Fall 2010

Math 805 Special functions B129 VV

11:00 AM MWF

theme: The polynomials of the "Askey scheme"

Hermite, krawtchouk, ..., Askey Wilson

We Consider

- basic structure
- How the polynomials appear in the representations of various algebras
- How they appear in combinatorics

I wont follow any text but the lectures should be self contained. A good reference is [21] which is available free on line.

Contents

General orthogonal polynomials

pos def matrices

3-term recurrence

the bilinear form

moments

Christoffel-Darboux formula

interlacing of zeros

the Cauchy interlacing thm

kernel polynomials

Associated polynomials

Gauss quadrature

Continued fractions

The Krawtchouk polynomials

3-term rec

forward / backward shift operators

Rodrigues type formula

generating functions

orthogonality

the Krawtchouk polys and Lie algebra sl_2

A characterization of the Krawtchouk polys using the notion of a Leonard pair

Krawtchouk polys and Hamming graphs

the terminating branch of Askey-scheme: an approach using the notion of a Leonard system.

Classification of Leonard systems

uniform approach to orthogonality using corresp LS

uniform approach to 3-term rec using corresp LS

Leonard systems of q-Racah type

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the tridiagonal relations and the q-Onsager algebra

the Askey-Wilson relations in $\mathbb{Z}_3$-symmetric form

Connections to the quantum group $U_q sl_2$ in its equitable presentation

Connections to the q-tetrahedron algebra $\boxtimes_q$

Special functions and orthogonal polynomials

References

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Hypergeometric series

$\mathbb{F}$ = field

$\forall\, a \in \mathbb{F}$ define

$$(a)_n = \underbrace{a(a+1)(a+2)\cdots(a+n-1)}_{n \text{ terms}} \qquad n = 0, 1, 2, \ldots$$

By convention

$$(a)_0 = 1$$

Pick nonneg integers r, s and scalars

$$a_1, a_2, \ldots, a_r ; \qquad b_1, b_2, \ldots, b_s \qquad \in \mathbb{F}$$

We define

$$ {}_rF_s\left(\begin{matrix} a_1\ a_2 \cdots a_r \\ b_1\ b_2 \cdots b_s \end{matrix} \,\middle|\, x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \frac{x^n}{n!} \qquad (1)$$

We call (1) a <u>hypergeometric</u> <u>series</u> in the variable x

Notational conventions

Since $\mathbb{F}$ is arbitrary we dont have the notion of a limit.

Hence we always assume that at least one of $a_1, a_2, \ldots, a_r$

is among $0, -1, -2, \ldots$

In this case the series terminates after finitely many steps

Example :

$$ {}_2F_1\left(\begin{matrix}-2 & a \\ & b\end{matrix}\middle/ x\right) = 1 + \frac{-2\ a}{b}\frac{x}{1!} + \frac{-2(-1)\,a(a+1)}{b(b+1)}\frac{x^2}{2!} $$

$$ = 1 - \frac{2a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 $$

Referring to (1), by a <u>trivial term</u> we mean a term where numerator is 0.

Obs

$$ n\text{th term triv} \longrightarrow (n+1)^{st} \text{ term triv} \qquad \forall n $$

We always assume $b_1, b_2, \ldots b_s$ and char(F) are such that the denominator in each nontrivial term is nonzero.

We view (1) as a (finite) sum over all nontrivial terms.

A sum

For an integer $n \geq 0$ find in closed form

$$ {}_2F_1\left({-n \;\; a \atop b} \middle/ 1 \right) $$

Ex

$$ {}_2F_1\left({0 \;\; a \atop b} \middle/ 1 \right) = 1 $$

$$ {}_2F_1\left({-1 \;\; a \atop b} \middle/ 1 \right) = 1 - \frac{a}{b} = \frac{b-a}{b} $$

$$ {}_2F_1\left({-2 \;\; a \atop b} \middle/ 1 \right) = 1 - \frac{2a}{b} + \frac{a(a+1)}{b(b+1)} $$

$$ = \frac{b^2+b - 2a(b+1) + a^2+a}{b(b+1)} $$

$$ = \frac{(b-a)^2 + b - a}{b(b+1)} $$

$$ = \frac{(b-a)(b-a+1)}{b(b+1)} $$

$$ = \frac{(b-a)_2}{(b)_2} $$

LEM 1 (Chu-Vandermonde)

$$ {}_2F_1\left({-n \;\; a \atop b} \middle/ 1 \right) = \frac{(b-a)_n}{(b)_n} \qquad n = 0, 1, 2 \ldots $$

Pf: Induction on n

Before giving a formal pf. I will do the case $n=3$ to give the idea

$$ {}_2F_1\left(\left.{-3,\ a \atop b}\right| 1\right) = 1 - \frac{3a}{b} + \frac{3a_2}{b_2} - \frac{a_3}{b_3} $$

$$ = 1 - \frac{2a}{b} + \frac{a_2}{b_2} \qquad (2) $$

$$ + $$

$$ -\frac{a}{b} + \frac{2a_2}{b_2} - \frac{a_3}{b_3} \qquad (3) $$

Obs (2) equals

$$ {}_2F_1\left(\left.{-2,\ a \atop b}\right| 1\right) $$

$$ = \frac{(b-a)_2}{(b)_2} \qquad (4) $$

by induction.

Obs (3) equals

$$ -\frac{a}{b} + \frac{2a(a+1)}{b(b+1)} - \frac{a(a+1)(a+2)}{b(b+1)(b+2)} $$

$$ = -\frac{a}{b}\left(1 - 2\,\frac{a+1}{b+1} + \frac{(a+1)(a+2)}{(b+1)(b+2)}\right) $$

$$ = -\frac{a}{b}\ {}_2F_1\left(\left.{-2,\ a+1 \atop b+1}\right| 1\right) $$

$$ = -\frac{a}{b}\ \frac{(b-a)_2}{(b+1)_2} \qquad (5) $$

by ind. Adding (4), (5) get

$$ \frac{(b-a)_3}{(b)_3} $$

Formal pf: Recall binomial identity

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

Obs

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}-n & a\\ & b\end{matrix}\middle|\, 1\right) &= \sum_{i=0}^{n} \frac{(-n)_i\, a_i}{b_i}\,\frac{1}{i!} \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{a_i}{b_i} \\
&= \sum_{i=0}^{n \to n-1} (-1)^i \binom{n-1}{i} \frac{a_i}{b_i} + \sum_{\substack{i=0\\ \downarrow\\ 1}}^{n} (-1)^i \binom{n-1}{i-1} \frac{a_i}{b_i} \\
&= \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \frac{a_i}{b_i} - \frac{a}{b} \sum_{i=1}^{n} (-1)^{i-1} \binom{n-1}{i-1} \frac{(a+1)_{i-1}}{(b+1)_{i-1}} \\
&= {}_2F_1\left(\begin{matrix}1-n & a\\ & b\end{matrix}\middle|\, 1\right) - \frac{a}{b}\, {}_2F_1\left(\begin{matrix}1-n & a+1\\ & b+1\end{matrix}\middle|\, 1\right) \\
&= \frac{(b-a)_{n-1}}{(b)_{n-1}} - \frac{a}{b}\,\frac{(b-a)_{n-1}}{(b+1)_{n-1}} \\
&= \frac{(b-a)_{n-1}}{(b)_n}\left(b+n-1-a\right) \\
&= \frac{(b-a)_n}{(b)_n} \qquad \square
\end{aligned}
$$

the Hermite polynomials

until further notice char $\mathbb{F} = 0$

Def 2 For $n = 0, 1, 2 \ldots$ define

$$H_n(x) = {}_2F_0\left(\begin{matrix} \frac{-n}{2} \quad \frac{1-n}{2} \\ - \end{matrix} \,\middle|\, \frac{-1}{x^2}\right)(2x)^n \qquad (6)$$

"nth Hermite poly in variable x"

n	H_n
0	1
1	$2x$
2	$4x^2 - 2$
3	$8x^3 - 12x$
4	$16x^4 - 48x^2 + 12$
$\vdots$	$\vdots$

Expanding (6)

$$H_n(x) = \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} (-1)^r \frac{(2x)^{n-2r}(-n)_{2r}}{r!} \qquad (7)$$

$=$ poly in x with degree n

Since $H_n = H_n(x)$ has degree n

$$\{H_i\}_{i=0}^{n} \quad \text{is basis for} \quad \text{Span}\{x^i\}_{i=0}^{n}$$

So

$$x H_n \in \text{Span}\{H_i\}_{i=0}^{n+1}$$

Find coeffs

check pattern for $n = 0, 1, 2 \ldots$

$$x H_0 = x = \frac{1}{2} H_1$$

$$x H_1 = 2x^2 = \frac{1}{2}(4x^2 - 2) + 1$$

$$= \frac{1}{2} H_2 + H_0$$

$$x H_2 = x(4x^2 - 2)$$

$$= \frac{1}{2}(8x^3 - 12x) + 2(2x)$$

$$= \frac{1}{2} H_3 + 2 H_1$$

LEM 3 The Hermite polynomials satisfy

$$x H_n(x) = \frac{1}{2} H_{n+1}(x) + n H_{n-1}(x) \qquad (8)$$

for $n = 0, 1, 2 \ldots$ where $H_{-1}(x) = 0$

pf Routine comparison of coeffs of like powers of x on either side of (8), using (7) $\square$

We now consider

$$\frac{d}{dx} H_n(x) \qquad n = 0, 1, 2, \ldots$$

Since $\mathbb{F}$ is arbitrary, we need to explain what $\frac{d}{dx}$ means

$\frac{d}{dx}$ is the unique $\mathbb{F}$-linear trans $\mathbb{F}[x] \to \mathbb{F}[x]$ such that

$$\frac{d}{dx} x^n = n x^{n-1} \qquad 1 \le n < \infty$$

and

$$\frac{d}{dx} 1 = 0$$

To find $\frac{d}{dx} H_n$, first find the pattern for small n.

n	H_n	$\frac{d}{dx} H_n$
1	$2x$	$2 = 2H_0$
2	$4x^2-2$	$8x = 4H_1$
3	$8x^3-12x$	$24x^2-12 = 6H_2$

LEM 4 The Hermite polynomials satisfy

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x) \qquad n = 0, 1, 2, \ldots$$

Lec 2 Wednesday Sept 8 9/8/10

Recall the Hermite polynomials

$$H_n(x) = {}_2F_0\left(\begin{matrix} \frac{-n}{2} & \frac{1-n}{2} \end{matrix} \,\middle|\, \frac{-1}{x^2} \right) (2x)^n$$

$$= \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} (-1)^r \frac{(2x)^{n-2r} (-n)_{2r}}{r!} \qquad (*)$$

Abbr $H_n = H_n(x)$, view $H_n \in \mathbb{F}[x]$

Abbr $D = \frac{d}{dx}$

LEM 4 $\quad D H_n = 2n H_{n-1} \qquad n = 0, 1, 2 \ldots$

pf Method 1: Apply D to $(*)$ term by term

Method 2: Use 3-term rec and induction on n

$n=0$ ✓
$n=1$ ✓

assume $n \ge 2$

By Lem 3

$$x H_{n-1} - (n-1) H_{n-2} = \frac{1}{2} H_n$$

Apply D:

$$\underbrace{H_{n-1} + x \underbrace{D H_{n-1}}_{\substack{2(n-1) H_{n-2} \\ \underbrace{2(n-1)\left(\frac{1}{2} H_{n-1} + (n-2) H_{n-3}\right)}}} - (n-1)\, 2(n-2) H_{n-3}}_{(n-1) H_{n-1}} = \frac{D H_n}{2}$$

$\square$

In view of Lem 4 we call D the "forward shift" operator for the Hermite polynomials.

We now give a "backward shift" operator

LEM 5

$$(2x - D)H_n = H_{n+1} \qquad n = 0, 1, 2 \ldots$$

pf By Lem 3 and Lem 4

$$LHS = 2\left(\tfrac{1}{2}H_{n+1} + nH_{n-1}\right) - 2nH_{n-1}$$

$$= RHS \qquad \square$$

Combining the forward and backward shift operators we get

$$(2x - D)D\, H_n = 2n\, H_n \qquad n = 0, 1, 2 \ldots$$

In other words $y = H_n$ is a solution to the differential equation

$$y'' - 2xy' + 2ny = 0$$

Notation We define an $\mathbb{F}$-linear trans

$$A: \begin{array}{ccc} \mathbb{F}[x] & \to & \mathbb{F}[x] \\ f & \to & xf \end{array}$$ "mult by x"

COR 6 For $n = 0, 1, 2 \ldots$ the Hermite poly H_n is an eigenvector for the operator

$$(2A - D)D$$

with eigenvalue $2n$

LEM 7 the operators A, D satisfy

$$DA - AD = \underset{\substack{\uparrow \\ \text{identity map on } \mathbb{F}[x]}}{I}$$ "Weyl relation"

pf For $0 \le n < \infty$ apply each side to x^n $\square$

For

$$A^* = (2A - D)D$$

A, A^* generate an algebra which is hom image of Weyl algebra. See Course notes from 2000

Next goal: a generating function for the Hermite polys

LEM 8 For $n = 0, 1, 2 \ldots$

$$H_n = \exp\left(-\frac{D^2}{4}\right)(2x)^n \qquad (*)$$

where we recall

$$\exp(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!}$$

pf. RHS of $*$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{D^{2r}}{2^{2r}} \frac{(2x)^n}{r!}$$

$$= \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} (-1)^r \, 2^{n-2r} \frac{D^{2r}(x^n)}{r!}$$

$$\left[\begin{aligned} &\text{For } 0 \le r \le n/2 \\ &D^{2r}(x^n) = n(n-1)\cdots(n-2r+1)\, x^{n-2r} \\ &\phantom{D^{2r}(x^n)} = (-n)_{2r}\, x^{n-2r} \end{aligned}\right]$$

$$= \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} (-1)^r \frac{(2x)^{n-2r}(-n)_{2r}}{r!}$$

$$= H_n \quad \checkmark$$

$\square$

let t denote an indeterminate that commutes with x

thm 9

$$\sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!} = \exp(2xt - t^2)$$

(each side is a formal sum)

pf

$$\text{LHS} = \sum_{n=0}^{\infty} \exp\left(-\frac{D^2}{4}\right) \frac{(2xt)^n}{n!}$$

$$= \exp\left(\frac{-D^2}{4}\right) \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!}$$

$$= \exp\left(-D^2/4\right) \exp(2xt)$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{D^{2r}}{2^{2r} r!} \exp(2xt)$$

$$\left[\begin{array}{l} D \exp(2xt) = 2t \exp(2xt) \text{ so} \\ D^{2r} \exp(2xt) = (2t)^{2r} \exp(2xt) \qquad 0 \le r < \infty \end{array} \right]$$

$$= \underbrace{\left(\sum_{r=0}^{\infty} \frac{(-1)^r t^{2r}}{r!} \right)}_{\exp(-t^2)} \exp(2xt)$$

$$= \exp(2xt - t^2)$$

$\square$

Next goal: Rodrigues formula for H_n

LEM 10 For $n = 0, 1, 2, \dots$

$$D^n \exp(-x^2) = (-1)^n H_n(x) \exp(-x^2)$$

pf Ind on n.

$n=0$ ✓

$n \geq 1$:

$$D^n \exp(-x^2) = D\, D^{n-1} \exp(-x^2)$$

ind on n

$$= (-1)^{n-1} D\Big(H_{n-1}(x) \exp(-x^2) \Big)$$

apply chain rule

$$= (-1)^{n-1} \Bigg(\underbrace{\underbrace{\big(D(H_{n-1}(x))\big)}_{2(n-1)H_{n-2}(x)} \exp(-x^2) + \underbrace{H_{n-1}(x) \underbrace{D\exp(-x^2)}_{-2x\exp(-x^2)}}_{-2\left(\frac{1}{2}H_n(x) + (n-1)H_{n-2}(x)\right)\exp(-x^2)}}_{-H_n(x)\exp(-x^2)} \Bigg)$$

$$= (-1)^n H_n(x) \exp(-x^2)$$

$\square$

thm 11 (Rodrigues) For $n = 0, 1, 2, \dots$

$$H_n(x) = (-1)^n \exp(x^2)\, D^n \exp(-x^2)$$

pf In Lem 10 multiply both sides by $\exp(x^2)$ and recall $\exp(x^2)\exp(-x^2) = 1$

$\square$

Next goal: An "addition formula" for the Hermite polynomials.

Given commuting indets x, y

Obs vector space $\mathbb{F}[x,y]$ has basis

$$H_r(x)\, H_s(y) \qquad 0 \le r, s < \infty \qquad (*)$$

Problem: For $0 \le n < \infty$ express $H_n(x+y)$ as a lin combination of $(*)$. Answer is called an "addition formula"

ex $n=2$

$$
\begin{aligned}
H_2(x+y) &= 4(x+y)^2 - 2 \\
&= 4x^2 + 8xy + 4y^2 - 2 \\
&= 4x^2 - 2 + 2\, 2x\, 2y + 4y^2 - 2 + 2 \\
&= H_2(x) H_0(y) + 2 H_1(x) H_1(y) + H_0(x) H_2(y) \\
&\quad + 2 H_0(x) H_0(y)
\end{aligned}
$$

We solve the above problem using the generating function.

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \exp(2xt - t^2)$$

So

$$\sum_{n=0}^{\infty} H_n(x+y)\frac{t^n}{n!} = \exp\left(2(x+y)t - t^2\right)$$

$$= \exp\left(2xt - t^2 + 2yt - t^2 + t^2\right)$$

$$= \exp(2xt - t^2)\exp(2yt - t^2)\exp(t^2)$$

$$= \left(\sum_{r=0}^{\infty} H_r(x)\frac{t^r}{r!}\right)\left(\sum_{s=0}^{\infty} H_s(y)\frac{t^s}{s!}\right)\left(\sum_{l=0}^{\infty}\frac{t^{2l}}{l!}\right) \quad (**)$$

So

$\frac{H_n(x+y)}{n!}$ is the coeff of t^n in $(**)$

Thm 12 For $n = 0, 1, 2, \ldots$

$$H_n(x+y) = \sum_{\substack{r,s,l \in \mathbb{Z} \\ 0 \le r,s,l \\ r+s+2l=n}} H_r(x)\,H_s(y)\,\frac{n!}{r!\,s!\,l!}$$

pf Immed from above disc. $\square$

Note: the binomial theorem gives an addition formula for the polys $\{x^n\}_{n=0}^{\infty}$

We have 2 bases for v.s. $\mathbb{F}[x]$.

$$\{x^n\}_{n=0}^{\infty} \qquad (*)$$

$$\{H_n\}_{n=0}^{\infty} \qquad (**)$$

Earlier we expressed $(**)$ in terms of $(*)$:

$$H_n = \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} (-1)^r (2x)^{n-2r} \frac{(-n)_{2r}}{r!}$$

We now express $(*)$ in terms of $(**)$

LEM 13 For $n = 0, 1, 2, \ldots$

$$x^n = \frac{1}{2^n} \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} H_{n-2r} \frac{(-n)_{2r}}{r!}$$

pf Recall

$$H_n = \exp\left(\frac{-D^2}{4}\right) (2x)^n$$

So

$$\begin{aligned}
(2x)^n &= \exp\left(\frac{D^2}{4}\right) H_n \\
&= \sum_{r=0}^{\infty} \frac{D^{2r}}{2^{2r}} \frac{H_n}{r!} \\
&= \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} \frac{D^{2r}}{2^{2r}} \frac{H_n}{r!}
\end{aligned}$$

$$\left[\begin{aligned} DH_n &= 2n H_{n-1} \quad \text{so} \\ D^{2r} H_n &= 2^{2r}\, n(n-1)\cdots(n-2r+1)\, H_{n-2r} \\ &= 2^{2r}(-n)_{2r}\, H_{n-2r} \end{aligned}\right]$$

$$= \sum_{\substack{r\in\mathbb{Z} \\ 0\le r\le n/2}} H_{n-2r}\,\frac{(-n)_{2r}}{r!}$$

The result follows. $\square$

Next goal: Fa $0 \le i, j < \infty$

express $H_i(x)\, H_j(x)$

as a linear combination of $\{H_n(x)\}_{n=0}^{\infty}$

thm 14 Fa $0 \le i, j < \infty$

$$H_i(x)\, H_j(x) = \sum_{l=0}^{\min(i,j)} H_{i+j-2l}(x)\, \frac{2^l (-i)_l (-j)_l}{l!}$$

pf Use generating functions

$$\left(\sum_{i=0}^{\infty} H_i(x) \frac{t^i}{i!} \right)\left(\sum_{j=0}^{\infty} H_j(x) \frac{u^j}{j!} \right)$$

$$= \exp(2xt - t^2)\, \exp(2xu - u^2)$$

$$= \exp(2xt + 2xu - t^2 - u^2)$$

$$= \exp(2x(t+u) - (t+u)^2 + 2tu)$$

$$= \exp(2x(t+u) - (t+u)^2)\, \exp(2tu)$$

$$= \left(\sum_{n=0}^{\infty} H_n(x) \frac{(t+u)^n}{n!} \right)\left(\sum_{l=0}^{\infty} \frac{2^l t^l u^l}{l!} \right) \qquad (*)$$

$$\sum_{n=0}^{\infty} H_n(x) \frac{(t+u)^n}{n!} = \sum_{n=0}^{\infty} H_n(x) \sum_{\substack{r,s\ge 0\\ r+s=n}} \frac{t^r u^s}{r!\,s!}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} H_{r+s}(x) \frac{t^r u^s}{r!\,s!}$$

So

$$(*) = \left(\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} H_{r+s}(x) \frac{t^r u^s}{r!\,s!} \right) \left(\sum_{l=0}^{\infty} z^l \frac{t^l u^l}{l!} \right)$$

$$\left[\begin{array}{l} \text{change vars} \\ r+l=i \\ s+l=j \\ \text{so} \quad r+s = i+j-2l \end{array} \right]$$

$$r,s \ge 0 \to l \le \min(i,j)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t^i u^j \sum_{l=0}^{\min(i,j)} \frac{H_{i+j-2l}(x)\, z^l}{(i-l)!\,(j-l)!\,l!}$$

For $0 \le i,j < \infty$ we compare coeffs of $t^i u^j$ to get

$$\frac{H_i(x)}{i!} \frac{H_j(x)}{j!} = \sum_{l=0}^{\min(i,j)} \frac{H_{i+j-2l}(x)\, z^l}{(i-l)!\,(j-l)!\,l!}$$

Now

$$H_i(x)\, H_j(x) = \sum_{l=0}^{\min(i,j)} \frac{H_{i+j-2l}(x)\, z^l}{l!} \underbrace{\frac{i!}{(i-l)!} \frac{j!}{(j-l)!}}_{(-i)_l\, (-j)_l}$$

$\square$

Next goal: the <u>discriminant</u> of the Hermite polynomial H_n

Given $f \in \mathbb{F}[x]$ with degree $n \geq 1$

factor f over the algebraic closure $\overline{\mathbb{F}}$

$$f = a(x - x_1)(x - x_2) \cdots (x - x_n) \qquad a \in \mathbb{F}, \quad x_i \in \overline{\mathbb{F}}$$

Define

$$\Delta(f) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^2$$

"discriminant of f"

Note

$$\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n \\ i \neq j}} (x_i - x_j) = (-1)^{\binom{n}{2}} \Delta(f)$$

Given nonzero $f, g \in \mathbb{F}[x]$

factor both over $\overline{\mathbb{F}}$

$$f = a(x-x_1)(x-x_2)\cdots(x-x_n)$$

$$g = b(x-y_1)(x-y_2)\cdots(x-y_m)$$

Define

$$Res(f,g) = \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i - y_j)$$

"resultant of f, g"

Obs

$$Res(g,f) = (-1)^{nm} Res(f,g)$$

For $1 \le i \le n$

$$g(x_i) = b\prod_{j=1}^{m}(x_i - y_j)$$

So

$$\prod_{i=1}^{n} g(x_i) = b^n \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i - y_j)$$

$$= b^n Res(f,g)$$

$$Res(f,g) = b^{-n}\prod_{i=1}^{n} g(x_i)$$

Similarly

$$Res(g,f) = a^{-m}\prod_{i=1}^{m} f(y_i)$$

LEM 15 Given $f \in \mathbb{F}[x]$ with degree $n \geq 1$

$$\Delta(f) = (-1)^{\binom{n}{2}} n^n \, Res(f', f)$$

where $f' = Df = \frac{df}{dx}$

pf Write

$$f = a(x-x_1)(x-x_2)\cdots(x-x_n)$$

For the moment fix i $(1 \leq i \leq n)$. Write

$$f = (x - x_i)\, h \qquad h = a \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (x - x_j)$$

Apply D:

$$f' = (x - x_i) h' + h$$

So

$$f'(x_i) = h(x_i)$$

$$= a \prod_{\substack{1 \leq j \leq n \\ j \neq i}} (x_i - x_j)$$

So

$$\underbrace{\prod_{i=1}^{n} f'(x_i)}_{b^n Res(f, f')} = \underbrace{a^n \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n \\ i \neq j}} (x_i - x_j)}_{(-1)^{\binom{n}{2}} \Delta(f)}$$

[b = leading coeff of f'
$= an$]

Obs

$$\deg f' = n-1$$

So

$$\mathrm{Res}(f, f') = \mathrm{Res}(f', f)(-1)^{\overbrace{n(n-1)}^{\text{even}}}$$

$$= \mathrm{Res}(f', f)$$

Result follows $\square$

LEM 16 For $n = 1, 2, 3, \ldots$

$$\operatorname{Res}(H_n, H_{n+1}) = \operatorname{Res}(H_{n-1}, H_n) \frac{(-1)^n n^n}{2^n} \qquad (*)$$

Moreover

$$\operatorname{Res}(H_0, H_1) = 1 \qquad (**)$$

pf Recall

$$x H_n = \tfrac{1}{2} H_{n+1} + n H_{n-1}$$

Factor H_n over $\overline{\mathbb{F}}$:

$$H_n = 2^n (x - x_1) \cdots (x - x_n)$$

For $1 \le i \le n$

$$\underbrace{x_i H_n(x_i)}_{=0} = \tfrac{1}{2} H_{n+1}(x_i) + n H_{n-1}(x_i)$$

So

$$H_{n+1}(x_i) = -2n H_{n-1}(x_i)$$

Now

$$\prod_{i=1}^{n} H_{n+1}(x_i) = (-1)^n 2^n n^n \underbrace{\prod_{i=1}^{n} H_{n-1}(x_i)}_{= \operatorname{Res}(H_n, H_{n-1})\, 2^{(n-1)n}}$$

$$\| $$

$$2^{(n+1)n} \operatorname{Res}(H_n, H_{n+1})$$

$$\operatorname{Res}(H_n, H_{n-1}) = \operatorname{Res}(H_{n-1}, H_n)(-1)^{n(n-1)} = \operatorname{Res}(H_{n-1}, H_n)$$

This gives $(*)$

Line $(**)$ holds since empty product is 1.

$\square$

Thm 17 For $n=1,2,\ldots$ the discriminant of H_n is

$$\Delta(H_n) = 2^{-\binom{n}{2}} \prod_{j=1}^{n} j^j$$

pf By Lem 15

$$\Delta(H_n) = (-1)^{\binom{n}{2}} n^n \operatorname{Res}(H_n', H_n)$$

$$H' = 2n H_{n-1}$$

$$= (-1)^{\binom{n}{2}} n^n \operatorname{Res}(H_{n-1}, H_n)$$

By Lem 16

$$\operatorname{Res}(H_{n-1}, H_n) = \frac{(-1)^{n-1}(n-1)^{n-1}}{2^{n-1}} \operatorname{Res}(H_{n-2}, H_{n-1})$$

and iterate

Now

$$\Delta(H_n) = (-1)^{\binom{n}{2}} n^n \prod_{j=1}^{n-1} \frac{(-1)^j j^j}{2^j}$$

Result follows. $\square$

Next goal: A combinatorial interpretation of H_n

In what follows

S is a finite set

$n = |S|$

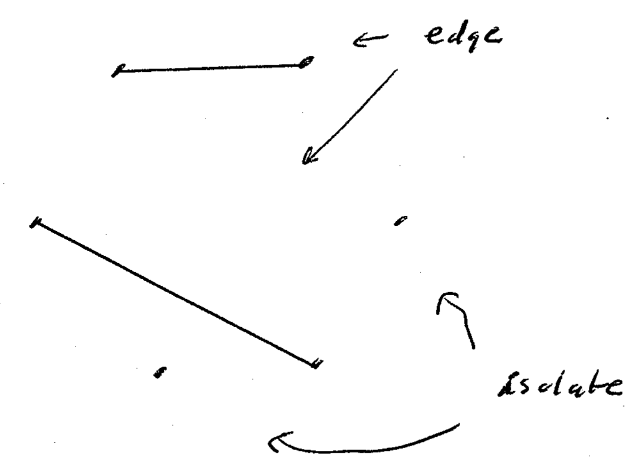
We consider (partial) matchings of S

Def 18 A <u>matching</u> of S is a partition of S into nonempty subsets with cardinality at most 2

size of subset	name
1	isolate
2	edge

Ex for $n = 6$

S:



Def 19 For commuting indeterminates x, y define $m_S \in \mathbb{F}[x,y]$ by

$$m_S = \sum_{\substack{m \\ \text{a matching} \\ \text{of } S}} x^{\#\text{isolates of } m} \, y^{\#\text{edges of } m}$$

Call m_S the (2 variable) matching polynomial of S

m_S only depends on $|S| = n$ so often write

m_n for m_S

Ex $n=3$

matchings of S:

$$m_3 = x^3 + xy + xy + xy$$

$$= x^3 + 3xy$$

LEM 20 For $0 \le r \le n/2$ the number of matchings of S with exactly r edges is

$$\frac{(-n)_{2r}}{2^r \, r!}$$

pf Assign the r edges one at a time

choices for edge 1 : $\binom{n}{2}$

" " " 2 : $\binom{n-2}{2}$

⋮

" " " r : $\binom{n-2r+2}{2}$

matchings of S with r edges

$$= \binom{n}{2}\binom{n-2}{2} \cdots \binom{n-2r+2}{2} \frac{1}{r!}$$

since edges are indistinguishable

$$= \frac{(-n)_{2r}}{2^r \, r!}$$

□

LEM 21 We have

$$m_n = \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} x^{n-2r} \, y^r \, \frac{(-n)_{2r}}{2^r \, r!}.$$

pf By L20 $\square$

COR 22 We have

$$H_n(x) = 2^n \, m_n\left(x, -\tfrac{1}{2}\right)$$

pf Recall

$$H_n(x) = \sum_{\substack{r \in \mathbb{Z} \\ 0 \le r \le n/2}} (2x)^{n-2r} \, \frac{(-1)^r \, (-n)_{2r}}{r!}$$

and use Lem 21. $\square$

LEM 23 The matching polynomial satisfies

$$m_n(\lambda x, \lambda^2 y) = \lambda^n m_n(x, y) \qquad \forall \lambda \in \mathbb{F}$$

pf For each matching m of S

$$\#\text{ isolates of } m \quad + \quad 2(\#\text{edges of } m) = n$$

$\square$

Def 24 Define $\mu_S \in \mathbb{F}[x]$ by

$$\mu_S = m_S(x, -1)$$

Call μ_S the (1 variable) <u>matching</u> <u>polynomial</u> of S

Often write

$$\mu_n = \mu_S \qquad n = |S|$$

thm 25 We have

$$H_n(x) = 2^{\frac{n}{2}} \mu_n(\sqrt{2}\, x),$$

$$\mu_n(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right)$$

"H_n and μ_n are the same up to normalization"

pf Use Cor 22 and Lem 23 with $\lambda = \sqrt{2}$ $\square$

Note we have

$$x\mu_n(x) = \mu_{n+1}(x) + n\mu_{n-1}(x)$$

$$D\mu_n(x) = n\mu_{n-1}(x)$$

$$(x - D)\mu_n(x) = \mu_{n+1}(x)$$

$$(x - D)D\mu_n(x) = n\mu_n(x)$$

the matching polynomial $\mu(K_n)$

n	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
0	1									
1	0	1								
2	-1	0	1							
3	0	-3	0	1						
4	3	0	-6	0	1					
5	0	15	0	-10	0	1				
6	-15	0	45	0	-15	0	1			
7	0	-105	0	105	0	-21	0	1		
8	105	0	-420	0	210	0	-28	0	1	
9	0	945	0	-1260	0	378	0	-36	0	1

Lec 4 Monday Sept 13

Next goal: A characterization of the Hermite polynomials

<u>Notation</u> By a <u>polynomial sequence</u> in $\mathbb{F}[x]$ we mean a sequence $\{p_n\}_{n=0}^{\infty}$ where $p_n \in \mathbb{F}[x]$ has degree exactly n for $0 \le n < \infty$.

For notational convenience we define $p_{-1} = 0$

A polynomial sequence $\{p_n\}_{n=0}^{\infty}$ is <u>monic</u> whenever p_n is monic for $0 \le n < \infty$.

Given a polynomial sequence $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ consider these properties

(3T): $\quad x p_n \in \mathrm{Span}(p_{n-1}, p_n, p_{n+1}) \quad n = 0, 1, 2, \ldots$

(D): $\quad D p_n \in \mathrm{Span}(p_{n-1}) \quad n = 0, 1, 2, \ldots$

We saw (3T), (D) hold for the Hermite polynomials $p_n = H_n$.

Are there other solutions? To get the answer we make some observations.

Given a poly sequence $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ that satisfies

(3T) + (0), define

$$\tilde{p}_n = \alpha_n p_n \qquad n = 0, 1, 2 \ldots$$

where $0 \neq \alpha_n \in \mathbb{F}$ is arbitrary. Then $\{\tilde{p}_n\}_{n=0}^{\infty}$ is a poly sequence in $\mathbb{F}[x]$ that satisfies (3T) + (0).

Also, let β, γ denote scalars in $\mathbb{F}$ such that $\beta \neq 0$, and def

$$p_n^{\vee} = p_n(\beta x - \gamma) \qquad n = 0, 1, 2 \ldots$$

Then $\{p_n^{\vee}\}_{n=0}^{\infty}$ is a poly sequence in $\mathbb{F}[x]$ that satisfies (3T) + (0)

Perhaps any poly sequence satisfying (3T) + (0) is related to the Hermite polys via the above "moves"? In view of our above comments WLOG we consider only monic poly sequences.

Thm 26 Assume $\mathbb{F}$ is alg closed and char 0.

A monic polynomial sequence $\{p_n\}_{n=0}^{\infty}$ in $\mathbb{F}[x]$ satisfies

(3T) + (D) iff (i) or (ii) hold below

(i) $\exists\ \alpha \in \mathbb{F}$ such that

$$p_n = (x-\alpha)^n \qquad n = 0, 1, 2, \ldots$$

(ii) $\exists\ \beta, \gamma \in \mathbb{F}$ with $\beta \neq 0$ such that

$$p_n = \frac{H_n(\beta x - \gamma)}{2^n \beta^n} \qquad n = 0, 1, 2, \ldots$$

pf $\Leftarrow$: clear

$\rightarrow$: We first claim

$$D p_n = n\, p_{n-1} \qquad 0 \le n < \infty$$

Assume $n \geq 1$ else triv.

By (D) $\exists\ \alpha_n \in \mathbb{F}$ s.t.

$$D p_n = \alpha_n p_{n-1}$$

$$LH = D\left(x^n + \text{Lower terms}\right)$$

$$= n x^{n-1} + L \text{ terms}$$

$$RHS = \alpha_n\left(x^{n-1} + L \text{ terms}\right)$$

so $\alpha_n = n$

<u>claim 2</u> $\exists\ a, b \in \mathbb{F}$ s.t.

$$x p_n = p_{n+1} + a p_n + n b\, p_{n-1} \qquad n = 0, 1, 2, \ldots \qquad (*)$$

<u>pf cl 2</u> By (3T) and since the p_n are monic

$$x p_n = p_{n+1} + a_n p_n + n\, b_n\, p_{n-1} \qquad n = 0, 1, 2 \ldots \qquad (**)$$

for some $a_n, b_n \in \mathbb{F}$.

Show a_n indep of n for $0 \le n < \infty$ and b_n indep of n for $1 \le n < \infty$

Apply D to $(**)$ for $n \ge 1$:

$$x \underset{\substack{\shortparallel \\ n p_{n-1}}}{p_n'} + p_n = \underset{\substack{\shortparallel \\ (n+1) p_n}}{p_{n+1}'} + a_n \underset{\substack{\shortparallel \\ n p_{n-1}}}{p_n'} + n b_n \underset{\substack{\shortparallel \\ (n-1) p_{n-2}}}{p_{n-1}'}$$

$$\left[\ x p_{n-1} = p_n + a_{n-1} p_{n-1} + (n-1) b_{n-1} p_{n-2}\ \right]$$

Get

$$0 = n(a_n - a_{n-1}) p_{n-1} + n(n-1)(b_n - b_{n-1}) p_{n-2}$$

But $p_0, p_1, p_2 \ldots$ are lin indep so

$$n(a_n - a_{n-1}) = 0 \qquad n = 1, 2, \ldots$$

$$n(n-1)(b_n - b_{n-1}) = 0 \qquad n = 2, 3, \ldots$$

So $a_0 = a_1 = a_2 = \cdots$

$b_1 = b_2 = b_3 = \cdots$ claim 2 proved.

Case $b = 0$

Setting $b = 0$ in $(*)$

$$p_{n+1} = (x-a)\,p_n \qquad n = 0,1,2\ldots$$

so

$$p_n = (x-a)^n \qquad n = 0,1,2\ldots$$

Case $b \neq 0$ Since $\mathbb{F}$ alg closed $\exists\, \beta \in \mathbb{F}$ $\beta \neq 0$

s.t.

$$\frac{1}{2\beta^2} = b$$

Define

$$\gamma = \beta a$$

then $(*)$ becomes

$$x\,p_n = p_{n+1} + \frac{\gamma}{\beta}\,p_n + \frac{n}{2\beta^2}\,p_{n-1} \qquad n = 0,1,2,\ldots \qquad (***)$$

Define

$$\tilde{p}_n = \frac{H_n(\beta x - \gamma)}{2^n \beta^n} \qquad n = 0,1,2,\ldots$$

To show $p_n = \tilde{p}_n$ we show $\tilde{p}_n$ satisfies the recursion $(***)$

Recall

$$x\,H_n = \tfrac{1}{2} H_{n+1} + n\,H_{n-1}$$

So

$$(\beta x-\gamma)\, H_n(\beta x-\gamma) = \tfrac{1}{2} H_{n+1}(\beta x-\gamma) + n H_{n-1}(\beta x-\gamma)$$

Divide each term by $2^n \beta^n$ and simplify:

$$x\,\tilde{p}_n = \tilde{p}_{n+1} + \frac{\gamma}{\beta}\,\tilde{p}_n + \frac{n}{2\beta^2}\,\tilde{p}_{n-1} \qquad n=0,1,2\ldots$$

p_n, $\tilde{p}_n$ satisfy the same recursion and have same initial conditions $p_0=1$, $\tilde{p}_0=1$, so

$$p_n=\tilde{p}_n \qquad n=0,1,2\ldots$$

$\square$

We continue to discuss the Hermite polynomials H_n

Next goal: orthogonality of H_n

Until further notice $\mathbb{F} = \mathbb{R}$

Consider $\mathbb{R}$-vector space $V = \mathbb{R}[x]$

Define a bilinear form $\langle \, , \rangle$ on V by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\, g(x)\, e^{-x^2}\, dx \qquad \forall f, g \in V$$

"weight function for Hermite polys"

$\langle \, , \rangle$ is symmetric:

$$\langle f, g \rangle = \langle g, f \rangle \qquad \forall f, g \in V$$

$\langle \, , \rangle$ is positive definite:

$$\langle f, f \rangle > 0 \qquad \forall\, 0 \neq f \in V$$

Recall the operator

$$A : V \to V, \quad f \mapsto xf$$

$\langle \, , \rangle$ is A-invariant:

$$\langle Af, g \rangle = \langle f, Ag \rangle \qquad \forall f, g \in V$$

I will use without proof just one result from Calculus:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi} \qquad (*)$$

thm 27 For $0 \le n, m < \infty$

$$\langle H_n, H_m \rangle = \delta_{nm} \, 2^n \, n! \, \sqrt{\pi}$$

pf First show

$$\int_{\mathbb{R}} H_r(x) e^{-x^2} dx = 0 \qquad r = 1, 2, 3 \ldots \qquad (**)$$

By Lem 10

$$H_r(x) e^{-x^2} = (-1)^r D^r e^{-x^2}$$

$$= (-1)^r D \left(D^{r-1} e^{-x^2}\right)$$

$$= -D\left(H_{r-1}(x) e^{-x^2}\right)$$

So

$$\int_{\mathbb{R}} H_r(x) e^{-x^2} dx = - H_{r-1}(x) e^{-x^2} \Big|_{x=-\infty}^{x=\infty}$$

$$= 0 - 0$$

$$= 0$$

Show for $m < n$

$$\langle H_n, H_m \rangle = 0$$

$H_m \in \mathrm{Span}(1, x, x^2, \dots, x^m)$

show

$$\langle H_n, x^m \rangle = 0$$

$$\langle H_n, x^m \rangle = \langle x^m H_n, 1 \rangle$$

$$\left[\begin{array}{l} x H_n \in \mathrm{Span}(H_{n-1}, H_n, H_{n+1}) \\ \text{so} \\ x^m H_n \in \mathrm{Span}(H_{n-m}, \dots, H_{n+m}) \end{array} \right]$$

$$= 0 \qquad \text{by } (**)$$

Show $\langle H_n, H_n \rangle = 2^n n! \sqrt{\pi}$

Ind on n

$n = 0$: done by $*$

$n \geq 1$:

$$\langle x H_{n-1}, H_n \rangle = \langle H_{n-1}, x H_n \rangle$$

$$\underbrace{\tfrac{1}{2} H_n + (n-1) H_{n-2}}_{\tfrac{1}{2}\langle H_n, H_n \rangle} \qquad \underbrace{\tfrac{1}{2} H_{n+1} + n H_{n-1}}_{n \langle H_{n-1}, H_{n-1} \rangle}$$

$$n \langle H_{n-1}, H_{n-1} \rangle = n\, 2^{n-1} (n-1)! \sqrt{\pi} = 2^{n-1} n! \sqrt{\pi}$$

ind

Next goal: moments of H_n

For $n = 0, 1, 2, \ldots$ define

$$u_n = \langle x^n, 1 \rangle \qquad \text{"nth moment of } H_n\text{"}$$
$$= \int_{\mathbb{R}} x^n e^{-x^2} \, dx$$

Obs

$$\langle x^i, x^j \rangle = u_{i+j} \qquad 0 \le i, j < \infty$$

LEM 28 For $n = 0, 1, 2, \ldots$

If n odd

$$u_n = 0$$

If n even

$$u_n = \sqrt{\pi} \; \frac{1}{2} \, \frac{3}{2} \, \frac{5}{2} \cdots \frac{n-1}{2}$$
$$= \frac{\sqrt{\pi} \; n!}{2^n \left(\frac{n}{2}\right)!}$$

pf Write x^n as lin comb of $\{H_r\}_{r=0}^{n}$

$$x^n = \sum_{r=0}^{n} \alpha_r H_r \qquad \alpha_r \in \mathbb{R}$$

$$u_n = \langle x^n, 1 \rangle$$
$$= \sum_{r=0}^{n} \alpha_r \underbrace{\langle H_r, 1 \rangle}_{0 \text{ if } r \ge 1}$$
$$= \alpha_0 \langle H_0, 1 \rangle$$
$$= \alpha_0 \sqrt{\pi}.$$

By Lem 13

$$\alpha_0 = \begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{n!}{2^n \left(\frac{n}{2}\right)!} & \text{if } n \text{ even} \end{cases}$$

$\square$

Def 29 For $n = 0, 1, 2 \ldots$

the nth <u>Hankel matrix</u> for Hermite polynomials is

$$\begin{pmatrix} u_0 & u_1 & u_2 & \cdots & u_n \\ u_1 & u_2 & & & \\ u_2 & & & & \vdots \\ \vdots & & \ddots & & \\ u_n & & \cdots & & u_{2n} \end{pmatrix}$$

This is the matrix of inner products

$$\left(\langle x^i, x^j \rangle \right)_{0 \le i, j \le n}$$

LEM 30 For $n = 0, 1, 2, \ldots$ the Hankel matrix from Def 28 is positive definite.

pf By constr and since $\langle , \rangle$ is pos def. $\square$

DEF 31 For $n = 0, 1, 2..$ Let D_n denote the determinant of the Hankel matrix for H_n

Obs $D_n > 0$ by LEM 30.

thm 32 For $n = 0, 1, 2..$

$$D_n = \prod_{i=0}^{n} \frac{\sqrt{\pi}\, i!}{2^i}$$

pf Let $S = S_n$ denote the transition matrix from $\{H_i\}_{i=0}^{n}$ to $\{x^i\}_{i=0}^{n}$. So

$$x^j = \sum_{i=0}^{n} S_{ij} H_i \qquad 0 \le j \le n$$

We found S in LEM 13

S is upper triangular with (i,i)-entry 2^{-i} for $0 \le i \le n$.

Obs

$$\text{Hankel}_n = \left(\langle x^i, x^j \rangle \right)_{0 \le i,j \le n}$$

$$= S^t \Big(\underbrace{\langle H_i, H_j \rangle_{0 \le i,j \le n}}_{\| \text{ m 27}} \Big) S$$

$$\text{diag} \left(\sqrt{\pi}\, 2^i\, i! \right)_{i=0}^{n}$$

Take det of each side

$$D_n = \det(S)^2 \prod_{i=0}^{n} \left(\sqrt{\pi}\; 2^i\, i! \right)$$

$$\det(S) = \prod_{i=0}^{n} 2^{-i}$$

Result follows. $\square$

Hankel matrix for Hermite

Obs

$$\frac{u_n}{\sqrt{\pi}} = \frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\,\frac{7}{2}\cdots\frac{n-1}{2} \qquad \text{for } n \text{ even}$$

$n=6$

	0	1	2	3	4	5	6
0	1	0	$\frac12$	0	$\frac12\frac32$	0	$\frac12\frac32\frac52$
1	0	$\frac12$	0	$\frac12\frac32$	0	$\frac12\frac32\frac52$	0
2	$\frac12$	0	$\frac12\frac32$	0	$\frac12\frac32\frac52$	0	$\frac12\frac32\frac52\frac72$
3	0	$\frac12\frac32$	0	$\frac12\frac32\frac52$	0	$\frac12\frac32\frac52\frac72$	0
4	$\frac12\frac32$	0	$\frac12\frac32\frac52$	0	$\frac12\frac32\frac52\frac72$	0	$\frac12\frac32\frac52\frac72\frac92$
5	0	$\frac12\frac32\frac52$	0	$\frac12\frac32\frac52\frac72$	0	$\frac12\frac32\frac52\frac72\frac92$	0
6	$\frac12\frac32\frac52$	0	$\frac12\frac32\frac52\frac72$	0	$\frac12\frac32\frac52\frac72\frac92$	0	$\frac12\frac32\frac52\frac72\frac92\frac{11}{2}$

this $\dfrac{\text{Hankl}_6}{\sqrt{\pi}}$

The ...

n	$D_n = \pi^{n/2}$ times
0	1
1	$\frac{1}{2}$
2	$\frac{1}{2^2}$
3	$\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{3}{2}$
4	$\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{3}{2}\,\frac{3}{2}$
5	$\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}$
6	$\frac{1}{2}\,\frac{1}{2}\,\frac{3}{2}\,\frac{3}{2}\,\frac{3}{2}\,\frac{3}{2}\,\frac{3}{2}\,\frac{5}{2}\,\frac{5}{2}$
7	
8	
9	
n	$\dfrac{1!\,2!\,3!\cdots n!}{2\;2^2\;2^3\cdots 2^n} = \prod_{j=1}^{n} \dfrac{j!}{2^j}$

$$u_n = \begin{cases} 0 & n \text{ odd} \\ \dfrac{n!\,\sqrt{\pi}}{2^n \left(\frac{n}{2}\right)!} & n \text{ even} \end{cases}$$

n	$\frac{u_n}{\sqrt{\pi}}$	$\frac{u_n}{\sqrt{\pi}}$
0	1	1
1	0	
2	$\frac{2}{2^2}$	$\frac{1}{2}$
3	0	
4	$\frac{4 \cdot 3}{2^4}$	$\frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 2}$
5	0	
6	$\frac{6 \cdot 5 \cdot 4}{2^6}$	$\frac{15}{8} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2}$
7	0	
8	$\frac{8 \cdot 7 \cdot 6 \cdot 5}{2^8}$	$\frac{105}{16} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2}$
9	0	
10	$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2^{10}}$	$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{945}{32}$

	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
H_0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}\frac{3}{2}$	0	$\frac{1}{2}\frac{3}{2}\frac{5}{2}$	0		
H_1		$\frac{1}{2}$	0	$\frac{1}{2}\frac{3}{2}$	0	$\frac{1}{2}\frac{3}{2}\frac{5}{2}$	0	$\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{7}{2}$		
H_2			$\frac{1}{2^2}$	0	$\frac{1}{2}\frac{3}{2}$	0	$\frac{45}{16}$	0		
H_3				$\frac{1}{2^3}$	0	$\frac{5}{2^3}$	0	$\frac{3\cdot5\cdot7}{2^5}$		
H_4					$\frac{1}{2^4}$	0	$\frac{3\cdot5}{2^5}$	0		
H_5						$\frac{1}{2^5}$	0	$\frac{7\cdot6}{2^7}$		
H_6							$\frac{1}{2^6}$	0		
H_7								$\frac{1}{2^7}$		
H_8										
H_9										

We continue to discuss the Hermite polys H_n

Assume $\mathbb{F} = \mathbb{R}$

Next goal: Write the Hermite polys in terms of their moments

Motivation:

Given $f \in \mathbb{R}[x]$

write
$$f = \sum_{i=0}^{n} \alpha_i x^i \qquad \alpha_i \in \mathbb{R} \qquad (*)$$

Obs for $j = 0, 1, 2 \ldots$
$$\langle f, x^j \rangle = \sum_{i=0}^{n} \alpha_i \underbrace{\langle x^i, x^j \rangle}_{= \mu_{i+j}}$$

" to compute $\langle f, x^j \rangle$ in $(*)$ replace x^i by μ_{i+j} for $0 \le i \le n$ "

thm 33 For $n=1,2,3\ldots$ the Hermite polynomial

$$H_n = \frac{2^n}{D_{n-1}} \det \begin{pmatrix} u_0 & u_1 & \cdots & & u_n \\ u_1 & & & & \\ \vdots & & \ddots & & \vdots \\ u_{n-1} & & \cdots & & u_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix}$$

pf: Obs RHS is polynomial in x with degree $\leq n$

Call it $p(x)$

In p the coeff of x^n is 2^n

In H_n ... ... 2^n

$$H_n - p \in \operatorname{Span}(x^j)_{j=0}^{n-1}$$

Show

$$\langle H_n - p, x^j \rangle = 0 \qquad 0 \leq j \leq n-1$$

We saw earlier

$$\langle H_n, x^j \rangle = 0$$

Also

$$\langle p, x^q \rangle = \frac{2^n}{D_{n-1}} \det \begin{pmatrix} & \text{same as} & \\ & \text{above} & \\ \hline u_q & u_{q+1} \quad \cdots & u_{q+n} \end{pmatrix}$$

[has duplicate rows]

$$= 0$$

So $\langle H_n - p, x^q \rangle = 0$

Now $H_n - p = 0$ since $\langle , \rangle$ is pos def and therefore nondegenerate on $\operatorname{Span}(x^q)_{q=0}^{n-1}$

So $H_n = p$ $\square$

General orthogonal polynomials

We will assume $\mathbb{F} = \mathbb{R}$ (Most of the Literature is about this case)

$\square$