2 \cdots c_n}$$

thm 41 (Christoffel- Darboux)

with above notation

For $n = 0, 1, 2 \ldots$

(i)
$$\sum_{i=0}^{n} p_i(x)\, p_i(y) \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$$
$$= \frac{p_{n+1}(x)\, p_n(y) - p_n(x)\, p_{n+1}(y)}{p_1(x) - p_1(y)} \quad \frac{b_1 b_2 \cdots b_n}{c_1 c_2 \cdots c_n}$$

(ii)
$$\sum_{i=0}^{n} p_i(x)^2 \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$$
$$= \left(p_{n+1}'(x)\, p_n(x) - p_n'(x)\, p_{n+1}(x) \right) \frac{b_0 b_1 \cdots b_n}{c_1 c_2 \cdots c_n}$$

$(f' = Df)$

pf (i) By disc above thm.

(ii) In (i) write $y = x + h$, simplify / cancel and then set $h = 0$

□

LEM 42 Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$

Assume

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2 \ldots$$

$$c_n b_{n-1} \neq 0 \qquad n = 1, 2 \ldots$$

Given arbitrary

$$0 \neq \gamma_n \in \mathbb{F} \qquad n = 0, 1, 2 \ldots$$

define

$$\tilde{p}_n = \frac{p_n}{\gamma_0 \gamma_1 \cdots \gamma_{n-1}} \qquad n = 0, 1, 2 \ldots$$

Then

$$x \tilde{p}_n = \tilde{c}_n \tilde{p}_{n-1} + \tilde{a}_n \tilde{p}_n + \tilde{b}_n \tilde{p}_{n+1} \qquad n = 0, 1, 2 \ldots$$

where

$$\tilde{c}_n = c_n \gamma_{n-1} \qquad n = 1, 2 \ldots$$

$$\tilde{a}_n = a_n \qquad n = 0, 1 \ldots$$

$$\tilde{b}_n = \frac{b_n}{\gamma_n} \qquad n = 0, 1, 2 \ldots$$

Moreover

$$\tilde{c}_n \tilde{b}_{n-1} = c_n b_{n-1} \qquad n = 1, 2 \ldots$$

pf ex

LEM 43 Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$

" " " $\{\tilde{p}_n\}_{n=0}^{\infty}$ "

Assume

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2, \ldots$$

$$x \tilde{p}_n = \tilde{c}_n \tilde{p}_{n-1} + \tilde{a}_n \tilde{p}_n + \tilde{b}_n \tilde{p}_{n+1} \qquad "$$

$$c_n b_{n-1} \neq 0, \qquad \tilde{c}_n \tilde{b}_{n-1} \neq 0 \qquad n = 1, 2, \ldots$$

TFAE

(i) $\tilde{a}_n = a_n$ for $n = 0, 1, 2, \ldots$ and

$\tilde{c}_n \tilde{b}_{n-1} = c_n b_{n-1}$ for $n = 1, 2, \ldots$

(ii) $\tilde{p}_n$ is a nonzero scalar multiple of p_n $\quad n = 0, 1, 2 \ldots$

pf ex

$\square$

Continue to discuss general orthog polynomials

$\mathbb{F} = \mathbb{R}$

Next goal: interlacing of zeros

Given $0 \neq f \in \mathbb{R}[x]$

factor f over $\mathbb{C}$

$$f = \alpha \prod_{i=1}^{n} (x - x_i) \qquad \alpha \in \mathbb{R} \qquad x_i \in \mathbb{C}$$

A root x_i is <u>simple</u> whenever

$$x_i \neq x_j \qquad (1 \leq j \leq n, \ j \neq i)$$

x_i is simple iff

$$f'(x_i) \neq 0 \qquad f' = \partial f$$

DEF 44 Given NONO $f, g \in \mathbb{R}[x]$

Assume $\deg f$, $\deg g$ differ by 1

WLOG $\deg f = \deg g + 1$

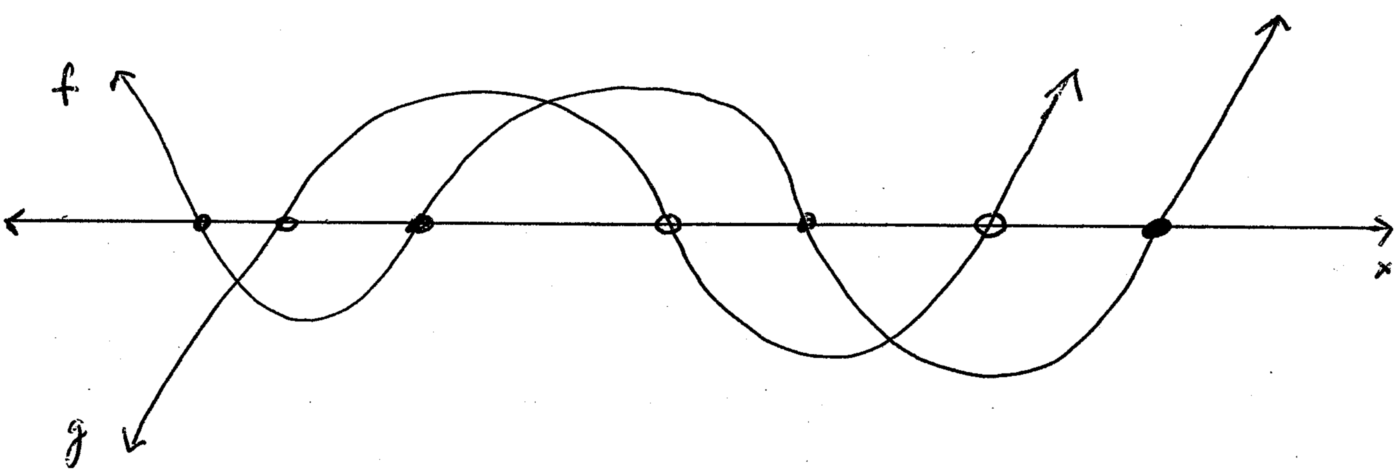
We say <u>the roots of f, g interlace</u> whenever

(i) For f and g all roots are simple and real

(ii) For $1 \le i \le \deg g$

$(i+1)^{st}$ Largest root of f $<$ i^{th} Largest root of g $<$ i^{th} Largest root of f

ex



Recall from Calculus

Given $f \in \mathbb{R}[x]$

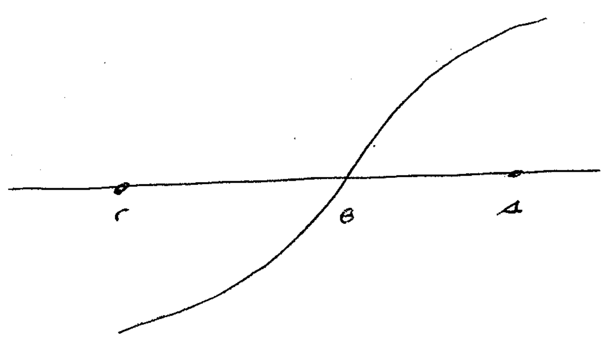
Given $r, s \in \mathbb{R}$ $(r < s)$

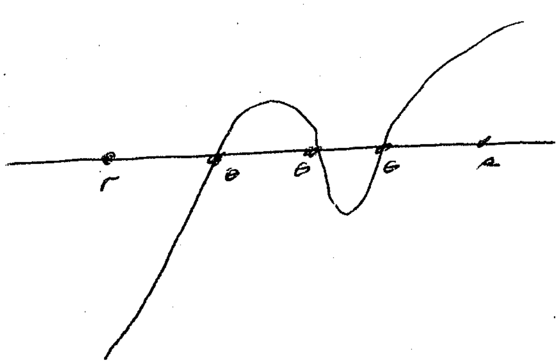
Suppose

$$f(r)f(s) < 0$$

then $\exists\ \theta \in \mathbb{R}$ such that

$$r < \theta < s \quad \text{and} \quad f(\theta) = 0$$





etc.

Given $0 \neq f \in \mathbb{R}[x]$ $\quad n = \deg f$

Assume all roots of f are simple and real

Define $x_i = i$th largest root of f $\quad (1 \le i \le n)$

LEM 45 Given f, $\{x_i\}_{i=1}^n$ as above

Given monic $g \in \mathbb{R}[x]$ with $\deg g = n \mp 1$

(i) Suppose $\deg g = n-1$. Then the roots of f, g interlace iff

$$(-1)^i g(x_i) < 0 \qquad 1 \le i \le n$$

(ii) Suppose $\deg g = n+1$. Then the roots of f, g interlace iff

$$(-1)^i g(x_i) > 0 \qquad 1 \le i \le n$$

pf ex

thm 46 Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$

such that

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2, \ldots \qquad (*)$$

$$p_0 = 1, \qquad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \qquad n = 1, 2, \ldots$$

Then the roots of p_n, p_{n+1} interlace for $n = 0, 1, 2, \ldots$

pf WLOG the p_n are monic. So

$$b_n = 1 \qquad n = 0, 1, 2 \ldots$$

$$c_n > 0 \qquad n = 1, 2, \ldots$$

Ind on n

$n = 0$ ✓

$n \geq 1$: By ind its p_{n-1}, p_n interlace

So roots of p_n are simple and real

define $x_i = i$th largest root of p_n $\qquad 1 \leq i \leq n$

By LEM 45 (ii) suffices to show

$$(-1)^i p_{n+1}(x_i) > 0 \qquad 1 \leq i \leq n$$

Let i be given. By LEM 45 (i)

$$(-1)^i p_{n-1}(x_i) < 0$$

Apply each side of $(*)$ to x_i and use $p_n(x_i) = 0$:

$$0 = \underbrace{c_n}_{>0} p_{n-1}(x_i) + p_{n+1}(x_i)$$

have opp sign. So $(-1)^i p_{n+1}(x_i) > 0$

Until further notice $\mathbb{F}$ arb

Notation For $d = 0, 1, 2 ..$

$\mathrm{Mat}_{d+1}(\mathbb{F})$ denotes the $\mathbb{F}$-algebra of all

$(d+1) \times (d+1)$ matrices with entries in $\mathbb{F}$

We index the rows/cols by $0, 1, 2, \ldots, d$.

$\mathbb{F}^{d+1}$ denotes the $\mathbb{F}$-vector space of all

$(d+1) \times 1$ matrices with entries in $\mathbb{F}$.

Index rows by $0, 1, \ldots d$.

obs $\mathrm{Mat}_{d+1}(\mathbb{F})$ acts on $\mathbb{F}^{d+1}$ by left mult.

Given tridiagonal matrix $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$

$$M = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ 0 & & & c_d & a_d \end{pmatrix}$$

M called <u>irreducible</u> whenever

$$c_n b_{n-1} \neq 0 \qquad 1 \le n \le d$$

LEM 47 Given $d = 0, 1, 2 \ldots$

Given irred tridiag $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$

(i) $I, M, M^2, \ldots, M^d$ are lin indep.

(ii) min poly of M = char poly of M

pf (i) For $0 \le n \le d$ row 0 of M^n has form

$$\begin{array}{ccccccccc} & & & & \neq 0 & & & & \\ (& *, & *, & \ldots, & \overset{\downarrow}{*}, & 0, & 0, & \ldots & 0) \\ & 0, & 1 & & n & & & & \end{array}$$

(ii) By (i) $\square$

Next goal: Another determinant formula for $\{p_n\}_{n=0}^{\infty}$

Given polynomial sequence $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2, \ldots$$

$$p_0 = 1, \qquad p_{-1} = 0$$

$$c_n b_{n-1} \neq 0 \qquad 1 \le n < \infty$$

Recall lin trans

$$A : \mathbb{F}[x] \to \mathbb{F}[x]$$
$$f \mapsto xf$$

Rel basis $\{p_n\}_{n=0}^{\infty}$ matrix rep A is

$$A : \begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & 0 \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \ddots \end{pmatrix}^t \qquad (*)$$

For $n = 0, 1, 2, \ldots$ let $T_n \in \mathrm{Mat}_{n+1}(\mathbb{F})$ denote the submatrix of $*$ corresp to rows/cols $0, 1, 2, \ldots, n$

Thm 48 For $n = 0, 1, 2, \ldots$

$$p_{n+1} = \frac{\det(xI - T_n)}{b_0 b_1 \cdots b_n} \qquad *$$

where

$$T_n = \begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & 0 \\ & c_2 & a_2 & \ddots & \\ & 0 & \ddots & \ddots & b_{n-1} \\ & & & c_n & a_n \end{pmatrix}$$

pf let $\tilde{p}_{n+1}$ = RHS of $*$

show $p_{n+1} = \tilde{p}_{n+1}$

show

$$x\tilde{p}_n = c_n \tilde{p}_{n-1} + a_n \tilde{p}_n + b_n \tilde{p}_{n+1} \qquad (**)$$

Assume $n \geq 2$ else trivial

Observe

$$xI - T_n = \left(\begin{array}{c|c|c} xI - T_{n-2} & \begin{matrix} 0 \\ \vdots \\ -b_{n-2} \end{matrix} & 0 \\ \hline 0 \quad\quad -c_{n-1} & x - a_{n-1} & -b_{n-1} \\ \hline 0 & -c_n & x - a_n \end{array}\right)$$

Find $\det(xI - T_n)$ by expanding along last row

$$\det(xI - T_n) = (x - a_n)\det(xI - T_{n-1}) -$$

$$(-c_n)\,\underbrace{\det\left(\begin{array}{c|c} xI - T_{n-2} & 0 \\ \hline 0 \quad\quad -c_{n-1} & -b_{n-1} \end{array}\right)}_{-b_{n-1}\det(xI - T_{n-2})}$$

So

$$\frac{\det(xI - T_n)}{b_0 b_1 \cdots b_{n-1}} = (x-a_n)\frac{\det(xI - T_{n-1})}{b_0 b_1 \cdots b_{n-1}} - c_n b_{n-1}\frac{\det(xI - T_{n-2})}{b_0 b_1 \cdots b_{n-1}}$$

So

$$b_n \tilde{p}_{n+1} = (x-a_n)\,\tilde{p}_n - c_n \tilde{p}_{n-1}$$

Gives (**)

Now $p_{n+1} = \tilde{p}_{n+1}$ by routine induction on n. $\blacksquare$

COR 49 With above notation, for $n=0,1,2,\ldots$

the zeros of p_{n+1} are the eigenvalues of T_n.

pf By thm 48 p_{n+1} is scalar multiple of the char poly of T_n $\square$

Ref to thm 48 the matrix T_n not symmetric

However we do have the following.

For $n = 0, 1, 2 \ldots$ define

$$k_n = \frac{b_0 b_1 \cdots b_{n-1}}{c_1 c_2 \cdots c_n}$$

So

$$k_0 = 1$$

$$k_n c_n = k_{n-1} b_{n-1} \qquad 1 \le n < \infty$$

Define matrix

$$K_n = \operatorname{diag}(k_0, k_1, \ldots, k_n) \qquad n = 0, 1, 2 \ldots$$

LEM 50 With above notation. for $n = 0, 1, 2 \ldots$

$K_n T_n$ is symmetric.

In other words

$$T_n^t = K_n T_n K_n^{-1}$$

pf $K_n T_n$ is tridiagonal

$$(K_n T_n)_{i,i-1} \overset{?}{=} (K_n T_n)_{i-1,i}$$

$$\quad \text{"} \qquad\qquad\qquad \text{"}$$

$$k_i c_i \qquad\qquad k_{i-1} b_{i-1}$$

DEF 51 A matrix $M \in \mathrm{Mat}_n(\mathbb{F})$ is symmetrizable whenever $\exists$ diagonal matrix $\Delta \in \mathrm{Mat}_n(\mathbb{F})$ s.t.

$$\Delta M \Delta^{-1} \text{ is symmetric.}$$

LEM 52 With above notation assume $\mathbb{F} = \mathbb{R}$ and

$$c_n b_{n-1} > 0 \qquad n = 0, 1, 2 \ldots$$

Then T_n is symmetrizable for $n = 0, 1, 2 \ldots$

pf Obs $k_n > 0$ so $\sqrt{k_n}$ exists in $\mathbb{R}$

Define

$$\Delta_n = \mathrm{diag}(\sqrt{k_0}, \sqrt{k_1}, \ldots, \sqrt{k_n})$$

So $\Delta_n^2 = K_n$

Obs

$$\Delta_n T_n \Delta_n^{-1} = \Delta_n^{-1} \Delta_n^2 T_n \Delta_n^{-1}$$

$$= \underbrace{\Delta_n^{-1}}_{\text{sym}} \underbrace{K_n T_n}_{\text{sym}} \Delta_n^{-1}$$

$$= \text{sym}$$

$\square$

Note 53 With above notation assume

$\mathbb{F} = \mathbb{R}$, $\qquad c_n b_{n-1} > 0 \qquad n = 1, 2, \ldots$

We saw earlier the roots of p_{n+1} are real and simple.

Here is another proof:

By Th 48

p_{n+1} = nonzero scalar multiple of char poly of T_n

By Lem 47

char poly of T_n = min poly of T_n

By Lem 52

roots of min poly of T_n are in $\mathbb{R}$ and mutually distinct.

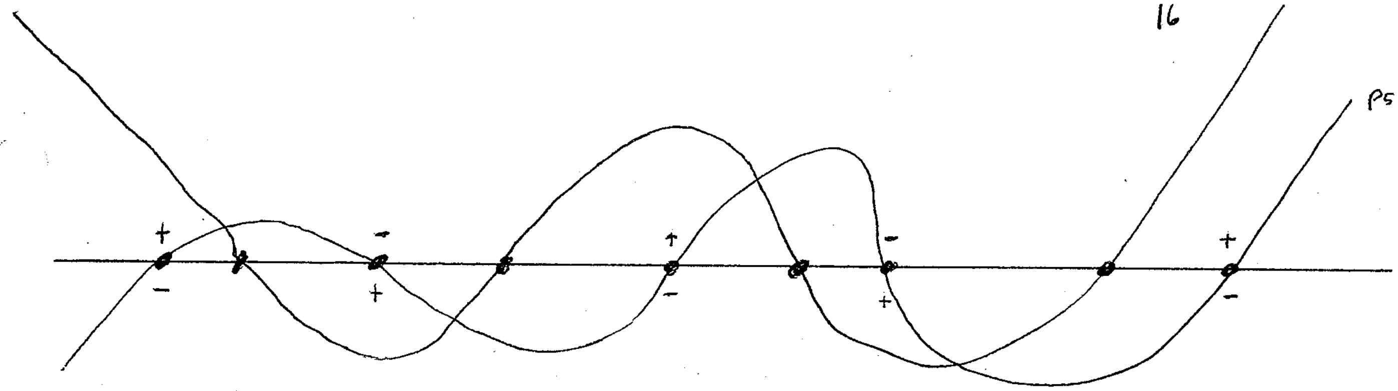
Therefore the zeros of p_{n+1} are real and simple. $\square$

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1}$$

$b_n = 1$

$c_n > 0$

n	p_n
0	1
1	$\frac{x - a_0}{b_0}$
2	$\frac{x^2 - x(a_0 + a_1) + a_0 a_1 - b_0 c_1}{b_0 b_1}$



$\square$

Lecture 8 Wednesday Sept 22

The interlacing argument in Note 53 is a special case of the following

Cauchy interlacing theorem

Fix integer $n \geq 1$

Fix $M \in Mat_n(\mathbb{C})$

Assume

conj-transpose $\overline{M}^t = M$ "Hermitean"

Recall eigvals of M are real.

List in order

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1$$

In M remove any row and corresp col to get a principle submatrix H

Obs H is Hermitean; List eigvals

$$\mu_{n-1} \leq \mu_{n-2} \leq \cdots \leq \mu_2 \leq \mu_1$$

Thm 54 (Cauchy) With above notation

$$\lambda_n \leq \mu_{n-1} \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \mu_1 \leq \lambda_1$$

pf WLOG

$$M = \left(\begin{array}{c|c} H & y \\ \hline \bar{y}^t & a \end{array}\right) \qquad a \in \mathbb{R}, \quad y \in \mathbb{C}^{n-1}$$

Since H is Hermitian,

$\exists \quad U \in Mat_{n-1}(\mathbb{C})$ s.t

$$\bar{U}^t U = I \qquad \text{"unitary"}$$

and

$$U^{-1} H U = \underbrace{diag(u_1, u_2, \ldots u_{n-1})}_{D}$$

Define

$$w = U^{-1} y \qquad w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \end{pmatrix}$$

Special case: Assume $u_{n-1} < u_{n-2} < \cdots < u_2 < u_1$

and $w_i \neq 0 \quad 1 \le i \le n-1$

Define

$$V = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & 1 \end{array}\right)$$

Obs

$$\bar{V}^t V = I$$

and

$$V^{-1} M V = \left(\begin{array}{c|c} U^{-1} & 0 \\ \hline 0 & 1 \end{array}\right) \underbrace{\left(\begin{array}{c|c} H & y \\ \hline \bar{y}^t & a \end{array}\right) \left(\begin{array}{c|c} U & 0 \\ \hline 0 & 1 \end{array}\right)}_{\left(\begin{array}{c|c} HU & y \\ \hline \bar{y}^t U & a \end{array}\right)}$$

$$\underbrace{}_{\left(\begin{array}{c|c} U^{-1} H U & U^{-1} y \\ \hline \bar{y}^t U & a \end{array}\right)}$$

$$= \left(\begin{array}{c|c} D & w \\ \hline \bar{w}^t & a \end{array}\right)$$

Let $f(x)$ = char poly of M

So
$$f(x) = \prod_{i=1}^{n} (x - \lambda_i)$$

Also
$$f(x) = \det(xI - M)$$

$$= \det(xI - V^{-1}MV)$$

$$= \det \left(\begin{array}{c|c} xI - D & -w \\ \hline -\bar{w}^t & x - a \end{array} \right)$$

[expand along last row]

$$= (x-a)\prod_{i=1}^{n-1}(x-u_i) - \sum_{i=1}^{n-1} |w_i|^2 (x-u_1)\cdots(x-u_{i-1})(x-u_{i+1})\cdots(x-u_{n-1})$$

So for $1 \le i \le n-1$

$$f(u_i) = -|w_i|^2 \prod_{\substack{1 \le j \le n-1 \\ j \ne i}} (u_i - u_j)$$

$$(-1)^i f(u_i) > 0 \qquad 1 \le i \le n-1$$

Now by LEM 45 (ii)

$$\lambda_n < u_{n-1} < \lambda_{n-1} < \cdots < \lambda_2 < \mu_1 < \lambda_1$$

done for special case

General case

We take limits

Define a sequence $\{\varepsilon_r\}_{r=1}^{\infty}$ s.t.

$$\varepsilon_r \in \mathbb{R} \setminus \{-w_1, -w_2, \dots, -w_{n-1}\}, \qquad 1 \le r < \infty$$

$$\varepsilon_r > 0 \qquad 1 \le r < \infty$$

$$\lim_{r \to \infty} \varepsilon_r = 0$$

For $1 \le i \le n-1$ and $1 \le r < \infty$ def

$$\mu_i^{(r)} = \mu_i - i\,\varepsilon_r$$

obs

$$\mu_{n-1}^{(r)} < \mu_{n-2}^{(r)} < \cdots < \mu_2^{(r)} < \mu_1^{(r)}$$

For $1 \le i \le n-1$ and $1 \le r < \infty$ def

$$w_i^{(r)} = w_i + \varepsilon_r$$

obs

$$w_i^{(r)} \neq 0$$

For $r = 1, 2, \ldots$

$$w^{(r)} = \begin{pmatrix} w_1^{(r)} \\ w_2^{(r)} \\ \vdots \\ w_{n-1}^{(r)} \end{pmatrix}$$

$$y^{(r)} = U w^{(r)}$$

$$D^{(r)} = \operatorname{diag}\left(u_1^{(r)}, u_2^{(r)}, \ldots, u_{n-1}^{(r)} \right)$$

$$H^{(r)} = U D^{(r)} U^{-1}$$

$$M^{(r)} = \left(\begin{array}{c|c} H^{(r)} & y^{(r)} \\ \hline \overline{y^{(r)}}^{t} & a \end{array} \right)$$

$M^{(r)}$ is Hermi; List eigvals

$$\lambda_n^{(r)} \le \lambda_{n-1}^{(r)} \le \cdots \le \lambda_2^{(r)} \le \lambda_1^{(r)}$$

Applying the special case to $M^{(r)}$, $H^{(r)}$ we get

$$\lambda_n^{(r)} < \mu_{n-1}^{(r)} < \lambda_{n-1}^{(r)} < \cdots < \lambda_2^{(r)} < \mu_1^{(r)} < \lambda_1^{(r)}$$

Now take Limits

$$\lim_{r\to\infty} \mu_i^{(r)} = \mu_i \qquad (1 \le i \le n-1)$$

$\lim_{r\to\infty} M^{(r)} = M$ so

$$\lim_{r\to\infty} \lambda_i^{(r)} \to \lambda_i \qquad (1 \le i \le n)$$

So $\lambda_n \le \mu_{n-1} \le \lambda_{n-1} \le \cdots \le \lambda_2 \le \mu_1 \le \lambda_1$ $\square$

Next goal: Kernel polynomials

Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ such that

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2 \ldots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \qquad n = 1, 2 \ldots$$

For $n = 0, 1, 2 \ldots$ define

$$K_n(x, y) = \sum_{i=0}^{n} \frac{p_i(x) p_i(y)}{z_i} \in \mathbb{R}[x, y]$$

$$z_i = \frac{c_1 c_2 \cdots c_i}{b_0 b_1 \cdots b_{i-1}}$$

"Kernel polynomials"

the kernel polynomials came up in Christoffel-Darboux

We now give another interp

Consider the bilin form $\langle , \rangle$ for $\{p_n\}_{n \ge 0}^{\infty}$

Normalize so $u_0 = \langle 1, 1 \rangle = 1$

Recall

$$\langle p_n, p_m \rangle = \delta_{nm} z_n \qquad 0 \le n, m < \infty$$

Problem Fix integer $n \geq 0$

Fix $\theta \in \mathbb{R}$

Maximize

$$f(\theta)$$

subject to

$$f \in \mathbb{R}[x],$$

$$\deg f \leq n,$$

$$\langle f, f \rangle = 1.$$

Sol: Write

$$f = \sum_{i=0}^{n} \alpha_i p_i \qquad \alpha_i \in \mathbb{R}$$

$$1 = \langle f, f \rangle$$

$$= (\alpha_0, \alpha_1, \ldots, \alpha_n) \underbrace{\begin{pmatrix} z_0 & & 0 \\ & z_1 & \\ & & \ddots \\ 0 & & & z_n \end{pmatrix}}_{Z} \underbrace{\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\alpha}$$

$$= \alpha^t Z \alpha$$

Maximize

$$f(\theta) = \sum_{i=0}^{n} \alpha_i p_i(\theta)$$

$$= (\alpha_0, \alpha_1, \ldots, \alpha_n) \underbrace{\begin{pmatrix} p_0(\theta) \\ p_1(\theta) \\ \vdots \\ p_n(\theta) \end{pmatrix}}_{p_\theta}$$

$$= \alpha^t p_\theta$$

Obs

$$p_\theta \neq 0$$

Since $p_0 = 1$

Problem becomes:

Maximize

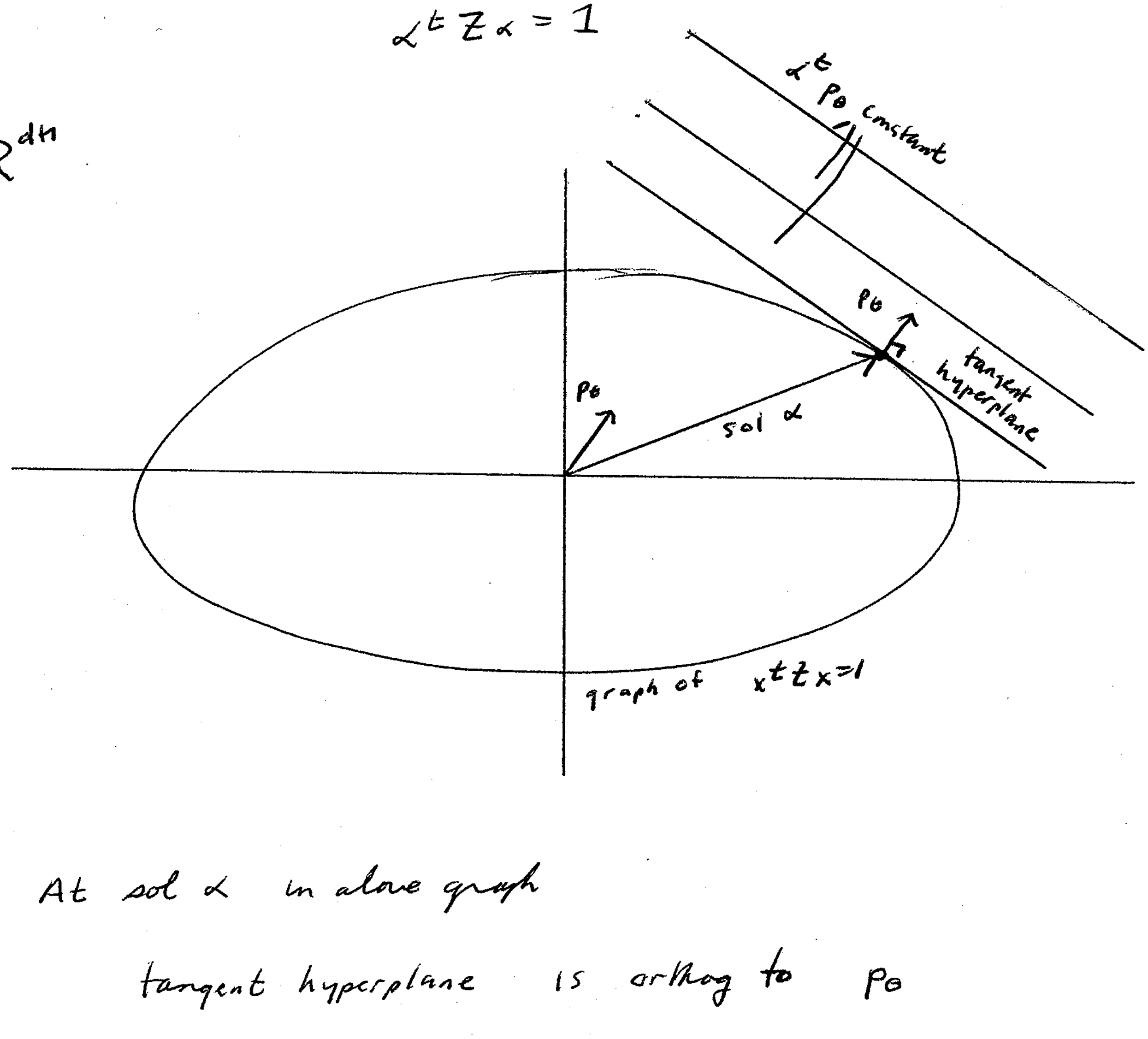
$$\alpha^t p_\theta$$

subject to

$$\alpha \in \mathbb{R}^{n+1}$$

$$\alpha^t Z \alpha = 1$$

$\mathbb{R}^{d+1}$



At sol α in above graph

tangent hyperplane is orthog to p_θ

" " " $Z\alpha$

So

$Z\alpha$, p_θ are lin dep

Since $p_\theta \neq 0$ $\exists\, \lambda \in \mathbb{R}$ s.t.

$$Z\alpha = \lambda\, p_\theta$$

so

$$\alpha = \lambda\, Z^{-1} p_\theta$$

Find λ:

$$\begin{aligned} 1 &= \alpha^t Z \alpha \\ &= \lambda^2\, p_\theta^t Z^{-1} p_\theta \\ &= \lambda^2 \sum_{i=0}^{n} \frac{p_i(\theta)^2}{z_i} \\ &= \lambda^2 \underbrace{K_n(\theta,\theta)}_{V_0} \end{aligned}$$

so

$$\lambda = \frac{\varepsilon}{\sqrt{K_n(\theta,\theta)}} \qquad \varepsilon \in \{1,-1\}$$

"pos square root"

Find α (up to sign)

$$\begin{aligned} \alpha &= \lambda\, Z^{-1} p_\theta \\ &= \frac{\varepsilon}{\sqrt{K_n(\theta,\theta)}}\, Z^{-1} p_\theta \end{aligned}$$

find $f(\theta)$

$$f(\theta) = \alpha^t p_\theta$$

$$= \frac{\varepsilon}{\sqrt{K_n(\theta,\theta)}} \underbrace{p_\theta^t Z^{-1} p_\theta}_{= \sum_{i=0}^{n} \frac{p_i(\theta)^2}{z_i} = K_n(\theta,\theta)}$$

$$= \varepsilon \sqrt{K_n(\theta,\theta)}$$

$f(\theta)$ is maximal at $\varepsilon = 1$

$$f(\theta) = \sqrt{K_n(\theta,\theta)}$$

find f

$$f = \sum_{i=0}^{n} \alpha_i p_i$$

$$= \frac{1}{\sqrt{K_n(\theta,\theta)}} \underbrace{\sum_{i=0}^{n} \frac{p_i(\theta)\, p_i}{z_i}}_{= K_n(\theta,x)}$$

$$f = \frac{K_n(\theta,x)}{\sqrt{K_n(\theta,\theta)}}$$

$\square$

Lecture 9 Friday Sept 24 9/24/10

Until further notice $\mathbb{F}$ arb

Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ that satisfy

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2, \ldots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n-1} \neq 0 \qquad n = 1, 2, \ldots$$

So

$$p_0 = 1$$

$$p_1 = \frac{x - a_0}{b_0}$$

$$p_2 = \frac{(x-a_0)(x-a_1) - b_0 c_1}{b_0 b_1}$$

$$\vdots$$

Since $p_{-1} = 0$ the parameter c_0 could be arbitrary — lets view as indet

Consider more general initial conditions

<u>Problem</u>: Describe $\{f_n\}_{n=-1}^{\infty}$ that satisfy

$$x f_n = c_n f_{n-1} + a_n f_n + b_n f_{n+1} \qquad n = 0, 1, 2 \ldots$$

f_0; f_{-1} arb

c_n, a_n, b_n as above.

$n = 0$: $\quad x f_0 = c_0 f_{-1} + a_0 f_0 + b_0 f_1$

$$f_1 = \underbrace{\frac{x - a_0}{b_0}}\, f_0 + \underbrace{1}_{\text{called}}\, \frac{-c_0 f_{-1}}{b_0}$$

$n=1$

$$x f_1 = c_1 f_0 + a_1 f_1 + b_1 f_2$$

$$f_2 = \frac{x-a_1}{b_1} f_1 - \frac{c_1}{b_1} f_0$$

$$= \underbrace{\frac{(x-a_0)(x-a_1) - b_0 c_1}{b_0 b_1}}_{p_2} f_0 + \underbrace{\frac{x-a_1}{b_1}}_{\text{call it } q_2} \frac{-c_0 f_{-1}}{b_0}$$

In gen

$$f_n = p_n f_0 + q_n \frac{-c_0 f_{-1}}{b_0} \qquad n = 0, 1, 2 \ldots$$

where

$$x q_n = c_n q_{n-1} + a_n q_n + b_n q_{n+1} \qquad n = 1, 2, \ldots$$

$$q_0 = 0 \qquad q_1 = 1$$

For $n = 1, 2 \ldots$

$$q_n \in \mathbb{F}[x]$$

$$\deg q_n = n-1$$

$$\text{coeff of } x^{n-1} \text{ in } q_n \text{ is } \frac{1}{b_1 b_2 \cdots b_{n-1}}$$

Def 55 With above notation call

$\{q_n\}_{n=1}^{\infty}$ the <u>numerator</u> polynomials for $\{p_n\}_{n=0}^{\infty}$

or .. <u>associated</u> .. ..

(In Literature often see

$$q_n = p_n^*$$

but we use p_n^* for something else)

Another view

$$\{p_n\}_{n=0}^{\infty} \iff \begin{pmatrix} a_0 & b_0 & & & \bigcirc \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ \bigcirc & & \ddots & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}$$

$\Downarrow$ delete top row/col

$$\{q_n\}_{n=1}^{\infty} \iff \begin{pmatrix} a_1 & b_1 & & & \bigcirc \\ c_2 & a_2 & b_2 & & \\ & c_3 & \ddots & \ddots & \\ \bigcirc & & \ddots & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}$$

Lets iterate

$$\begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & b_2 & \\ & & c_3 & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

Def 56 Given $\{p_n\}_{n=0}^{\infty}$ as above

For $s = 0, 1, 2 \ldots$

let $\{p_n(x;s)\}_{n=0}^{\infty}$ denote the polynomials

associated with

$$\begin{pmatrix} a_s & b_s & & & 0 \\ c_{s+1} & a_{s+1} & b_{s+1} & & \\ & c_{s+2} & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & \end{pmatrix}$$

So

$$p_n(x;0) = p_n(x) \qquad n = 0, 1, 2 \ldots$$

$$p_n(x;1) = q_{n+1}(x) \qquad n = 0, 1, 2 \ldots$$

Call $\{p_n(x;s)\}_{n=0}^{\infty}$ the <u>associated polynomials</u>

<u>of order s</u>

Next goal: find formula for

$$\{p_n(x;s)\}_{n=0}^{\infty} \quad \text{in terms of} \quad \{p_n\}_{n=0}^{\infty}, \ \{q_n\}_{n=1}^{\infty}$$

We need a det

LEM 57 Given $\{p_n\}_{n=0}^{\infty}$ as above

$$\det\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \frac{c_1 c_2 \cdots c_n}{b_1 b_2 \cdots b_n} \qquad n=0,1,2,\ldots$$

pf Using the 3-term rec

$$\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}\begin{pmatrix} 0 & \frac{-c_n}{b_n} \\ 1 & \frac{x-a_n}{b_n} \end{pmatrix}$$

$n=1,2,3,\ldots$

So

$$\det\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \det\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}\frac{c_n}{b_n}$$

$n=1,2,3\ldots$

$$\det\begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix} = \det\begin{pmatrix} 1 & \frac{x-a_0}{b_0} \\ 0 & 1 \end{pmatrix} = 1$$

result follows. $\square$

Thm 58 Given $\{p_n\}_{n=0}^{\infty}$ as above

For $s = 1, 2, \dots$

$$p_n(x; s) = \frac{\det \begin{pmatrix} p_{s-1} & p_{n+s} \\ q_{s-1} & q_{n+s} \end{pmatrix}}{\det \begin{pmatrix} p_{s-1} & p_s \\ q_{s-1} & q_s \end{pmatrix}} \qquad n = 0, 1, 2 \dots$$

pf Suf to show that for $n = s, s+1, \dots$

$$p_{n-s}(x; s) = \frac{\det \begin{pmatrix} p_{s-1} & p_n \\ q_{s-1} & q_n \end{pmatrix}}{\det \begin{pmatrix} p_{s-1} & p_s \\ q_{s-1} & q_s \end{pmatrix}} \qquad *$$

Let $h_n =$ RHS of $*$

One checks

$$x h_n = c_n h_{n-1} + a_n h_n + b_n h_{n+1} \qquad n = s, s+1, \dots$$

$$h_s = 1, \qquad h_{s-1} = 0$$

The $\{p_{n-s}(x; s)\}_{n=s}^{\infty}$ satisfy the same 3-term rec and initial conditions so

$$h_n(x) = p_{n-s}(x, s) \qquad n = s, s+1 \dots$$

Result follows. $\square$

Lecture 10 Monday Sept 27 9/27/10

$\mathbb{F}$ arb

Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0,1,2\ldots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n-1} \neq 0 \qquad n = 1,2,\ldots$$

Associated polys $\{q_n\}_{n=1}^{\infty}$ satisfy

$$x q_n = c_n q_{n-1} + a_n q_n + b_n q_{n+1} \qquad n = 1,2,3\ldots$$

$$q_1 = 1, \quad q_0 = 0$$

thm 59 With above notation

$$\left\langle \frac{p_n(x) - p_n(y)}{x-y}, 1 \right\rangle = \frac{u_0}{b_0} q_n(y)$$

$$n = 0,1,2,\ldots$$

$$u_0 = \langle 1,1 \rangle$$

pf For $n = 0,1,2\ldots$ def

$$Q_n(y) = \frac{b_0}{u_0} \left\langle \frac{p_n(x) - p_n(y)}{x-y}, 1 \right\rangle$$

Show $Q_n(y) = q_n(y)$

Obs $Q_0(y) = 0, \qquad Q_1(y) = 1$

Check 3-term rec

$$y\, Q_n \stackrel{?}{=} c_n Q_{n-1} + a_n Q_n + b_n Q_{n+1} \qquad n = 1, 2, \dots$$

$$\text{LHS} = \left\langle y \frac{p_n(x) - p_n(y)}{x-y}, 1 \right\rangle$$

$$\text{RHS} = c_n \left\langle \frac{p_{n-1}(x) - p_{n-1}(y)}{x-y}, 1 \right\rangle + a_n \left\langle \frac{p_n(x) - p_n(y)}{x-y}, 1 \right\rangle + b_n \left\langle \frac{p_{n+1}(x) - p_{n+1}(y)}{x-y}, 1 \right\rangle$$

$$= \left\langle \frac{x\, p_n(x) - y\, p_n(y)}{x-y}, 1 \right\rangle$$

$$\text{RHS} - \text{LHS} = \left\langle \frac{x-y}{x-y} p_n(x), 1 \right\rangle$$

$$= \left\langle p_n(x), p_0(x) \right\rangle$$

$$= 0 \qquad \text{since } n \geq 1$$

We have shown

$$Q_n(y) = q_n(y) \qquad n = 1, 2, 3 \dots$$

$\square$

We will return to $\dfrac{p_n(x) - p_n(y)}{x-y}$

in a moment

LEM 60 (Associated Christoffel-Darboux)

with above notation

(i) For $n = 1, 2, \ldots$

$$\sum_{i=1}^{n} p_i(x)\, q_i(y)\, \frac{b_1 b_2 \cdots b_{i-1}}{c_2 c_3 \cdots c_i} =$$

$$\frac{c_1}{x-y} + \frac{b_1 b_2 \cdots b_n}{c_2 c_3 \cdots c_n} \frac{p_{n+1}(x) q_n(y) - p_n(x)\, q_{n+1}(y)}{x-y}$$

(ii) For $n = 1, 2, \ldots$

$$\sum_{i=1}^{n} p_i(x)\, q_i(x)\, \frac{b_1 b_2 \cdots b_{i-1}}{c_2 c_3 \cdots c_i} =$$

$$\left(q_{n+1}'(x)\, p_n(x) - q_n'(x)\, p_{n+1}(x) \right) \frac{b_1 \cdots b_n}{c_2 \cdots c_n}$$

$$f' = Df$$

pf (i)

$$\sum_{i=1}^{n} p_i(x)\, q_i(y)\, \frac{b_1 b_2 \cdots b_{i-1}}{c_2 \cdots c_i} =$$

$$\frac{1}{x-y} \sum_{i=1}^{n} (x-y)\, p_i(x)\, q_i(y)\, \frac{b_1 \cdots b_{i-1}}{c_2 \cdots c_i}$$

$$= \frac{1}{x-y} \sum_{i=1}^{n} \Big(c_i p_{i-1}(x) + a_i p_i(x) + b_i p_{i+1}(x) \Big)\, q_i(y)\, \frac{b_1 \cdots b_{i-1}}{c_2 \cdots c_i}$$

$$- \frac{1}{x-y} \sum_{i=1}^{n} p_i(x) \Big(c_i q_{i-1}(y) + a_i q_i(y) + b_i q_{i+1}(y) \Big)\, \frac{b_1 \cdots b_{i-1}}{c_2 \cdots c_i}$$

[cancel where possible]

$$= \frac{c_1}{x-y} + \frac{b_1 \cdots b_n}{c_2 \cdots c_n}\, \frac{p_{n+1}(x)\, q_n(y) - p_n(x)\, q_{n+1}(y)}{x-y}$$

(ii) In (i) set
$y = x + h$

RHS becomes

$$\frac{-c_1}{h} + \frac{b_1 \cdots b_n}{c_2 \cdots c_n} \frac{p_{n+1}(x)\left(q_n(x) + h q_n'(x) + \cdots\right) - p_n(x)\left(q_{n+1}(x) + h q_{n+1}'(x)\right)}{-h}$$

$$= \underbrace{\frac{-c_1}{h} + \frac{b_1 \cdots b_n}{c_2 \cdots c_n} \frac{p_n(x) q_{n+1}(x) - p_{n+1}(x) q_n(x)}{h}}_{\overset{\shortparallel}{0} \text{ by } L57}$$

$$+ \frac{b_1 \cdots b_n}{c_2 \cdots c_n} \left(q_{n+1}'(x) p_n(x) - q_n'(x) p_{n+1}(x) \right)$$

$$+ h(\ ) + h^2(\ ) \cdots$$

Now let $h \to 0$

RHS becomes

$$\left(q_{n+1}'(x) p_n(x) - q_n'(x) p_{n+1}(x) \right) \frac{b_1 \cdots b_n}{c_2 \cdots c_n}$$

$\square$

thm 61 For $\{p_n\}_{n=0}^{\infty}$ as above

$$\frac{p_{n+1}(x) - p_{n+1}(y)}{x-y} = \sum_{r=0}^{n} \frac{p_r(x) \overbrace{p_{n-r}(y; r+1)}^{\text{assoc polys order } r+1}}{b_r} \qquad (*)$$

$n = 0, 1, 2 \ldots$

pf $n=0$: $\frac{1}{b_0} = \frac{1}{b_0}$ ✓

$n \geq 1$: By thm 58

$$\text{RHS of } * = \sum_{r=0}^{n} \frac{p_r(x)}{b_r} \frac{\det\begin{pmatrix} p_r(y) & p_{n+1}(y) \\ q_r(y) & q_{n+1}(y) \end{pmatrix}}{\det\begin{pmatrix} p_r(y) & p_{r+1}(y) \\ q_r(y) & q_{r+1}(y) \end{pmatrix}} \Leftarrow \overset{L57}{\frac{c_1 c_2 \cdots c_r}{b_1 b_2 \cdots b_r}}$$

$$= \frac{1}{b_0} \sum_{r=0}^{n} p_r(x) \frac{b_0 b_1 \cdots b_{r-1}}{c_1 c_2 \cdots c_r} \det\begin{pmatrix} p_r(y) & p_{n+1}(y) \\ q_r(y) & q_{n+1}(y) \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{1}{b_0} \sum_{r=0}^{n} p_r(x) p_r(y) \frac{b_0 \cdots b_{r-1}}{c_1 \cdots c_r} & p_{n+1}(y) \\ \\ \frac{1}{c_1} \sum_{r=1}^{n} p_r(x) q_r(y) \frac{b_1 \cdots b_{r-1}}{c_2 \cdots c_r} & q_{n+1}(y) \end{pmatrix}$$

Apply C-D (Reg + Assoc)

$$= \det \begin{pmatrix} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x-y} \frac{b_1 \cdots b_n}{c_1 \cdots c_n} & p_{n+1}(y) \\ \\ \frac{1}{x-y} + \frac{p_{n+1}(x) q_n(y) - p_n(x) q_{n+1}(y)}{x-y} \frac{b_1 \cdots b_n}{c_1 \cdots c_n} & q_{n+1}(y) \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & p_{n+1}(y) \\ \frac{1}{x-y} & q_{n+1}(y) \end{pmatrix}$$

$$+ \frac{p_{n+1}(x)}{x-y} \underbrace{\frac{b_1 \cdots b_n}{c_1 \cdots c_n} \det \begin{pmatrix} p_n(y) & p_{n+1}(y) \\ q_n(y) & q_{n+1}(y) \end{pmatrix}}_{= 1}$$

$$= \frac{p_{n+1}(x) - p_{n+1}(y)}{x-y}$$

$\square$

LEM 62 Given $\{p_n\}_{n=0}^{\infty}$ as above.

Assume $\mathbb{F} = \mathbb{R}$ and

$$c_n b_{n-1} > 0 \qquad n = 1, 2, \ldots$$

Then the roots of p_n, q_n interlace for $n = 1, 2, \ldots$

pf Recall

$$\deg p_n = n$$

$$\deg q_n = n-1$$

WLOG p_n monic $\rightarrow$ q_n also monic

By LEM 57

$$0 < \det \begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix}$$

$$= p_n q_{n+1} - q_n p_{n+1}$$

Recall roots of p_n are simple and real

Let

$$\lambda_i = i\text{th Largest root of } p_n \qquad 1 \le i \le n$$

For $1 \le i \le n$

$$0 < \underbrace{p_n(\lambda_i)}_{=0} q_{n+1}(\lambda_i) - q_n(\lambda_i) p_{n+1}(\lambda_i)$$

So

$$q_n(\lambda_i)\, p_{n+1}(\lambda_i) < 0$$

Recall roots of p_n, p_{n+1} interlace so

$$(-1)^i p_{n+1}(\lambda_i) > 0 \qquad 1 \le i \le n$$

Now

$$(-1)^i q_n(\lambda_i) < 0 \qquad 1 \le i \le n$$

Now roots of p_n, q_n interlace by LEM 45 (i) $\square$

Next goal: ...

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$\mathbb{F} = \mathbb{R}$ Given $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2 \ldots$$

$$p_0 = 1 \qquad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \qquad n = 1, 2, \ldots$$

Corresp bil form $\langle , \rangle$ on $\mathbb{R}[x]$ satisfies

$$\langle p_n, p_m \rangle = \delta_{nm} \mu_0 \frac{c_1 c_2 \cdots c_n}{b_0 b_1 \cdots b_{n-1}} \qquad 0 \le n, m < \infty$$

Fix integer $d \ge 0$

Consider $V = \mathrm{Span}\{x^i\}_{i=0}^{d}$

V has basis $x^i \quad 0 \le i \le d$

and orthog basis $p_i \quad 0 \le i \le d$

In a moment we display another orthog basis for V

Consider $V p_{d+1} = \mathrm{Span}\{x^i p_{d+1}\}_{i=0}^{d}$

Obs the sum $V + V p_{d+1}$ is direct

$V + V p_{d+1}$ has basis $\{x^i\}_{i=0}^{2d+1}$

In gen $\langle V, V p_{d+1} \rangle \neq 0$

But $\langle 1, V p_{d+1} \rangle = \langle V, p_{d+1} \rangle$

$= 0$.

Recall roots of p_{d+1} are simple and real — call them $\{\theta_i\}_{i=0}^{d}$

$$p_{d+1} = \frac{1}{b_0 b_1 \cdots b_d} \prod_{i=0}^{d} (x-\theta_i)$$

For $0 \le i \le d$ define $e_i \in \mathbb{R}[x]$

$$e_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{x-\theta_j}{\theta_i-\theta_j}$$

e_i has degree d

coeff of x^d in e_i is $\prod_{\substack{0 \le j \le d \\ j \ne i}} (\theta_i-\theta_j)^{-1}$

Obs

$$e_i(\theta_j) = \delta_{ij} \qquad 0 \le i, j \le d$$

So $\{e_i\}_{i=0}^{d}$ lin indep

So $\{e_i\}_{i=0}^{d}$ is basis for V

Note

$$(x-\theta_i) e_i \in V p_{d+1} \qquad (0 \le i \le d)$$

$$e_i e_j \in V p_{d+1} \quad \forall\, i \ne j \qquad (0 \le i, j \le d)$$

$$e_i^2 - e_i \in V p_{d+1} \qquad (0 \le i \le d)$$

$$f = \sum_{i=0}^{d} f(\theta_i) e_i \qquad \forall f \in V$$

In part

$$1 = \sum_{i=0}^{d} e_i$$

LEM 63 With above notation

$$\langle e_i, e_j \rangle = 0 \text{ if } i \neq j \qquad (0 \leq i, j \leq d)$$

pf

$$\langle e_i, e_j \rangle = \langle 1, \underbrace{e_i e_j}_{\in V_{p d r_0}} \rangle$$

$$= 0 \qquad \square$$

Lecture 11 Wednesday Sept 29 9/29/10

Continue to discuss Gauss quadrature

$\mathbb{F} = \mathbb{R}$ Given $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2..$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \qquad n = 1, 2, ..$$

Corresp $\langle \, , \, \rangle$ on $\mathbb{R}[x]$ satisfies

$$\langle p_n, p_m \rangle = \delta_{nm} \frac{\mu_0}{k_n} \qquad 0 \le n, m < \infty$$

$$k_n = \frac{b_0 b_1 \cdots b_{n-1}}{c_1 c_2 \cdots c_n}, \qquad \mu_0 = \langle 1, 1 \rangle .$$

Fix integer $d \ge 0$

p_{d+1} has roots $\{\theta_i\}_{i=0}^{d}$

For $0 \le i \le d$ define

$$e_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{x - \theta_j}{\theta_i - \theta_j}$$

We saw

$\{e_i\}_{i=0}^{d}$ is orthog basis for V $\qquad V = \text{Span}\{x^i\}_{i=0}^{d}$

thm 64 With above notation,

Given scalars $\{m_i\}_{i=0}^d$ in $\mathbb{R}$

TFAE

(i) $\langle 1, f \rangle = \sum_{i=0}^d f(\theta_i) m_i \qquad \forall f \in V$

(ii) $m_i = \langle 1, e_i \rangle \qquad (0 \le i \le d)$

Suppose (i), (ii) hold. Then

$$m_i = \langle e_i, e_i \rangle > 0 \qquad (0 \le i \le d)$$

pf (i) → (ii)

$$\langle 1, e_i \rangle = \sum_{j=0}^d \underbrace{e_i(\theta_j)}_{\delta_{ij}} m_j$$

$$= m_i$$

(ii) → (i) We saw

$$f = \sum_{i=0}^d f(\theta_i) e_i$$

so

$$\langle 1, f \rangle = \sum_{i=0}^d f(\theta_i) \langle 1, e_i \rangle$$

$$= \sum_{i=0}^d f(\theta_i) m_i$$

Suppose (i), (ii)

$$m_i = \langle 1, e_i \rangle$$

$$= \langle e_0 + e_1 + \cdots + e_d, e_i \rangle$$

$$= \langle e_i, e_i \rangle \qquad \text{by L63}$$

$$> 0 \qquad \text{since } \langle , \rangle \text{ pos def.}$$

□

Def 65 With above notation

for $0 \le i \le d$ define

$$m_i = \langle 1, e_i \rangle$$

"ith Christoffel number"

Th 66 With above notation

(i) $\langle 1, f \rangle = \sum_{i=0}^{d} f(\theta_i) m_i \qquad \forall f \in V + V_{pdm}$

(ii) $\langle f, g \rangle = \sum_{i=0}^{d} f(\theta_i) g(\theta_i) m_i \qquad \forall f, g \in V$

pf (i) for $f \in V$ done by Th 64.

For $f \in V_{pdm}$

$$\underbrace{\langle 1, f \rangle}_{0} \stackrel{?}{=} \sum_{i=0}^{d} \underbrace{f(\theta_i)}_{0} m_i \qquad \text{ok}$$

(ii) Obs

$$\langle f, g \rangle = \langle 1, fg \rangle$$

and $fg \in V + V_{pdm}$

Now apply (i).

$\square$

LEM 67 With above notation

for $0 \le i \le d$

(i) $$e_i = \frac{p_{d+1}}{(x-\theta_i)\, p'_{d+1}(\theta_i)} \qquad (f' = Df)$$

(ii) $$m_i = \left\langle 1, \frac{p_{d+1}}{(x-\theta_i)\, p'_{d+1}(\theta_i)} \right\rangle$$

(iii) $$m_i = \left\| \frac{p_{d+1}}{(x-\theta_i)\, p'_{d+1}(\theta_i)} \right\|^2$$

pf (i) Recall

$$p_{d+1} = \alpha \prod_{i=0}^{d} (x-\theta_i) \qquad \alpha = \frac{1}{b_0 b_1 \cdots b_d}$$

Write

$$p_{d+1} = (x-\theta_i)\, g$$

$$g = \alpha \prod_{\substack{0 \le j \le d \\ j \ne i}} (x-\theta_j)$$

$$p'_{d+1} = (x-\theta_i)\, g' + g$$

$$p'_{d+1}(\theta_i) = g(\theta_i) = \alpha \prod_{\substack{0 \le j \le d \\ j \ne i}} (\theta_i-\theta_j)$$

Result follows.

(ii) By (i) and $m_i = \langle 1, e_i \rangle$

(iii) By (i) and $m_i = \langle e_i, e_i \rangle$

$\square$

Recall our two orthogonal bases for V

$$\{p_n\}_{n=0}^{d} \qquad \{e_i\}_{i=0}^{d}$$

LEM 68

(i) $$p_n = \sum_{i=0}^{d} p_n(\theta_i)\, e_i \qquad 0 \le n \le d$$

(ii) $$e_i = \frac{m_i}{\mu_0} \sum_{n=0}^{d} p_n(\theta_i)\, k_n\, p_n \qquad 0 \le i \le d$$

(iii) $$\langle e_i, p_n \rangle = p_n(\theta_i)\, m_i \qquad 0 \le i, n \le d$$

pf (i) $\deg p_n \le d$

(iii) $$\langle e_i, p_n \rangle = \sum_{j=0}^{d} p_n(\theta_j) \underbrace{\langle e_i, e_j \rangle}_{= \delta_{ij} m_i}$$

$$= p_n(\theta_i)\, m_i$$

(ii) Write

$$e_i = \sum_{m=0}^{d} \alpha_{mi}\, p_m$$

$$\underbrace{\langle e_i, p_n \rangle}_{= p_n(\theta_i) m_i} = \sum_{m=0}^{d} \alpha_{mi} \underbrace{\langle p_m, p_n \rangle}_{= \delta_{nm} \frac{\mu}{k_n}}$$

$$= \alpha_{ni} \frac{\mu}{k_n}$$

$$\alpha_{ni} = \frac{m_i}{\mu} k_n\, p_n(\theta_i)$$

$\square$

thm 69 With the above notation

(i)
$$\sum_{i=0}^{d} p_n(\theta_i)\, p_m(\theta_i)\, m_i = \delta_{nm} \frac{\mu_0}{k_n} \qquad 0 \le n, m \le d$$
"row orthog"

(ii)
$$\sum_{n=0}^{d} p_n(\theta_i)\, p_n(\theta_j)\, k_n = \delta_{ij} \frac{\mu_0}{m_i} \qquad 0 \le i, j \le d$$
"column orthog"

pf (i) In $\langle p_n, p_m \rangle = \delta_{nm} \frac{\mu_0}{k_n}$

expand p_n, p_m using L 68 (i)

and use $\langle e_i, e_j \rangle = \delta_{ij} m_i$

(ii) In $\langle e_i, e_j \rangle = \delta_{ij} m_i$

expand e_i, e_j using L 68 (ii)

and use $\langle p_n, p_m \rangle = \delta_{nm} \frac{\mu_0}{k_n}$

□

Special cases of thm 69

part (i)

$n=m$:
$$\sum_{i=0}^{d} p_n(\theta_i)^2 m_i = \frac{u_0}{k_n} \qquad 0 \le n \le d$$

$n=m=0$:
$$\sum_{i=0}^{d} m_i = \mu_0$$

part (ii)

$i=j$:
$$\sum_{n=0}^{d} p_n(\theta_i)^2 k_n = \frac{\mu_0}{m_i} \qquad 0 \le i \le d$$

(Gives a way to compute m_i)

With above notation

recall Kernel polynomials

$$K_d(x,y) = \sum_{n=0}^{d} p_n(x)\, p_n(y)\, k_n$$

In th 69 (ii)

$$LHS = \sum_{n=0}^{d} p_n(\theta_i)\, p_n(\theta_j)\, k_n$$

$$= K_d(\theta_i, \theta_j)$$

$$= K_{d+1}(\theta_i, \theta_j) \quad \text{since} \quad p_{d+1}(\theta_i) = 0, \; p_{d+1}(\theta_j) = 0$$

So thm 69 (ii) asserts

$$K_d(\theta_i, \theta_j) = K_{d+1}(\theta_i, \theta_j) = \delta_{ij} \frac{\mu_0}{m_i} \qquad 0 \le i, j \le d$$

What does Christoffel-Darboux assert?

For $0 \le i, j \le d$, $i \ne j$

$$K_d(\theta_i, \theta_j) = \frac{\overset{=0}{p_{d+1}(\theta_i)}\, p_d(\theta_j) - p_d(\theta_i)\, \overset{=0}{p_{d+1}(\theta_j)}}{p_1(\theta_i) - p_1(\theta_j)} \; \frac{b_1 \cdots b_d}{c_1 \cdots c_d}$$

$$= 0 \qquad \text{"Nothing new"}$$

$$K_{d+1}(\theta_i, \theta_j) = \frac{p_{d+2}(\theta_i)\, \overset{=0}{p_{d+1}(\theta_j)} - \overset{=0}{p_{d+1}(\theta_i)}\, p_{d+2}(\theta_j)}{p_1(\theta_i) - p_1(\theta_j)} \; \frac{b_1 \cdots b_{d+1}}{c_1 \cdots c_{d+1}}$$

$$= 0 \qquad \text{"nothing new"}$$

For $0 \le i \le d$

$$K_d(\theta_i, \theta_i) = \left(p'_{d+1}(\theta_i)\, p_d(\theta_i) - p'_d(\theta_i)\, \underbrace{p_{d+1}(\theta_i)}_{=0} \right) \frac{b_0 b_1 b_2 \cdots b_d}{c_1 c_2 \cdots c_d}$$

$$= p'_{d+1}(\theta_i)\, p_d(\theta_i)\, \frac{b_0 b_1 \cdots b_d}{c_1 \cdots c_d}$$

$$K_{d+1}(\theta_i, \theta_i) = \left(p'_{d+2}(\theta_i)\, \underbrace{p_{d+1}(\theta_i)}_{=0} - p'_{d+1}(\theta_i)\, p_{d+2}(\theta_i) \right) \frac{b_0 b_1 b_2 \cdots b_{d+1}}{c_1 c_2 \cdots c_{d+1}}$$

$$= -\, p'_{d+1}(\theta_i)\, p_{d+2}(\theta_i)\, \frac{b_0 b_1 b_2 \cdots b_{d+1}}{c_1 c_2 \cdots c_{d+1}}$$

COR 70 With above notation

For $0 \le i \le d$

$$\frac{\mu_0}{m_i} = p'_{d+1}(\theta_i)\, p_d(\theta_i)\, \frac{b_0 b_1 \cdots b_d}{c_1 \cdots c_d}$$

$$= -\, p'_{d+1}(\theta_i)\, p_{d+2}(\theta_i)\, \frac{b_0 b_1 \cdots b_{d+1}}{c_1 \cdots c_{d+1}}$$

$\square$

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Cor 71 With above notation for $0 \le i \le d$

$$\frac{m_0}{m_i} = \frac{p_d(\theta_i)}{c_1 c_2 \cdots c_d} \prod_{\substack{0 \le j \le d \\ j \ne i}} (\theta_i - \theta_j)$$

pf Recall

$$b_0 b_1 \cdots b_d \, p_{d+1} = \prod_{j=0}^{d} (x - \theta_j)$$

So

$$b_0 b_1 \cdots b_d \, p_{d+1}'(\theta_i) = \prod_{\substack{0 \le j \le d \\ j \ne i}} (\theta_i - \theta_j)$$

Now use Cor 70 $\square$

Next goal: An interp of $\{m_i\}_{i=0}$ using partial fractions 1/24/10

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For $f \in V$ consider

$$\frac{f}{p_{d+1}} = \frac{b_0 b_1 \cdots b_d f}{\prod_{i=0}^{d} (x-\theta_i)}$$

$$= \sum_{i=0}^{d} \frac{\sigma_i}{x-\theta_i} \qquad \sigma_i \in \mathbb{R}$$

" partial fraction decomp "

To get σ_i mult both sides by $x-\theta_i$

$$\frac{b_0 b_1 \cdots b_d f}{\prod_{\substack{0 \le j \le d \\ j \ne i}} (x-\theta_j)} = \sigma_i + \sum_{\substack{0 \le j \le d \\ j \ne i}} \sigma_j \frac{x-\theta_i}{x-\theta_j}$$

and set $x = \theta_i$

$$\frac{b_0 b_1 \cdots b_d f(\theta_i)}{\prod_{\substack{0 \le j \le d \\ j \ne i}} (\theta_i-\theta_j)} = \sigma_i$$

LEM 72 For $f \in V$

$$\frac{f}{p_{d+1}} = b_0 \cdots b_d \sum_{i=0}^{d} \frac{1}{x-\theta_i} \frac{f(\theta_i)}{\prod_{\substack{0 \le j \le d \\ j \ne i}} (\theta_i-\theta_j)}$$

$\square$

With ref to L 72 we now take

$$f = q_{d+1} \qquad \text{(assoc poly)}$$

and get something nice:

thm 73 With above notation

$$\frac{q_{d+1}}{p_{d+1}} = \frac{b_0}{\mu_0} \sum_{i=0}^{d} \frac{m_i}{x - \theta_i}$$

pf Apply L 72 with $f = q_{d+1}$

For $0 \le i \le d$ find $q_{d+1}(\theta_i)$

By L 57

$$\det \begin{pmatrix} p_d & p_{d+1} \\ q_d & q_{d+1} \end{pmatrix} = \frac{c_1 c_2 \cdots c_d}{b_1 b_2 \cdots b_d}$$

set $x = \theta_i$, $p_{d+1}(\theta_i) = 0$

$$p_d(\theta_i)\, q_{d+1}(\theta_i) = \frac{c_1 c_2 \cdots c_d}{b_1 b_2 \cdots b_d}$$

So

$$\frac{q_{d+1}}{p_{d+1}} = b_0 b_1 \cdots b_d \sum_{i=0}^{d} \frac{1}{x - \theta_i} \overbrace{\frac{c_1 \cdots c_d}{b_1 \cdots b_d} \frac{1}{p_d(\theta_i)} \frac{1}{\prod_{\substack{0 \le j \le d \\ j \ne i}} (\theta_i - \theta_j)}}^{\text{Cor 71} \;\; \frac{m_i}{\mu_0}}$$

$$= \frac{b_0}{\mu_0} \sum_{i=0}^{d} \frac{m_i}{x - \theta_i}$$

$\square$

$\square$

Lecture 12 Friday Oct 1 10/1/10

[illegible] a kernel [illegible]

$\mathbb{F}$ arb

Given $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1} \qquad n = 0, 1, 2, \ldots$$

$$p_0 = 1, \qquad p_{-1} = 0$$

$$c_n b_{n-1} \neq 0 \qquad n = 1, 2, \ldots$$

· Next goal: consider

$$c_n + a_n + b_n$$

LEM 74 $\forall a \in \mathbb{F}$ TFAE

(i) $p_n(a) = 1$ $\quad n = 0, 1, 2 \ldots$

(ii) The following matrix has const row sum a

$$\begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & 0 \\ & c_2 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

pf (i) → (ii) By 3-term rec

$$\underset{1}{\underset{\|}{a p_n(a)}} = \underset{1}{\underset{\|}{c_n p_{n-1}(a)}} + \underset{1}{\underset{\|}{a_n p_n(a)}} + b_n \underset{1}{\underset{\|}{p_{n+1}(a)}}$$

(ii) → (i) Ind on n

$p_0(a) = 1$ ✓

$$p_1(a) = \frac{a - a_0}{b_0} = 1$$

For $n \geq 1$

$$p_{n+1}(a) = \frac{(a - a_n) p_n(a) - c_n p_{n-1}(a)}{b_n}$$

$$= \frac{a - a_n - c_n}{b_n}$$

$$= \frac{b_n}{b_n}$$

$$= 1$$

□

Given $a \in \mathbb{F}$

assume $p_n(a) \neq 0$ $\quad n = 0, 1, 2, \ldots$

For $n = 0, 1, 2 \ldots$ define

$$f_n = \frac{p_n}{p_n(a)}$$

So that $f_n(a) = 1$

LEM 75 With above notation

$$x f_n =$$

$$c_n \frac{p_{n-1}(a)}{p_n(a)} f_{n-1} + a_n f_n + b_n \frac{p_{n+1}(a)}{p_n(a)} f_{n+1}$$

$$n = 0, 1, 2, \ldots$$

pf routine $\square$

Given $a \in \mathbb{F}$

assume

$$a = c_n + a_n + b_n \qquad n = 1, 2, \dots$$

but no assumption about $n = 0$

thus

$$\begin{pmatrix} a_0 & b_0 & & 0 \\ c_1 & a_1 & b_1 & \\ & c_2 & & \\ 0 & & \ddots & \end{pmatrix}$$

has row sum a for $n = 1, 2, \dots$ but not nec for $n = 0$

Define $\quad c_0 = a - b_0 - a_0 \qquad$ "excess"

" Weak Row Sum a "

The above situation occurs naturally as we will see.

EXAMPLE 76 Assume $p_n(a) = 1$ $n = 0, 1, 2, \ldots$

Then the associated polynomials $\{q_n\}_{n=1}^{\infty}$ satisfy

WRS a

LEM 77 Assume $\{p_n\}_{n=0}^{\infty}$ satisfies WRS

Then

$$p_n(a) = 1 + \frac{c_0}{b_0} q_n(a) \qquad n = 0, 1, 2, \ldots$$

where $\{q_n\}_{n=1}^{\infty}$ are assoc polys and $q_0 = 0$

excess $c_0 = a - b_0 - a_0$

pf

Define $\tilde{p}_n = 1 + \frac{c_0}{b_0} q_n(a)$

One checks

$$\tilde{p}_0(a) = 1 = p_0(a)$$

$$\tilde{p}_1(a) = \frac{a - a_0}{b_0} = p_1(a)$$

Since $\{q_n\}_{n=1}^{\infty}$ satisfies the same 3-term rec as $\{p_n\}_{n=0}^{\infty}$

we get

$$a \tilde{p}_n(a) = c_n \tilde{p}_{n-1}(a) + a_n \tilde{p}_n(a) + b_n \tilde{p}_{n+1}(a) \qquad n = 1, 2, \ldots$$

So $\tilde{p}_n(a) = p_n(a)$ $n = 0, 1, 2, \ldots$

$\square$

Recall Kernel polys

$$K_n(x,y) = \sum_{i=0}^{n} p_i(x)\, p_i(y)\, k_i$$

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$$

LEM 78 For $n = 0, 1, 2, \ldots$

Given $f \in \mathbb{F}[x]$ with $\deg f \le n$

$$\langle f(x), K_n(x,y) \rangle = \mu_0 f(y)$$

$$\mu_0 = \langle 1, 1 \rangle$$

pf Write

$$f(x) = \sum_{i=0}^{n} \alpha_i p_i(x) \qquad \alpha_i \in \mathbb{F}$$

$$\langle f(x), K_n(x,y) \rangle =$$

$$\left\langle \sum_{i=0}^{n} \alpha_i p_i(x), \sum_{j=0}^{n} p_j(x) p_j(y) k_j \right\rangle$$

$$= \sum_{i=0}^{n} \alpha_i p_i(y) k_i \underbrace{\| p_i(x) \|^2}_{\mu_0/k_i}$$

$$= \mu_0 \sum_{i=0}^{n} \alpha_i p_i(y)$$

$$= \mu_0 f(y)$$

$\square$

Fix $a \in \mathbb{F}$

Consider $K_n(x,a) \in \mathbb{F}[x]$

$$K_n(x,a) = \sum_{i=0}^{n} p_i(x)\, p_i(a)\, k_i$$

$\deg K_n(x,a) \le n$

coeff of x^n in $K_n(x,a)$ is $\dfrac{p_n(a)\, k_n}{b_0 b_1 \cdots b_{n-1}} = \dfrac{p_n(a)}{c_1 c_2 \cdots c_n}$

LEM 79 For $n = 0, 1, 2, \ldots$

$$p_n(x) = \frac{K_n(x,a) - K_{n-1}(x,a)}{p_n(a)\, k_n}$$

$k_{-1} = 0$

provided $p_n(a) \neq 0$

pf Routine $\square$

thm 80 Given $a \in \mathbb{F}$. For $n = 0, 1, 2 ..$

$$x K_n(x, a) =$$

term	coeff
$K_{n+1}(x,a)$	$c_{n+1} \dfrac{p_n(a)}{p_{n+1}(a)}$
$K_n(x,a)$	$a - c_{n+1} \dfrac{p_n(a)}{p_{n+1}(a)} - b_n \dfrac{p_{n+1}(a)}{p_n(a)}$
$K_{n-1}(x,a)$	$b_n \dfrac{p_{n+1}(a)}{p_n(a)}$

provided $p_n(a) \neq 0$, $p_{n+1}(a) \neq 0$

pf
$$x K_n(x,a) = (x-a) K_n(x,a) + a K_n(x,a)$$

$$(x-a) K_n(x,a) = (x-a) \frac{p_{n+1}(x) p_n(a) - p_n(x) p_{n+1}(a)}{p_1(x) - p_1(a)} \frac{b_1 b_2 \cdots b_n}{c_1 c_2 \cdots c_n}$$

$$\left[\frac{x-a}{p_1(x) - p_1(a)} = b_0 \right]$$

$$= \left(p_{n+1}(x) p_n(a) - p_n(x) p_{n+1}(a) \right) \frac{b_0 \cdots b_n}{c_1 \cdots c_n}$$

$$= \left(\frac{K_{n+1}(x,a) - K_n(x,a)}{p_{n+1}(a) K_{n+1}} p_n(a) - \frac{K_n(x,a) - K_{n-1}(x,a)}{p_n(a) K_n} p_{n+1}(a) \right) \frac{b_0 \cdots b_n}{c_1 \cdots c_n}$$

Result follows

$\square$

Note 81 With ref to Thm 80

assume $p_n(a) \neq 0$ $n = 0, 1, 2, \ldots$

Then $\{K_n(x, a)\}_{n=0}^{\infty}$ satisfy WRS a

the excess $= b_0 p_1(a)$

$= a - a_0$

In thm 80 consider the following special case.

Assume $p_n(a) \neq 0$ $n = 0, 1, 2 ..$

After adjusting as in LEM 75, WLOG

$$p_n(a) = 1 \qquad n = 0, 1, 2 ...$$

Now for $n = 0, 1, 2..$

$$K_n(x, a) = \sum_{i=0}^{n} p_i(x) \underbrace{p_i(a)}_{1} k_i$$

$$= \sum_{i=0}^{n} p_i(x) k_i$$

LEM 82 With above notation / assumptions

$$x K_n(x, a) =$$

$$K_{n-1}(x, a)\, b_n + K_n(x, a)\left(a - c_{n+1} - b_n \right) + K_{n+1}(x, a)\, c_{n+1}$$

$$n = 0, 1, 2, .. \qquad K_{-1} = 0$$

pf Set $p_n(a) = 1$ $n = 0, 1, 2..$ in thm 80 $\square$

We now bring in $\langle , \rangle$

No longer assume $p_n(a) = 1$

thm 83 For $a \in \mathbb{F}$

$$\left\langle (x-a) K_n(x,a),\ K_m(x,a) \right\rangle =$$

$$- \delta_{nm}\, \mu_0\, p_n(a)\, p_{n+1}(a)\, \frac{b_0 b_1 \cdots b_n}{c_1 c_2 \cdots c_n}$$

$$n, m = 0, 1, 2, \ldots$$

pf WLOG $m \le n$

$$LHS = \left\langle (x-a) \frac{p_{n+1}(x) p_n(a) - p_n(x) p_{n+1}(a)}{p_1(x) - p_1(a)} \frac{b_1 \cdots b_n}{c_1 \cdots c_n},\ K_m(x,a) \right\rangle$$

$$\left[\frac{x-a}{p_1(x) - p_1(a)} = b_0 \right]$$

$$= \left\langle p_{n+1}(x) p_n(a) - p_n(x) p_{n+1}(a),\ \sum_{i=0}^{m} p_i(x) p_i(a) k_i \right\rangle \frac{b_0 \cdots b_n}{c_1 \cdots c_n}$$

$$= \begin{cases} 0 & m < n \\ -\underbrace{\| p_n(x) \|^2_{K_n}}_{\mu_0} p_n(a)\, p_{n+1}(a) \dfrac{b_0 b_1 \cdots b_n}{c_1 \cdots c_n} & m = n \end{cases}$$

Result follows. $\square$

Cor 84 With ref to Th 83

assume $p_n(a) \neq 0$ $n = 0, 1, 2 \ldots$

Define

$$(,) : \mathbb{F}[x] \times \mathbb{F}[x] \to \mathbb{F}$$
$$f \quad g \to \langle (x-a) f, g \rangle$$

Then $\{ K_n(x, a) \}_{n=0}^{\infty}$ are orthog rel $(,)$ $\square$

pf By Th 83.

$\square$

Next goal: Continued fractions and orthog polys.

$\mathbb{F}$ arb

Given two sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ from $\mathbb{F}$

Define

$$C_0 = b_0$$

$$C_1 = b_0 + \frac{a_1}{b_1}$$

$$C_2 = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}}$$

$$C_3 = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3}}}$$

$$\vdots$$

For $n = 0, 1, 2 \ldots$ call C_n the <u>nth convergent</u> of the continued fraction

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \qquad *$$

$$= b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots$$

We view the continued fraction as a formal expression of form *.

In general we do not care if $\{C_n\}_{n=0}^{\infty}$ converges to a limit.

For $n = 0, 1, 2 \ldots$ simplify C_n

$$C_0 = \frac{b_0}{1} = \frac{A_0}{B_0}$$

$$C_1 = \frac{b_0 b_1 + a_1}{b_1} = \frac{A_1}{B_1}$$

$$C_2 = \frac{b_0 b_1 b_2 + b_0 a_2 + a_1 b_2}{b_1 b_2 + a_2} = \frac{A_2}{B_2}$$

$$\vdots$$

LEM 85 For $n = 1, 2, \ldots$

(i) $$A_n = b_n A_{n-1} + a_n A_{n-2} \qquad (A_{-1} = 1)$$

(ii) $$B_n = b_n B_{n-1} + a_n B_{n-2} \qquad (B_{-1} = 0)$$

pf Ind on n

$n = 1$ ✓

$n \geq 2$:

To get C_n from C_{n-1} replace b_{n-1} by $b_{n-1} + \frac{a_n}{b_n}$

$$C_{n-1} = \frac{A_{n-1}}{B_{n-1}} = \frac{b_{n-1} A_{n-2} + a_{n-1} A_{n-3}}{b_{n-1} B_{n-2} + a_{n-1} B_{n-3}} \quad \text{by ind}$$

$A_{n-2}, A_{n-3}, B_{n-2}, B_{n-3}$ do not inv b_{n-1}

$$C_n = \frac{A_n}{B_n} = \frac{\left(b_{n-1} + \frac{a_n}{b_n}\right) A_{n-2} + a_{n-1} A_{n-3}}{\left(b_{n-1} + \frac{a_n}{b_n}\right) B_{n-2} + a_{n-1} B_{n-3}}$$

$$= \frac{(b_{n-1} b_n + a_n) A_{n-2} + a_{n-1} b_n A_{n-3}}{(b_{n-1} b_n + a_n) B_{n-2} + a_{n-1} b_n B_{n-3}}$$

$$= \frac{(b_{n-1} b_n + a_n) A_{n-2} + b_n \left(A_{n-1} - b_{n-1} A_{n-2}\right)}{(b_{n-1} b_n + a_n) B_{n-2} + b_n \left(B_{n-1} - b_{n-1} B_{n-2}\right)}$$

$$= \frac{b_n A_{n-1} + a_n A_{n-2}}{b_n B_{n-1} + a_n B_{n-2}} \quad ✓$$

$\square$

LEM 86 With above notation

$$A_n B_{n-1} - B_n A_{n-1} = (-1)^{n+1} a_1 a_2 \cdots a_n \qquad n = 1, 2, \ldots$$

pf Ind on n

$n=1$ ✓

$n \geq 2$:

$$A_n B_{n-1} - B_n A_{n-1} = \left(b_n A_{n-1} + a_n A_{n-2} \right) B_{n-1}$$

$$- \left(b_n B_{n-1} + a_n B_{n-2} \right) A_{n-1}$$

$$= - a_n \left(\underbrace{A_{n-1} B_{n-2} - B_{n-1} A_{n-2}}_{\overset{\text{ind}}{=} (-1)^n a_1 \cdots a_{n-1}} \right)$$

$\square$

Given monic $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ s.t.

$$x p_n = c_n p_{n-1} + a_n p_n + p_{n+1} \qquad n = 0,1,2.. \qquad \ast$$

$$p_0 = 1 \qquad p_{-1} = 0$$

$$c_n \neq 0 \qquad n = 1,2..$$

[above a_n not same as in CF]

Find CF s.t.

$$B_n = p_n \qquad \forall n$$

Write $\ast$ as

$$p_{n+1} = (x - a_n) p_n - c_n p_{n-1} \qquad n = 0,1,2..$$

So

$$p_n = \underbrace{(x - a_{n-1})}_{\text{"}b_n\text{"}} p_{n-1} - \underbrace{c_{n-1}}_{\text{"}a_n\text{"}} p_{n-2} \qquad n = 1,2..$$

Compare with L 85

"b_0" and "a_1" play no role in B_n $\quad n = 0,1,2..$

so take

"b_0" $= 0$ $\qquad$ View "a_1" as free

DESIRED CF IS

$$0 + \cfrac{-c_0}{x-a_0 + \cfrac{-c_1}{x-a_1 + \cfrac{-c_2}{x-a_2 + \cfrac{-c_3}{\ddots}}}} \qquad \bigstar$$

$$= -\frac{c_0}{x-a_0-}\ \frac{c_1}{x-a_1-}\ \frac{c_2}{x-a_2-} \cdots$$

thm 87 For the CF $\bigstar$, for $n = 0,1,2..$

the nth convergent $C_n = A_n / B_n$ where

B_n = orig p_n

$A_n = -c_0 \, q_n$

↑ assoc poly

pf For each equation LHS, RHS satisfies

same recursion and init conditions $\square$

Next topic: Gauss hypergeometric differential equation

Consider hypergeometric series

$$ {}_rF_s\left(\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{matrix} \Big/ x\right) $$

What happens if we apply $D = \frac{d}{dx}$?

What happens if we replace some a_i (resp b_i) by $a_i \pm 1$ (resp $b_i \pm 1$) ?

LEM 88

$$ \frac{d}{dx} F\left(\begin{matrix} a_1 \cdots a_r \\ b_1 \cdots b_s \end{matrix} \Big/ x\right) = \frac{a_1 \cdots a_r}{b_1 \cdots b_s} F\left(\begin{matrix} a_1+1, a_2+1, \ldots a_r+1 \\ b_1+1, \cdots b_s+1 \end{matrix} \Big/ x\right) $$

provided $b_1 b_2 \cdots b_s \neq 0$

pf Compare coeffs of x^n for $0 \le n < \infty$

$$ LHS = \frac{(a_1)_{n+1} \cdots (a_r)_{n+1}}{(b_1)_{n+1} \cdots (b_s)_{n+1}} \frac{n+1}{(n+1)!} $$

$$ = \frac{a_1 \cdots a_r}{b_1 \cdots b_s} \frac{(a_1+1)_n \cdots (a_r+1)_n}{(b_1+1)_n \cdots (b_s+1)_n} \frac{1}{n!} $$

$$ = RHS $$

$\square$

LEM 89

$$F\left(\begin{matrix} a_1+1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle/ x\right) - F\left(\begin{matrix} a_1 \cdots a_r \\ b_1 \cdots b_s \end{matrix} \middle/ x\right)$$

$$= \frac{a_2 a_3 \cdots a_r}{b_1 \cdots b_s} \times F\left(\begin{matrix} a_1+1, a_2+1, \ldots a_r+1 \\ b_1+1, \ldots, b_s+1 \end{matrix} \middle/ x\right)$$

provided $b_1 \cdots b_s \neq 0$

pf Compare coeffs of x^n for $n = 0, 1, 2 \ldots$

WLOG $n \geq 1$ else both coeff 0

LHS =

$$\frac{(a_1+1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n \, n!} - \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n \, n!}$$

$$= \frac{(a_2)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{(a_1+1)_{n-1}}{(n-1)!}$$

$$= \frac{a_2 \cdots a_r}{b_1 \cdots b_s} \frac{(a_1+1)_{n-1} \cdots (a_r+1)_{n-1}}{(b_1+1)_{n-1} \cdots (b_s+1)_{n-1}} \frac{1}{(n-1)!}$$

$= RHS$

$\square$

COR 90

$$\left(a_1 + x\frac{d}{dx}\right) F\left(\begin{matrix} a_1, a_2, \ldots a_r \\ b_1, \ldots b_s \end{matrix} \Big/ x\right) = a_1 F\left(\begin{matrix} a_1+1, a_2, \ldots a_r \\ b_1 \ldots b_s \end{matrix} \Big/ x\right)$$

pf WLOG $b_1 \cdots b_s \neq 0$ else both sides equal a_1

Obs

$$a_1\left(F\left(\begin{matrix} a_1+1, a_2, \ldots a_r \\ b_1 \ldots b_s \end{matrix} \Big/ x\right) - F\left(\begin{matrix} a_1 \ldots a_r \\ b_1 \ldots b_s \end{matrix} \Big/ x\right)\right)$$

$$= \frac{a_1 \cdots a_r}{b_1 \cdots b_s} \; x \; F\left(\begin{matrix} a_1+1 \; a_2+1 \cdots a_r+1 \\ b_1+1 \cdots b_s+1 \end{matrix} \Big/ x\right)$$

$$= x \frac{d}{dx} F\left(\begin{matrix} a_1 \ldots a_r \\ b_1 \ldots b_s \end{matrix} \Big/ x\right) \qquad \square$$

Given above result, we might wonder about

$$b_0 + x\frac{d}{dx}$$

LEM 91

$$\left(b_1 + x \frac{d}{dx} \right) F\left(\begin{matrix} a_1, \dots a_r \\ b_1+1, b_2, \dots, b_s \end{matrix} \Big/ x \right) = b_1 F\left(\begin{matrix} a_1, \dots, a_r \\ b_1 \dots b_s \end{matrix} \Big/ x \right)$$

pf Compare coeff of x^n fn $n = 0, 1, 2 \dots$

$$LHS = b_1 \frac{(a_1)_n \cdots (a_r)_n}{(b_1+1)_n (b_2)_n \cdots (b_s)_n} \frac{1}{n!} +$$

$$\frac{(a_1)_n \cdots (a_r)_n}{(b_1+1)_n (b_2)_n \cdots (b_s)_n} \frac{n}{n!}$$

$$= \frac{(a_1)_n \cdots (a_r)_n}{(b_1+1)_n (b_2)_n \cdots (b_s)_n}$$

$$= b_1 \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n}$$

$\square$

thm 92 $F\left(\begin{matrix} a_1 \ldots a_r \\ b_1 \ldots b_s \end{matrix} \middle/ x\right)$ satisfies the differential equation

$$\left(a_1 + x\frac{d}{dx}\right)\left(a_2 + x\frac{d}{dx}\right) \cdots \left(a_r + x\frac{d}{dx}\right) F\left(\begin{matrix} a_1 \ldots a_r \\ b_1 \ldots b_s \end{matrix} \middle/ x\right)$$

$$=$$

$$\left(b_1 + x\frac{d}{dx}\right)\left(b_2 + x\frac{d}{dx}\right) \cdots \left(b_s + x\frac{d}{dx}\right) \frac{d}{dx} F\left(\begin{matrix} a_1 \ldots a_r \\ b_1 \ldots b_s \end{matrix} \middle/ x\right)$$

pf WLOG $b_1 \ldots b_s \neq 0$ else both sides 0

$$LHS = F\left(\begin{matrix} a_1+1, a_2+1, \ldots, a_r+1 \\ b_1\, b_2 \ldots b_s \end{matrix} \middle/ x\right) a_1 \cdots a_r$$

$$RHS = \left(b_1 + x\frac{d}{dx}\right)\left(b_2 + x\frac{d}{dx}\right) \cdots \left(b_s + x\frac{d}{dx}\right) \frac{a_1 \cdots a_r}{b_1 \cdots b_s} F\left(\begin{matrix} a_1+1, \ldots a_r+1 \\ b_1+1 \ldots b_s+1 \end{matrix} \middle/ x\right)$$

$$= F\left(\begin{matrix} a_1+1 \ldots a_r+1 \\ b_1 \ldots b_s \end{matrix} \middle/ x\right) a_1 \cdots a_r$$

th 93 (Gauss)

$$\left(x(1-x)\frac{d^2}{dx^2} + \left(c - x(1+a+b)\right)\frac{d}{dx} - ab \right) {}_2F_1\left(\begin{matrix} a\, b \\ c \end{matrix} \middle/ x\right) = 0$$

pf By th 92

$$\left(\left(a + x\frac{d}{dx}\right)\left(b + x\frac{d}{dx}\right) - \left(c + x\frac{d}{dx}\right)\frac{d}{dx} \right) {}_2F_1\left(\begin{matrix} a\, b \\ c \end{matrix} \middle/ x\right) = 0$$

Now elim all instances of $\frac{d}{dx}x$ using

$$\frac{d}{dx}x - x\frac{d}{dx} = 1$$

$\square$