Algebraic Graph Theory Notes

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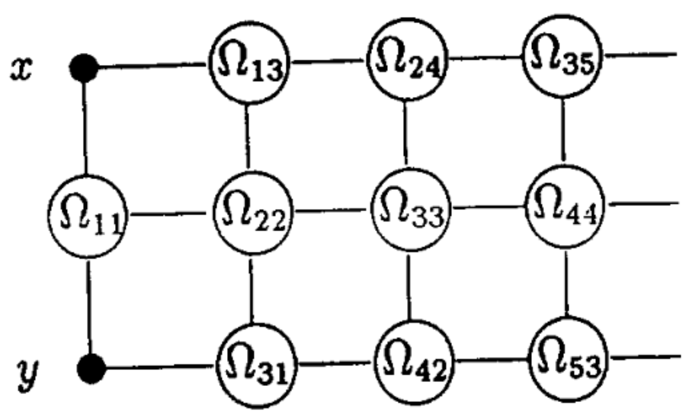
1 Combinatorially 2 Homogenous Graphs

We fix $\Gamma = (X, R)$ to be a bipartite distance regular graph with diameter $D \geq 3$ and valency $k \geq 3$ for the rest of this section.
Definition 14: For $x, y \in X$ with $\partial(x, y) = 2$, we set $\Omega_{i,j}(x, y) = \Omega_{i,j} = \Gamma_i(x) \cap \Gamma_j(y)$, that is the set of vertices at distance i from x and distance j from y. Let $\mathcal{L} = \{ij : 1 \leq i, j \leq D, i + j \geq 2, |i - j| = 0, 2\}$.



Observe that $\Omega_{i,j} \neq 0 \implies ij \in \mathcal{L}$ since $\partial(x, y) = 2$. Recall that b_i is the number of vertices at distance $i+1$ from x and 1 from y, and c_i is the number of vertices at distance $i-1$ from x and 1 from y. Then we have:
Lemma15: a) For $i < D$, $z \in \Omega_{i+1,i-1}$ is adjacent to $c_{i+1} - c_{i-1}$ vertices in $\Omega_{i,i}$, c_{i-1} vertices in $\Omega_{i,i-2}$ and b_{i+1} vertices in $\Omega_{i+2,i}$. We have analogous equalities for $\Omega_{i-1,i+1}$.
b) For $i < D$ and $z \in \Omega_{i,i}$ if δ_i is the number of neighbours in $\Omega_{i-1,i-1}$, then z has $b_i - c_i + \delta_i$ neighbours in $\Omega_{i+1,i+1}, c_i - \delta_i$ vertices in $\Omega_{i-1,i+1}$ and by definition δ_i vertices in $\Omega_{i-1,i-1}$.
c)$z \in \Omega_{D,D}$ has k neighbours in $\Omega_{D-1,D-1}$ where k is the valency.
The proofs follow from the definitions of the c_i and b_i and $\Omega_{i,j}$.

Theorem 16: The following are equivalent:
$a)\forall i, z \in \Omega_{ii}\gamma_i = |\Omega_{1,1} \cap \Gamma_{i-1}(z)|$ is independent of z.
$b)\forall i, z \in \Omega_{ii}\delta_i = |\Omega_{i-1,i-1} \cap \Gamma_1(z)|$ is independent of z.
$c)\forall ij, mn \in \mathcal{L}$ and for $z \in \Omega_{i,j}$ the scalar $|\Omega_{m,n} \cap \Gamma(z)|$ is independent of z.
$d)span\{\sum_{z\in\Omega_{i,j}} \hat{z} : ij \in \mathcal{L}\}$ is $A-$invariant.

If the four conditions above are true then $\forall i, 1 \leq i < D, \gamma_i$ is non zero and $\delta_i = c_{i-1}\gamma_i/\gamma_{i-1}$.
Proof: $a \implies b$ follows from counting the number of ordered pairs uv such that $u \in \Omega_{1,1} \cap \Gamma_{i-1}(z)$ and $v \in \Omega_{i-1,i-1} \cap \Gamma(z)$. There are γ_i choices of u and given u there are c_{i-1} choices of v. On the other hand there are δ_i choice for v and given v there are γ_{i-1} choices for u. Therefore we get the equation $\delta_i = c_{i-1}\gamma_i/\gamma_{i-1}$ above, and this shows that δ_i is independent of z if the γ_i's are. $b \implies c$ follows from lemma 15. $c \implies d$ follows from the definitions. Lastly $d \implies a$ follows by observing that $|\Omega_{1,1} \cap \Gamma_{i-1}(z)|$ is the z coordinate of $A_{i-1}(\sum_{z\in\Omega_{1,1}} \hat{z})$.

We can now define the notion of combinatorially 2 homogenous.
Theorem 17: The following are equivalent:
$a)$ the conditions of theorem 13 (from the previous section) hold $\forall i, 1 \leq i \leq D-1$.
$b)\forall x, y$ with $\partial(x, y) = 2$ the conditions of theorem 16 hold.
$c)\exists x, y$ with $\partial(x, y) = 2$ such that the conditions of theorem 16 hold.

When any of the conditions of theorem 17 hold, we say that Γ is combinatorially 2 homogenous.