

General orthog polynomials

We assume $\mathbb{F} = \mathbb{R}$

We need a fact about pos def matrices:

Fix integer $n > 0$

Let $M =$ real symmetric $n \times n$ matrix

Recall the eigenvalues of M are in \mathbb{R}

M called pos def whenever these equals > 0

In this case $\det M > 0$

Let V denote a vector space over \mathbb{R} with $\dim n$

Fix a basis $\{v_i\}_{i=1}^n$ for V

Define a bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ s.t

$$\langle v_i, v_j \rangle = M_{ij} \quad \text{for } 1 \leq i, j \leq n$$

$\langle \cdot, \cdot \rangle$ sym.

Recall $\langle \cdot, \cdot \rangle$ is pos def whenever

$$\langle u, u \rangle > 0 \quad \forall u \neq 0 \in V$$

TFAE (ex)

(i) M is pos def

(ii) $\langle \cdot, \cdot \rangle$ is pos def

(iii) $\exists N \in \text{Mat}_n(\mathbb{R})$ s.t. $M = N^t N$

(iv) \exists basis $\{u_i\}_{i=1}^n$ for V s.t. $\langle u_i, u_j \rangle = \delta_{ij}$
for $1 \leq i, j \leq n$. "orthonormal basis"
wrt $\langle \cdot, \cdot \rangle$

Thm 34 With above notation TFAE

(i) M is pos def

(ii) $\exists t \leq n$

$$\det \left((M_{ij})_{1 \leq i, j \leq t} \right) > 0$$

"principal minor"

pf (i) \rightarrow (ii)

$\langle \cdot, \cdot \rangle$ is pos def on V

So $\exists t \leq n$ the restriction of $\langle \cdot, \cdot \rangle$ to $\text{Span}(v_i)_{i=1}^t$

is pos def.

So $(M_{ij})_{1 \leq i, j \leq t}$ is pos def

So $\det((M_{ij})_{1 \leq i, j \leq t}) > 0$

(ii) \rightarrow (i) Ind on n

n=1 clear

n ≥ 2 : By ind

$(M_{ij})_{1 \leq i, j \leq n-1}$ is pos def

Define $W = \text{Span}(v_i)_{i=1}^{n-1}$

Restr of $\langle \cdot, \cdot \rangle$ to W is pos def

So \exists basis $\{u_i\}_{i=1}^{n-1}$ for W s.t.

$$\langle u_i, u_j \rangle = \delta_{ij} \quad 1 \leq i, j \leq n-1$$

$\det M \neq 0$ so $\langle \cdot, \cdot \rangle$ nondeg on V

6

Define $W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \forall w \in W\}$

$\dim W^\perp = 1$ since $\langle \cdot, \cdot \rangle$ nondeg

$W \cap W^\perp = 0$ since $\langle \cdot, \cdot \rangle$ is pos def on W

Now $V = W + W^\perp$ (orthog dis sum)

Pick $0 \neq v \in W^\perp$

Put $a = \langle v, v \rangle$

show $a > 0$

Obs $u_1, u_2, \dots, u_{n-1}, v$ is basis for V

Matrix of inner products is

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & a \end{array} \right)$$

Let S denote transition matrix from orig basis $\{v_i\}_{i=1}^n$ to above basis. Then

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & a \end{array} \right) = S^t M S$$

Apply det

$$a = (\det S)^2 \det M$$

\downarrow \downarrow

> 0 > 0

$$\exists b \in \mathbb{R} \quad b^2 = a$$

Define

$$u_n = \frac{v}{b}$$

So

$$\langle u_n, u_n \rangle = \frac{\langle v, v \rangle}{a} = 1$$

Now $\{u_i\}_{i=1}^n$ is an orthonormal basis for V w.r.t. $\langle \cdot, \cdot \rangle$

So M is pos def. □

For general orthogonal polynomials there are 3 natural starting points:

(i) 3-term recurrence

(ii) the bilinear form

(iii) moments

(i) \rightarrow (ii) Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ s.t.

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0, 1, 2, \dots$$

$$p_0 = 1 \quad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \quad n=1, 2, \dots$$

Obs for $n=0, 1, 2, \dots$

p_n has degree n

$$\text{coeff of } x^n \text{ is } \frac{1}{b_0 b_1 \dots b_{n-1}}$$

Obs

$\{p_n\}_{n=0}^{\infty}$ is a basis for $\mathbb{R}[x]$

Fix $u_0 \in \mathbb{R}$ $u_0 > 0$

Define a bilinear form

$$\langle \cdot, \cdot \rangle : \mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}$$

such that

$$\langle p_n, p_m \rangle = \delta_{nm} u_0 \frac{c_0 c_2 \dots c_n}{b_0 b_1 \dots b_{n-1}} \quad 0 \leq n, m < \infty$$

By constr $\langle \cdot, \cdot \rangle$ is symmetric and positive def.

Basis $\{p_n\}_{n=0}^{\infty}$ is orthogonal w.r.t $\langle \cdot, \cdot \rangle$ and

$$\|p_n\|^2 = \frac{c_n}{b_{n-1}} \|p_{n-1}\|^2 \quad n = 1, 2, \dots$$

$$\|p_0\|^2 = u_0$$

LEM 35 The above form $\langle \cdot, \cdot \rangle$ is A -invariant.

In other words

$$\langle xf, g \rangle = \langle f, xg \rangle \quad \forall f, g \in \mathbb{R}[x]$$

pf wlog

$$f = p_n$$

$$g = p_m$$

??

$$\langle x_{p_n}, p_m \rangle = \langle c_n p_{n-1} + a_n p_n + b_n p_{n+1}, p_m \rangle$$

*

$$\langle p_n, x_{p_m} \rangle = \langle p_n, c_m p_{m-1} + a_m p_m + b_m p_{m+1} \rangle$$

**

Case $|n-m| \geq 2$: *, ** both 0

Case $n=m$: * = ** ✓

Case $|n-m|=1$: wlog $m=n-1$

$$* = c_n \|p_{n-1}\|^2$$

$$** = b_{n-1} \|p_n\|^2$$

So * = **

□

(ii) \rightarrow (iii)

Given symmetric, pos definite, A -invariant bilinear form

$$\langle \cdot, \cdot \rangle : \mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}$$

For $n=0,1,2,\dots$ define

$$u_n = \langle x^n, 1 \rangle$$

Since $\langle \cdot, \cdot \rangle$ is A -inv

$$\langle x^i, x^j \rangle = u_{i+j} \quad 0 \leq i, j < \infty$$

So for $n=0,1,2,\dots$ the Hankel matrix

$$\begin{pmatrix} u_0 & u_1 & u_2 & \dots & u_n \\ u_1 & u_2 & & & \vdots \\ u_2 & & & & \\ \vdots & & & & \\ u_n & \dots & & & u_{2n} \end{pmatrix} \quad *$$

is pos def.

Define D_n to be the det of $*$. Then

$$D_n > 0$$

$$n=0,1,2,\dots$$

(iii) \rightarrow (ii)

12

Given a sequence of real numbers $\{u_n\}_{n=0}^{\infty}$

For $n=0,1,2,\dots$ we define the n th Hankel matrix
as in * and let D_n denote its det.

Assume $D_n > 0$ $n=0,1,2,\dots$

Define an \mathbb{R} -linear map

$$\phi: \mathbb{R}[x] \rightarrow \mathbb{R}$$

set

$$\phi(x^n) = u_n \quad \text{for } n=0,1,2,\dots$$

Define

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{R}[x] \times \mathbb{R}[x] &\rightarrow \mathbb{R} \\ f \quad g &\rightarrow \phi(fg) \end{aligned}$$

Then $\langle \cdot, \cdot \rangle$ is a bilinear form which is

symmetric and A -inv

LEM 36 Above bil form \langle, \rangle is pos def.

pf Given $0 \neq f \in \mathbb{R}[x]$ show $\|f\|^2 > 0$

Write

$$f = a_0 + a_1 x + \dots + a_n x^n$$

$$\|f\|^2 = (a_0, \dots, a_n) \begin{pmatrix} \text{Hankel}_n \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$$

show Hankel_n is pos def

Its principle minors are

$$D_0, D_1, \dots, D_n$$

and hence all positive.

Now Hankel_n is pos def by

thm 34

□

(ii) \rightarrow (i) Given a symmetric, pos def, A -inv 14

bilinear form

$$\langle \cdot, \cdot \rangle : \mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}$$

Pick any sequence of nmo real numbers $\{b_n\}_{n=0}^{\infty}$, ($b_0=1$)

Define

$$x_n = \frac{x}{b_n} \quad n=0,1,2,\dots$$

Apply Gram-Schmidt to the basis

$$\{x_0, x_1, x_2, \dots, x_n, \dots\}_{n=0}^{\infty}$$

to get a sequence of polys $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$

By construction, for $n=0,1,2,\dots$

p_n has degree exactly n

$$\text{coeff of } x^n \text{ in } p_n \text{ is } \frac{1}{b_0 b_1 \dots b_n}$$

Also

$$\langle p_n, p_m \rangle = 0 \quad \text{if } n \neq m \quad 0 \leq n, m < \infty$$

$$\|p_n\|^2 > 0 \quad n=0,1,2,\dots$$

LEM 37 The above polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a 3-term recurrence

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0,1,2,\dots \quad (*)$$

($p_{-1} = 0$)

where

$$c_n b_{n+1} > 0 \quad n=1,2,\dots \quad (**)$$

pf Write $x p_n$ as lin comb of p_0, p_1, p_2, \dots

$$x p_n = \sum_{i=0}^{n+1} d_i p_i$$

In LHS coef of x^{n+1} is $\frac{1}{b_0 b_1 \dots b_{n+1}}$

-- RHS -- -- is $\frac{d_{n+1}}{b_0 b_1 \dots b_n}$

So $d_{n+1} = b_n$

Show $d_i = 0$ ($0 \leq i < n+1$)

$$\underbrace{\langle p_i, x p_n \rangle}_{\substack{|| \\ d_i || p_i ||^2 \\ \vee \\ 0}} = \underbrace{\langle x p_i, p_n \rangle}_{\substack{|| \\ 0}} \quad \begin{array}{l} \in \text{Span}(p_0, p_1, \dots, p_{i-1}) \\ \text{use } i < n+1 \end{array}$$

So $d_i = 0$

We now have (*),

show (**):

$$\langle x p_{n-1}, p_n \rangle = \langle p_{n-1}, x p_n \rangle$$

Expand each side using (*). Get

$$b_{n-1} \underbrace{|| p_n ||^2}_{\neq 0} = c_n \underbrace{|| p_{n-1} ||^2}_{\neq 0}$$

c_n, b_{n-1}
have same
sign

□

Continue to discuss general orthog polynomials

Thm 38 Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ that satisfy a 3-term recurrence

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0,1,2,\dots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \quad n=1,2,\dots$$

Let $\{u_n\}_{n=0}^{\infty}$ and $\{D_n\}_{n=0}^{\infty}$ denote the corresp moments and Hankel determinants. Then

$$(i) \quad p_n = \frac{1}{b_0 b_1 \dots b_{n-1}} \frac{1}{D_n} \det$$

$$\begin{pmatrix} u_0 & u_1 & \dots & u_n \\ u_1 & & & \vdots \\ \vdots & & & \\ u_{n-1} & \dots & & u_{2n-1} \\ 1 & x & x^2 & \dots & x^n \end{pmatrix}$$

$$n=1,2,\dots$$

$$(ii) \quad D_n / D_{n-1} = u_0 c_1 c_2 \dots c_n b_0 b_1 \dots b_{n-1} \quad n=1,2,\dots$$

$$(iii) \quad D_n = u_0^{n+1} (c_1 b_0)(c_2 b_1) \dots (c_n b_{n-1})^2 (c_n b_n) \quad n=0,1,2,\dots$$

$$n=0,1,2,\dots$$

pf (i) Similar to pf of Thm 33

(ii) Assume $n \geq 1$ else trivial.

2

By (i)

$$\langle p_n, x^n \rangle = \frac{1}{b_0 b_1 \dots b_{n-1}} \frac{D_n}{D_{n-1}}$$

By our earlier discussion

$$\begin{aligned} u_0 \frac{c_1 c_2 \dots c_n}{b_0 b_1 \dots b_{n-1}} &= \|p_n\|^2 \\ &= \langle p_n, p_n \rangle \\ &= \left\langle p_n, \frac{x^n}{b_0 b_1 \dots b_{n-1}} + L \text{ terms} \right\rangle \\ &= \frac{\langle p_n, x^n \rangle}{b_0 b_1 \dots b_{n-1}} \end{aligned}$$

Therefore

$$\frac{D_n}{D_{n-1}} = u_0 c_1 c_2 \dots c_n b_0 b_1 \dots b_{n-1}$$

(iii) By (ii)

□

More on pos def matrices

Fix integer $n \geq 1$

Fix $M \in \text{Mat}_n(\mathbb{R})$

Consider condition

(SDD) For $1 \leq i \leq n$

$$M_{ii} > \sum_{\substack{1 \leq j \leq n \\ i \neq j}} |M_{ij}|$$

"strict diagonal dominance"

LEM 39 With above notation assume M is SDD
 let $\theta \in \mathbb{C}$ denote an equal of M (i.e. root
 of char. polynomial of M).

Then the real part

$$\text{Re}(\theta) > 0$$

$$\left[\begin{array}{l} \theta = a + bi \quad a, b \in \mathbb{R} \quad i^2 = -1 \\ \text{Re}(\theta) = a, \quad \text{Im}(\theta) = b \end{array} \right]$$

pf Pick $0 \neq v \in \mathbb{C}^n$ s.t. $Mv = \theta v$

Write $v = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$

Pick i ($1 \leq i \leq n$) s.t.

$$|d_i| \geq |d_j| \quad (1 \leq j \leq n)$$

Obs: $|d_i| > 0$ since $v \neq 0$

In $Mv = \theta v$ consider row i

$$\sum_{j=1}^n M_{ij} d_j = \theta d_i$$

Rewrite as

$$(\theta - M_{ii}) d_i = \sum_{\substack{j=1 \\ j \neq i}}^n M_{ij} d_j$$

Now

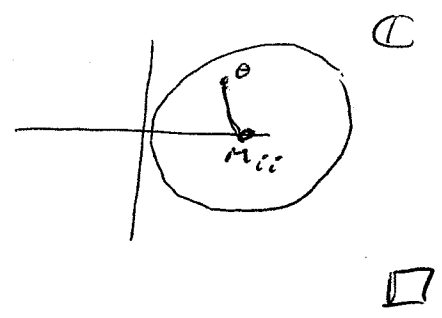
$$\begin{aligned}
|\theta - M_{ii}| |d_i| &= |(\theta - M_{ii}) d_i| \\
&= \left| \sum_{\substack{j=1 \\ j \neq i}}^n M_{ij} d_j \right| \\
&\leq \sum_{\substack{j=1 \\ j \neq i}}^n |M_{ij}| |d_j| \\
&\leq \sum_{\substack{j=1 \\ j \neq i}}^n |M_{ij}| |d_i| \\
&< M_{ii} |d_i|
\end{aligned}$$

So

$$|\theta - M_{ii}| < M_{ii}$$

So

$$\operatorname{Re}(\theta) > 0$$



thm 40 Given $M \in \text{Mat}_n(\mathbb{R})$

Assume M is symmetric and SDD

Then M is pos def.

pf For each eigenvalue θ of M

$\theta \in \mathbb{R}$ since M is symmetric

$\therefore \theta = \text{Re}(\theta) > 0$ by LEM 39.

Now M is pos def.



[wont use th 40 right away]

Next goal: Christoffel - Darboux identity 6

Given polynomial sequence $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ s.t.

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0,1,2,\dots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n+1} \neq 0 \quad n=1,2,\dots$$

\mathbb{F} arb

Write

$$z_n = \frac{c_1 c_2 \dots c_n}{b_0 b_1 \dots b_{n-1}} \quad n=0,1,2,\dots$$

For $n=0,1,2,\dots$ find

$$\sum_{i=0}^n \frac{p_i(x) p_i(y)}{z_i}$$

$[x, y \text{ commuting variables}]$

View

$$\begin{aligned} \sum_{i=0}^n \frac{p_i(x) p_i(y)}{z_i} &= \frac{1}{x-y} \sum_{i=0}^n \frac{(x-y) p_i(x) p_i(y)}{z_i} \\ &= \frac{1}{x-y} \sum_{i=0}^n \left((c_i p_{i+1}(x) + a_i p_i(x) + b_i p_{i-1}(x)) \frac{p_i(y)}{z_i} \right. \\ &\quad \left. - \frac{p_i(x)}{z_i} (c_i p_{i+1}(y) + a_i p_i(y) + b_i p_{i-1}(y)) \right) \end{aligned}$$

$$= \frac{1}{x-y} \left(\sum_{i=1}^n p_{i-1}(x) p_i(y) \underbrace{\left(\frac{c_i}{z_i} - \frac{b_{i-1}}{z_{i-1}} \right)}_{u_0} + \sum_{i=1}^n p_i(x) p_{i-1}(y) \underbrace{\left(\frac{b_{i-1}}{z_{i-1}} - \frac{c_i}{z_i} \right)}_{u_0} - p_n(x) p_{n+1}(y) \frac{b_n}{z_n} + p_{n+1}(x) p_n(y) \frac{b_n}{z_n} \right)$$

$$= \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x-y} \frac{b_n}{z_n}$$

$$\left[\begin{aligned} x p_0 &= a_0 p_0 + b_0 p_1 & p_0 &= 1 \\ p_1 &= \frac{x-a_0}{b_0} \\ p_1(x) - p_1(y) &= \frac{x-y}{b_0} \\ \frac{b_n}{(x-y) z_n} &= \frac{1}{p_1(x) - p_1(y)} \frac{b_n}{b_0} \frac{1}{z_n} \\ &= \frac{1}{p_1(x) - p_1(y)} \frac{b_1 b_2 \dots b_n}{c_1 c_2 \dots c_n} \end{aligned} \right]$$

$$= \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{p_1(x) - p_1(y)} \frac{b_1 b_2 \dots b_n}{c_1 c_2 \dots c_n}$$

Thm 41 (Christoffel-Darboux)

with above notation

For $n = 0, 1, 2, \dots$

$$(i) \quad \sum_{i=0}^n p_i(x) p_i(y) \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i}$$

=

$$\frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{p_n(x) - p_n(y)} \frac{b_1 b_2 \dots b_n}{c_1 c_2 \dots c_n}$$

$$(ii) \quad \sum_{i=0}^n p_i(x)^2 \frac{b_0 b_1 \dots b_{i-1}}{c_1 c_2 \dots c_i}$$

$$= \left(p_{n+1}'(x) p_n(x) - p_n'(x) p_{n+1}(x) \right) \frac{b_0 b_1 \dots b_n}{c_1 c_2 \dots c_n}$$

$$(f' = Df)$$

pf (i) By disc above thm.

(ii) In (i) write $y = x+h$, simplify/cancel and

then let $h \rightarrow 0$

□

LEM 42. Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$

Assume

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0, 1, 2, \dots$$

$$c_n b_{n+1} \neq 0 \quad n=1, 2, \dots$$

Given arbitrary

$$0 \neq \gamma_n \in \mathbb{F} \quad n=0, 1, 2, \dots$$

define

$$\tilde{p}_n = \frac{p_n}{\gamma_0 \gamma_1 \dots \gamma_{n-1}} \quad n=0, 1, 2, \dots$$

Then

$$x \tilde{p}_n = \tilde{c}_n \tilde{p}_{n+1} + \tilde{a}_n \tilde{p}_n + \tilde{b}_n \tilde{p}_{n-1} \quad n=0, 1, 2, \dots$$

where

$$\tilde{c}_n = c_n \gamma_n \quad n=1, 2, \dots$$

$$\tilde{a}_n = a_n \quad n=0, 1, 2, \dots$$

$$\tilde{b}_n = \frac{b_n}{\gamma_n} \quad n=0, 1, 2, \dots$$

Moreover

$$\tilde{c}_n \tilde{b}_{n+1} = c_n b_{n+1} \quad n=1, 2, \dots$$

pf ex

LEM 43

Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $F[x]$

... $\{\tilde{p}_n\}_{n=0}^{\infty}$...

Assume

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0, 1, 2, \dots$$

$$x \tilde{p}_n = \tilde{c}_n \tilde{p}_{n+1} + \tilde{a}_n \tilde{p}_n + \tilde{b}_n \tilde{p}_{n-1} \quad \dots$$

$$c_n b_{n+1} \neq 0, \quad \tilde{c}_n \tilde{b}_{n+1} \neq 0 \quad n=1, 2, \dots$$

TFAE

(i) $\tilde{a}_n = a_n$ for $n=0, 1, 2, \dots$ and

$$\tilde{c}_n \tilde{b}_{n+1} = c_n b_{n+1} \text{ for } n=1, 2, \dots$$

(ii) \tilde{p}_n is a nonzero scalar multiple of p_n
 $n=0, 1, 2, \dots$

pf ex



Continue to discuss general orthog polynomials

$$\mathbb{F} = \mathbb{R}$$

Next goal: interlacing of zeros

Given $0 \neq f \in \mathbb{R}[x]$

factor f over \mathbb{C}

$$f = \alpha \prod_{i=1}^n (x - x_i) \quad \alpha \in \mathbb{R} \quad x_i \in \mathbb{C}$$

A root x_i is simple whenever

$$x_i \neq x_j \quad (1 \leq j \leq n, j \neq i)$$

x_i is simple \Leftrightarrow

$$f'(x_i) \neq 0$$

$$f' = Df$$

DEF 44 Given non-zero $f, g \in \mathbb{R}[x]$

Assume $\deg f, \deg g$ differ by 1

wlog $\deg f = \deg g + 1$

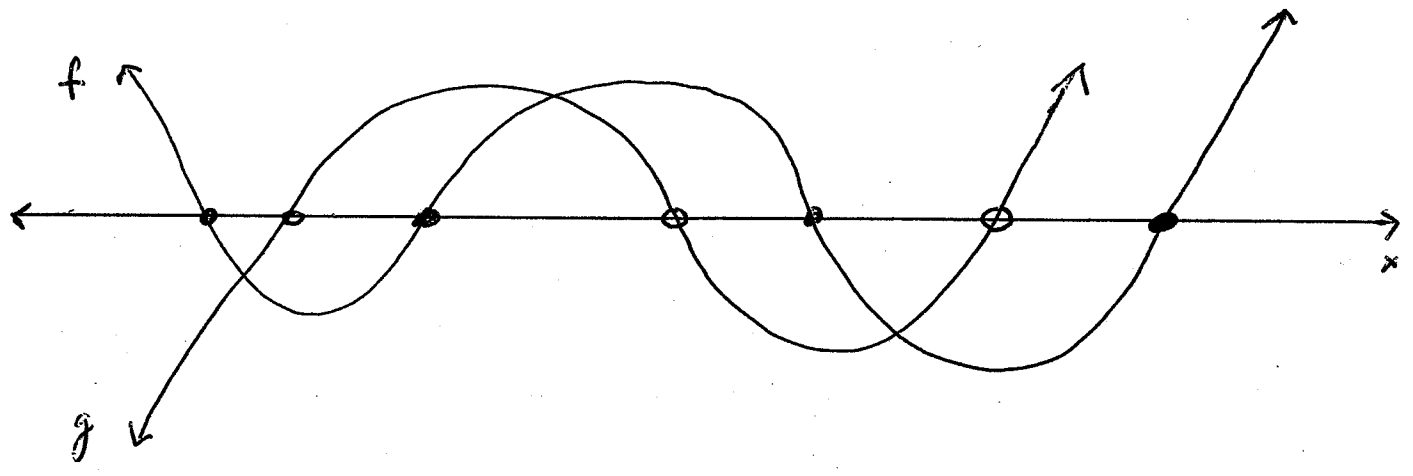
We say the roots of f, g interlace whenever

(i) For f and g all roots are simple and real

(ii) For $1 \leq i \leq \deg g$

$$(i)^{\text{th}} \text{ Largest root of } f < i^{\text{th}} \text{ Largest root of } g < (i+1)^{\text{th}} \text{ Largest root of } f$$

ex



Recall from Calculus

Given $f \in \mathbb{R}[x]$

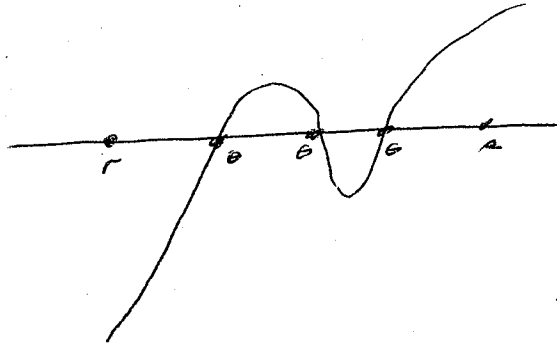
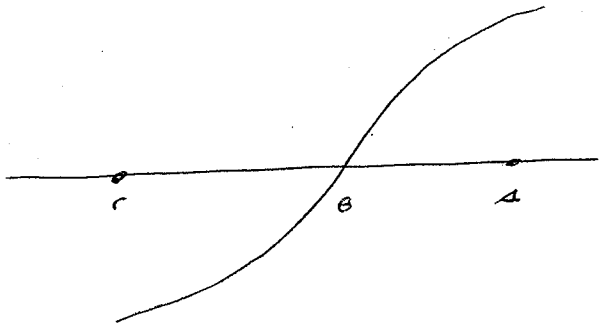
Given $r, a \in \mathbb{R}$ ($r < a$)

Suppose

$$f(r)f(a) < 0$$

then $\exists \theta \in \mathbb{R}$ such that

$$r < \theta < a \text{ and } f(\theta) = 0$$



etc.

Given $0 \neq f \in \mathbb{R}[x]$ $n = \deg f$

Assume all roots of f are simple and real

Define $x_i = i$ th largest root of f ($1 \leq i \leq n$)

LEM 45 Given $f, \{x_i\}_{i=1}^n$ as above

Given monic $g \in \mathbb{R}[x]$ with $\deg g = n \mp 1$

(i) Suppose $\deg g = n-1$. Then the roots of f, g
interlace \Leftrightarrow

$$(-1)^i g(x_i) < 0 \quad 1 \leq i \leq n$$

(ii) Suppose $\deg g = n+1$. Then the roots of f, g interlace

\Leftrightarrow

$$(-1)^i g(x_i) > 0 \quad 1 \leq i \leq n$$

pt ex

Thm 46 Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$

such that

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0, 1, 2, \dots \quad (*)$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n b_{n-1} > 0 \quad n=1, 2, \dots$$

Then the roots of p_n, p_{n+1} interlace for $n=0, 1, 2, \dots$

pf WLOG the p_n are monic. So

$$b_n = 1 \quad n=0, 1, 2, \dots$$

$$c_n > 0 \quad n=1, 2, \dots$$

Ind on n

$n=0$ ✓

$n \geq 1$: By ind its p_{n-1}, p_n interlace

So roots of p_n are simple and real

define $x_i = i$ th largest root of $p_n \quad 1 \leq i \leq n$

By LEM 45(ii) suffices to show

$$(-1)^i p_{n+1}(x_i) > 0 \quad 1 \leq i \leq n$$

Let i be given. By LEM 45(i)

$$(-1)^i p_{n-1}(x_i) < 0$$

Apply each side of (*) to x_i and use $p_n(x_i) = 0$:

$$0 = \underbrace{c_n p_{n+1}(x_i)}_+ + p_{n-1}(x_i)$$

... have opp sign. So $(-1)^i p_{n+1}(x_i) > 0$

Until further notice \mathbb{F} arb

Notation For $d = 0, 1, 2, \dots$

$\text{Mat}_d(\mathbb{F})$ denotes the \mathbb{F} -algebra of all $(d+1) \times (d+1)$ matrices with entries in \mathbb{F}

We index the rows/cols by $0, 1, 2, \dots, d$.

\mathbb{F}^{d+1} denotes the \mathbb{F} -vector space of all $(d+1) \times 1$ matrices with entries in \mathbb{F} .

Index rows by $0, 1, \dots, d$.

obs. $\text{Mat}_d(\mathbb{F})$ acts on \mathbb{F}^{d+1} by left mult.

So

$$\frac{\det(xI - T_n)}{b_0 b_1 \dots b_{n-1}} = (x - a_n) \frac{\det(xI - T_{n-1})}{b_0 b_1 \dots b_{n-2}} - \frac{c_n b_{n-1} \det(xI - T_{n-2})}{b_0 b_1 \dots b_{n-1}}$$

So

$$b_n \tilde{p}_{n+1} = (x - a_n) \tilde{p}_n - c_n \tilde{p}_{n-1}$$

Given (**)

Now $p_{n+1} = \tilde{p}_{n+1}$ by routine induction on n . □

COR 49

With above notation, for $n=0, 1, 2, \dots$

the zeros of p_{n+1} are the eigenvalues of T_n .

pf By Thm 48 p_{n+1} is scalar multiple of the

char poly of T_n □

Ref to thm 48 the matrix T_n not symmetric

However we do have the following.

For $n = 0, 1, 2, \dots$ define

$$k_n = \frac{b_0 b_1 \dots b_n}{c_1 c_2 \dots c_n}$$

So

$$k_0 = 1$$

$$k_n c_n = k_{n+1} b_{n+1}$$

$$1 \leq n < \infty$$

Define matrix

$$K_n = \text{diag}(k_0, k_1, \dots, k_n)$$

$$n = 0, 1, 2, \dots$$

LEM 50 With above notations for $n = 0, 1, 2, \dots$

$$K_n T_n \text{ is symmetric.}$$

In other words

$$T_n^t = K_n T_n K_n^{-1}$$

pf $K_n T_n$ is tri-diagonal

$$(K_n T_n)_{i,j} \stackrel{?}{=} (K_n T_n)_{j,i}$$

"

"
 $k_i b_i$

$k_i c_i$

DEF 51 A matrix $M \in \text{Mat}_n(\mathbb{F})$

is symmetrizable whenever \exists diagonal matrix

$\Delta \in \text{Mat}_n(\mathbb{F})$ st.

$\Delta M \Delta^{-1}$ is symmetric.

LEM 52 With above notation assume $\mathbb{F} = \mathbb{R}$

and

$$c_n b_n > 0$$

$$n = 0, 1, 2, \dots$$

Then T_n is symmetrizable for $n = 0, 1, 2, \dots$

pf

Obs $k_n > 0$ so

$\sqrt{k_n}$ exists in \mathbb{R}

Define

$$\Delta_n = \text{diag}(\sqrt{k_0}, \sqrt{k_1}, \dots, \sqrt{k_n})$$

$$\text{So } \Delta_n^2 = K_n$$

Obs

$$\begin{aligned} \Delta_n T_n \Delta_n^{-1} &= \Delta_n^{-1} \Delta_n^2 T_n \Delta_n^{-1} \\ &= \underbrace{\Delta_n^{-1}}_{\text{sym}} \underbrace{(K_n T_n)}_{\text{sym}} \Delta_n^{-1} \\ &= \text{sym} \end{aligned}$$

□

Note 53 With above notation assume

$$F = \mathbb{R}, \quad c_n b_{n-1} > 0 \quad n = 1, 2, \dots$$

We saw earlier the roots of p_{n+1} are real and simple.

Here is another proof:

By 4.48

p_{n+1} = nonzero scalar multiple of char poly of T_n

By Lem 47

char poly of T_n = min poly of T_n

By Lem 52

roots of min poly of T_n are in \mathbb{R} and mutually distinct.

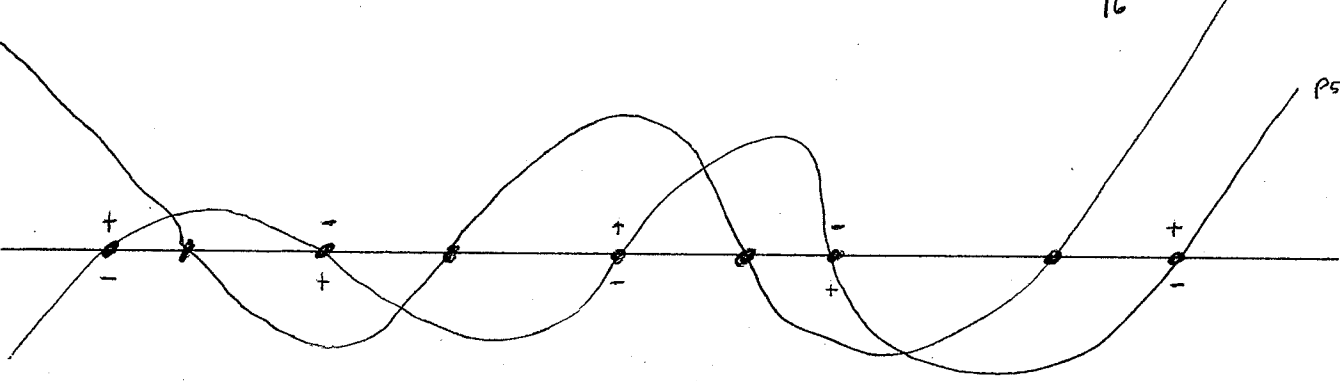
Therefore the zeros of p_{n+1} are real and simple. \square

$$x p_n = c_n p_{n-1} + a_n p_n + b_n p_{n+1}$$

$$b_n = 1$$

$$c_n > 0$$

n	p_n
0	1
1	$\frac{x - a_0}{b_0}$
2	$\frac{x^2 - x(a_0 + a_1) + a_0 a_1 - b_0 c_1}{b_0 b_1}$



The interlacing argument in Note 53 is a special case of the following

Cauchy interlacing theorem

Fix integer $n \geq 1$

Fix $M \in \text{Mat}_n(\mathbb{C})$

Assume

conj-transpose

$$\overline{M}^t = M$$

"Hermitian"

Recall eigs of M are real.

List in order

$$\lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_2 \leq \lambda_1$$

In M remove any row and corresp col to get a principle submatrix H

Obs H is Hermitian; List eigs

$$\mu_n \leq \mu_{n-1} \leq \dots \leq \mu_2 \leq \mu_1$$

Thm 54 (Cauchy)

With above notation

$$\lambda_n \leq \mu_{n-1} \leq \lambda_{n-1} \leq \dots \leq \lambda_2 \leq \mu_1 \leq \lambda_1$$

pf

WLOG

$$M = \left(\begin{array}{c|c} H & y \\ \hline \bar{y}^t & a \end{array} \right)$$

$$a \in \mathbb{R}$$

$$y \in \mathbb{C}^{n \times 1}$$

Since H is Hermitian,

$$\exists U \in \text{Mat}_{n \times n}(\mathbb{C}) \quad \text{s.t.}$$

$$\bar{U}^t U = I$$

"unitary"

and

$$U^t H U = \underbrace{\text{diag}(u_1, u_2, \dots, u_n)}_D$$

Define

$$w = U^t y$$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

Special case: Assume $u_{n-1} < u_{n-2} < \dots < u_2 < u_1$
 and $w_i \neq 0 \quad 1 \leq i \leq n-1$

Define

$$V = \left(\begin{array}{c|c} u & 0 \\ \hline 0 & 1 \end{array} \right)$$

Obs

$$\bar{V}^t V = I$$

and

$$\begin{aligned}
 V^{-1} M V &= \left(\begin{array}{c|c} u^t & 0 \\ \hline 0 & 1 \end{array} \right) \left(\begin{array}{c|c} H & y \\ \hline \bar{q}^t & a \end{array} \right) \left(\begin{array}{c|c} u & 0 \\ \hline 0 & 1 \end{array} \right) \\
 &= \left(\begin{array}{c|c} H u & y \\ \hline \bar{q}^t u & a \end{array} \right) \\
 &= \left(\begin{array}{c|c} u^t H u & u^t y \\ \hline \bar{q}^t u & a \end{array} \right) \\
 &= \left(\begin{array}{c|c} D & w \\ \hline \bar{w}^t & a \end{array} \right)
 \end{aligned}$$

Let $f(x) = \text{char poly of } M$

So
$$f(x) = \prod_{i=1}^n (x - \lambda_i)$$

Also

$$f(x) = \det(xI - M)$$

$$= \det(xI - V^{-1}MV)$$

$$= \det \left(\begin{array}{c|c} xI - D & -w \\ \hline -\bar{w}^t & x-a \end{array} \right)$$

[expand along last row]

$$= (x-a) \prod_{i=1}^{n-1} (x - u_i) - \sum_{i=1}^{n-1} |w_i|^2 (x - u_1) \dots (x - u_{i-1})(x - u_{i+1}) \dots (x - u_{n-1})$$

So for $1 \leq i \leq n-1$

$$f(u_i) = -|w_i|^2 \prod_{\substack{1 \leq j \leq n-1 \\ j \neq i}} (u_i - u_j)$$

$$(-1)^i f(u_i) > 0 \quad 1 \leq i \leq n-1$$

Now by LEM 45 (ii)

$$\lambda_n < u_{n-1} < \lambda_{n-1} < \dots < \lambda_2 < u_1 < \lambda_1$$

done for special case

General case

We take limits

Define a sequence $\{\epsilon_r\}_{r=1}^{\infty}$ s.t.

$$\epsilon_r \in \mathbb{R} \setminus \{-w_1, -w_2, \dots, -w_{n+1}\}, \quad 1 \leq r < \infty$$

$$\epsilon_r > 0$$

$$1 \leq r < \infty$$

$$\lim_{r \rightarrow \infty} \epsilon_r = 0$$

For $1 \leq i \leq n+1$ and $1 \leq r < \infty$ def

$$\mu_i^{(r)} = \mu_i - i \epsilon_r$$

obs

$$\mu_{n+1}^{(r)} < \mu_{n+2}^{(r)} < \dots < \mu_2^{(r)} < \mu_1^{(r)}$$

For $1 \leq i \leq n+1$ and $1 \leq r < \infty$ def

$$w_i^{(r)} = w_i + \epsilon_r$$

obs

$$w_i^{(r)} \neq 0$$

$$\omega^{(r)} = \begin{pmatrix} \omega_1^{(r)} \\ \omega_2^{(r)} \\ \vdots \\ \omega_n^{(r)} \end{pmatrix}$$

$$y^{(r)} = U \omega^{(r)}$$

$$D^{(r)} = \text{diag} \left(\mu_1^{(r)}, \mu_2^{(r)}, \dots, \mu_n^{(r)} \right)$$

$$H^{(r)} = U D^{(r)} U^T$$

$$M^{(r)} = \left(\begin{array}{c|c} H^{(r)} & y^{(r)} \\ \hline \overline{y^{(r)}}^T & a \end{array} \right)$$

$M^{(r)}$ is Herm? List equals

$$\lambda_n^{(r)} \leq \mu_n^{(r)} \leq \dots \leq \lambda_2^{(r)} \leq \lambda_1^{(r)}$$

Applying the special case to $M^{(r)}, H^{(r)}$ we get

$$\lambda_n^{(r)} < \mu_n^{(r)} < \lambda_n^{(r)} < \dots < \lambda_2^{(r)} < \mu_1^{(r)} < \lambda_1^{(r)}$$

Now take limits

$$\lim_{r \rightarrow \infty} \mu_i^{(r)} = \mu_i \quad (1 \leq i \leq n)$$

$$\lim_{r \rightarrow \infty} M^{(r)} = M \quad \text{so}$$

$$\lim_{r \rightarrow \infty} \lambda_i^{(r)} \rightarrow \lambda_i \quad (1 \leq i \leq n)$$

$$\text{So } \lambda_n \leq \mu_n \leq \lambda_n \leq \dots \leq \lambda_2 \leq \mu_1 \leq \lambda_1$$

□

Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{R}[x]$ such that

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n=0,1,2,\dots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n, b_{n-1} > 0 \quad n=1,2,\dots$$

For $n=0,1,2,\dots$ define

$$K_n(x,y) = \sum_{i=0}^n \frac{p_i(x) p_i(y)}{Z_i} \in \mathbb{R}[x,y]$$

$$Z_i = \frac{c_1 c_2 \dots c_i}{b_0 b_1 \dots b_{i-1}}$$

"kernel polynomials"

The kernel polynomials came up in Christoffel-Darboux

We now give another interp

Consider the bilin form $\langle \cdot, \cdot \rangle$ for $\{p_n\}_{n=0}^{\infty}$

Normalize so $u_0 = \langle 1, 1 \rangle = 1$

Recall

$$\langle p_n, p_m \rangle = \delta_{nm} Z_n \quad 0 \leq n, m < \infty$$

Problem

Fix integer $n \geq 0$

Fix $\theta \in \mathbb{R}$

Maximize

$$f(\theta)$$

subject to

$$f \in \mathbb{R}[x],$$

$$\deg f \leq n,$$

$$\langle f, f \rangle = 1.$$

Sol: Write

$$f = \sum_{i=0}^n \alpha_i p_i \quad \alpha_i \in \mathbb{R}$$

$$1 = \langle f, f \rangle$$

$$= (\alpha_0, \alpha_1, \dots, \alpha_n) \underbrace{\begin{pmatrix} z_0 & & 0 \\ & z_1 & \\ 0 & & \ddots \\ & & & z_n \end{pmatrix}}_Z \underbrace{\begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{pmatrix}}_d$$

$$= \alpha^t Z \alpha$$

Maximize

$$f(\theta) = \sum_{i=0}^n \alpha_i p_i(\theta)$$

$$= (\alpha_0, \alpha_1, \dots, \alpha_n) \underbrace{\begin{pmatrix} p_0(\theta) \\ p_1(\theta) \\ \vdots \\ p_n(\theta) \end{pmatrix}}_{p_\theta}$$

$$= \alpha^t p_\theta$$

Obs

$$p_0 \neq 0$$

Since $p_0 = 1$

Problem becomes:

Maximize

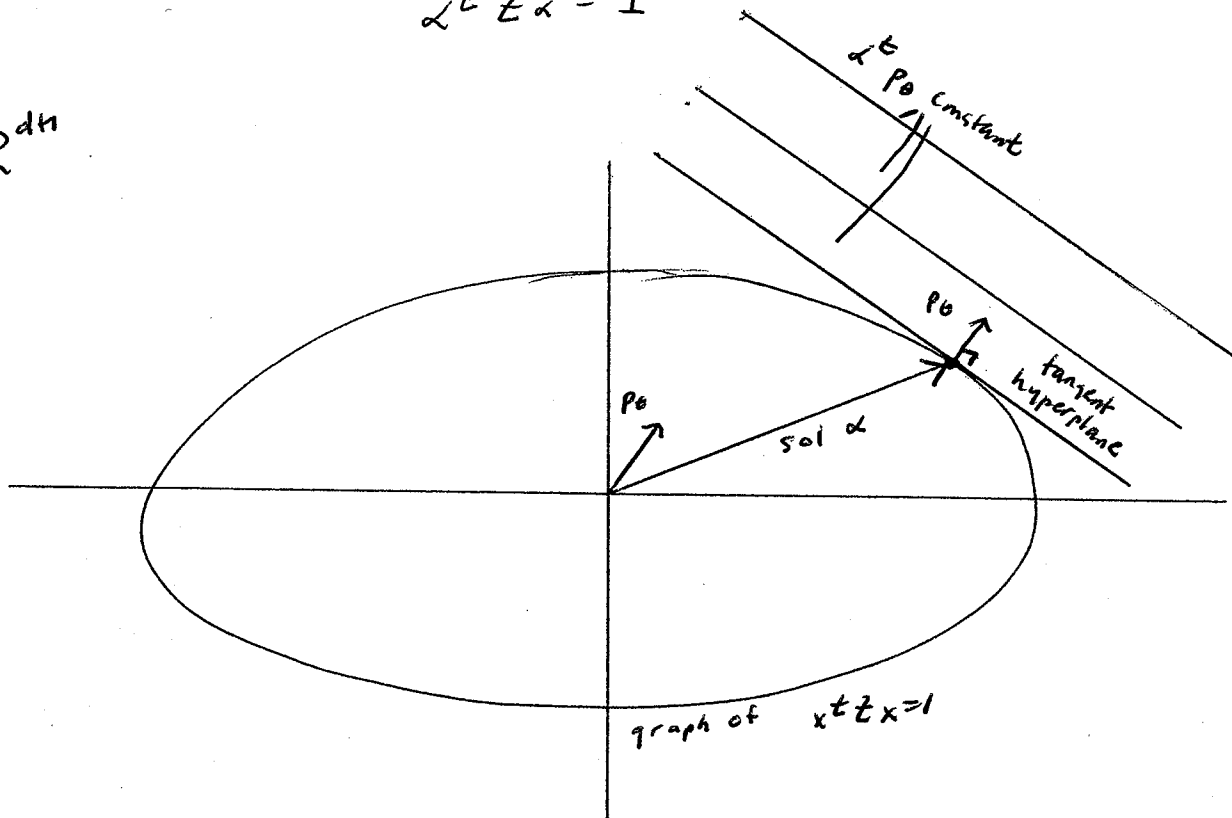
$$\alpha^T p_0$$

subject to

$$\alpha \in \mathbb{R}^{n+1}$$

$$\alpha^T z = 1$$

\mathbb{R}^{n+1}



At sol α in above graph

tangent hyperplane is orthog to p_0

... .. $z \alpha$

So

$z \alpha, p_0$ are lin dep

Since $p_0 \neq 0 \exists \lambda \in \mathbb{R}$ s.t.

10

$$Z\alpha = \lambda p_0$$

so

$$\alpha = \lambda Z^{-1} p_0$$

Find λ :

$$\begin{aligned} 1 &= \alpha^t Z \alpha \\ &= \lambda^2 p_0^t Z^{-1} p_0 \\ &= \lambda^2 \sum_{i=0}^n \frac{p_i(\omega)^2}{z_i} \\ &= \lambda^2 \underbrace{\kappa_n(\theta, \theta)}_{v_0} \end{aligned}$$

so

$$\lambda = \frac{\varepsilon}{\sqrt{\kappa_n(\theta, \theta)}}$$

$$\varepsilon \in \{1, -1\}$$

"pos square root"

Find α (up to sign)

$$\begin{aligned} \alpha &= \lambda Z^{-1} p_0 \\ &= \frac{\varepsilon}{\sqrt{\kappa_n(\theta, \theta)}} Z^{-1} p_0 \end{aligned}$$

find $f(\theta)$

$$f(\theta) = \alpha^t p_\theta$$

$$= \frac{\varepsilon}{\sqrt{\kappa_n(\theta, \theta)}} \underbrace{p_\theta^t Z^{-1} p_\theta}_{\sum_{i=0}^n \frac{p_i(\theta)^2}{z_i}} \quad \text{" } \kappa_n(\theta, \theta)$$

$$= \varepsilon \sqrt{\kappa_n(\theta, \theta)}$$

$f(\theta)$ is maximal at $\varepsilon = 1$

$$f(\theta) = \sqrt{\kappa_n(\theta, \theta)}$$

find f

$$f = \sum_{i=0}^n \alpha_i p_i$$

$$= \frac{1}{\sqrt{\kappa_n(\theta, \theta)}} \underbrace{\sum_{i=0}^n \frac{p_i(\theta) p_i}{z_i}}_{\text{" } \kappa_n(\theta, x)}$$

$$f = \frac{\kappa_n(\theta, x)}{\sqrt{\kappa_n(\theta, \theta)}}$$



Until further notice \mathbb{F} arb

Given polynomials $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ that satisfy

$$x p_n = c_n p_{n+1} + a_n p_n + b_n p_{n-1} \quad n = 0, 1, 2, \dots$$

$$p_0 = 1, \quad p_{-1} = 0$$

$$c_n, b_{n-1} \neq 0 \quad n = 1, 2, \dots$$

So

$$p_0 = 1$$

$$p_1 = \frac{x - a_0}{b_0}$$

$$p_2 = \frac{(x - a_0)(x - a_1) - b_0 c_1}{b_0 b_1}$$

⋮

Since $p_{-1} = 0$ the parameter c_{-1} could be arbitrary —
lets view as indit

Consider more general initial conditions

Problem: Describe $\{f_n\}_{n=0}^{\infty}$ that satisfy

$$x f_n = c_n f_{n+1} + a_n f_n + b_n f_{n-1} \quad n = 0, 1, 2, \dots$$

f_0, f_1 arb

c_n, a_n, b_n as above.

$$n=0: \quad x f_0 = c_0 f_1 + a_0 f_0 + b_0 f_{-1}$$

$$f_1 = \underbrace{\frac{x - a_0}{b_0}}_{\text{call it}} f_0 + \frac{1}{b_0} \frac{-c_0 f_1}{b_0}$$