

3. Some Algebra related to Q -poly DRGs

We would like to investigate the structure of an U -mod T -module for a Q -poly DRG. (not rec this)

To do this efficiently it is helpful to introduce the notion of a tridagonal pair of linear transformations.

We will use our results on TD pairs to get an action of the quantum q - $U_{\mathfrak{sl}_2}$ on the st. module of certain DRGs.

Until further notice \mathbb{F} is any field

Def 1 Let V denote a nonzero, finite dim'l vector space over \mathbb{F} . A tridiagonal pair (TP pair) on V is an ordered pair of linear transformations

$$A: V \rightarrow V, \quad A^*: V \rightarrow V$$

such that

(i) Each of A, A^* is diagonalizable on V

(ii) \exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A s.t.

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad 0 \leq i \leq d.$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) \exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* s.t.

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad 0 \leq i \leq \delta$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$

(iv) there does not exist a subspace W of V s.t.

$$A W \subseteq W, \quad A^* W \subseteq W, \quad W \neq 0, \quad W \neq V.$$

" V is irreducible as a module for A, A^* "

Note 2. Given a TD pair A, A^* on V . Then

A^*, A is a TD pair on V . Call these TD pairs duals

EX 3 ~~FRAC~~ Given a DRG $\Gamma = (X, R)$ diam 0

Assume $\{E_i\}_{i=0}^p$ is a \mathbb{Q} -poly ordering of the prim

idempotents of Γ . Fix $x \in X$ write $T = T(x)$ etc.

Let W denote an unred T -module. Then

the pair A, A^* acts on W as a TD pair

Pf. Let $r = \text{endpt of } W$
 $t = \text{dual endpt of } W$
 $g = \text{diameter } \dots$
 $d = \text{dual diameter}$

Ref to Def 1 take V to be W

(i) clear

(ii) take $V_i = E_{t+i} W$ $0 \leq i \leq d$

(iii) take $V_i^* = E_{r+i}^* W$ $0 \leq i \leq g$

(iv) Since W is unred as T -module, and since T
 is gen by A, A^* □

Referring to Def 1, it will turn out $d = \delta$.

EX 4. A Leonard pair is the same thing as

a TD pair for which the v_i, v_i^* all have $\text{dim } 1$.

We now give an example of a TD pair related
to quantum sps

Until further notice

assume \mathbb{F} is alg closed

Fix $0 \neq q \in \mathbb{F}$ s.t. q not a root of 1

Recall

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}$$

Let A_q denote the associative \mathbb{F} -algebra with 1,

defined by generators x, y and relations

$$x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 = 0$$

"cubic"

$$y^3 x - [3]_q y^2 x y + [3]_q y x y^2 - x y^3 = 0$$

"q-Serre
relations"

A_q called "positive part of $U_q \hat{\mathfrak{sl}}_2$ "

Recall q -Serre relations are special case of the TD relations,

EX 5 Let V denote a finite-diml U_q -module

on which neither of x, y is nilpotent.

then x, y act on V as a TD pair.

Pf $\forall \theta \in \mathbb{F}$ def

$$\begin{aligned} V(\theta) &= \theta\text{-eigenspace of } x \text{ on } V \\ &= \{v \in V \mid x \cdot v = \theta v\} \end{aligned}$$

Possibly $V(\theta) = 0$

We show that $\forall \theta \in \mathbb{F}$

$$y V(\theta) \subseteq V(q^{-2}\theta) + V(\theta) + V(q^2\theta) \quad (*)$$

To see (*) pick $v \in V(\theta)$ so $xv = \theta v$. Obs

$$\begin{aligned} 0 &= \left(x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 \right) v \\ &= x^3 y v - [3]_q \theta x^2 y v + [3]_q \theta^2 x y v - \theta^3 y v \\ &= (x - q^2 I)(x - I)(x - q^{-2} I) y v \end{aligned}$$

We assume q not a root of 1 so

$$q^2, 1, q^{-2} \text{ mut distinct}$$

so

$$V \in V(q^{-2}\theta) + V(\theta) + V(q^2\theta)$$

Since \mathbb{F} is alg closed and X is not nil on V

X has at least one non 0 eigen θ on V .

Consider sequence

$$\theta, q^{-2}\theta, q^{-4}\theta, \dots$$

these scalars mut distinct since q not a root of 1.

So they are not all eigen of X on V

So \exists eigen α of X on V s.t. $q^{-2}\alpha$ is not an

eigen of X on V .

Consider sequence

$$\alpha, \alpha q^2, \alpha q^4, \dots$$

\exists nonneg integer d s.t. αq^{2d} is eigen of X on V

$p_{i0} = i = d$ but not $i = d+1$.

Set

$$V_i = V(\alpha^2 z^i) \quad 0 \leq i \leq d$$

Obs

$$V_0 + V_1 + \dots + V_d$$

is γ -inv and γ -inv by constr. By (*)

$$\forall V_i \leq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d)$$

where $V_{-1} = 0, V_{d+1} = 0$. So $V_0 + \dots + V_d$ is γ -inv.

Now

$$V = \sum_{i=0}^d V_i$$

by irreducibility. Now X is diagonalizable on V

and Def 1 (iii) holds.

Int the roles of X & γ we find γ is diag on V , and

Def 1 (iii) holds,

Def 1 (iv) is from inv of V . Result follows. \square

Until further notice

F alg closed, char 0

View $\mathfrak{sl}_2(F)$ as the Lie algebra of all

2×2 trace 0 matrices over F , with Lie bracket

$$[r, s] = rs - sr.$$

Given $x \in \mathfrak{sl}_2(F)$, char poly of x has form

$$\lambda^2 - \alpha \quad \alpha \in F$$

(since x has trace 0)

Call x nilpotent if $\alpha = 0$ and semi simple if $\alpha \neq 0$

Given $x, y \in \mathfrak{sl}_2(F)$ obs x, y generate

$\mathfrak{sl}_2(F)$ iff $x, y, [x, y]$ are lin indep (since $\mathfrak{sl}_2(F)$

has dim 3 as V_3/F)

EX 6. Given semi-simple $x, y \in \mathfrak{sl}_2(F)$

that generate $\mathfrak{sl}_2(F)$. Let V denote a f.d.

irred $\mathfrak{sl}_2(F)$ -module. Then x, y act on V as a TD pair.
(Infact LP)

pf ex.

□

Ree's Onsager algebra \mathcal{O} is the Lie algebra over \mathbb{F}

defined by gens x, y and rels

$$[x, [x, [x, y]]] = 4[x, y]$$

$$[y, [y, [y, x]]] = 4[y, x]$$

Ex 7. Let V denote a f.d. irred \mathcal{O} -module.

then x, y act on V as a TD pair

pf Similar to pf of Ex 5.

□

Assume F arb

Given TD pair A, A^* on V

An ordering of the eigenspaces of A (resp A^*)

is called standard whenever it satisfies Def 1 (ii)

(resp. Def 1 (iii))

Obs if $\{V_i\}_{i=0}^d$ is a st. ordering of the eigenspaces

of A then so is $\{V_{d-i}\}_{i=0}^d$, and no other ordering is standard.

Sim for A^* .

An ordering of the primitive idempotents or eigenvalues

of A (resp A^*) is called standard if the corresp

ordering of the eigenspaces is standard.

Next goal: explain the relations of a TD system.

Given a TD system \mathbb{F} on V as in Def 8

Then the following are TD systems on V :

$$\mathbb{F}^* := \left(A^*, \{E_i^*\}_{i=0}^{\delta}; A, \{E_i\}_{i=0}^d \right) \quad \text{"dual"}$$

$$\mathbb{F}^{\downarrow} := \left(A; \{E_i\}_{i=0}^d; A^*, \{E_{\delta-i}^*\}_{i=0}^{\delta} \right) \quad \text{"1st inv"}$$

$$\mathbb{F}^{\Downarrow} := \left(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^{\delta} \right) \quad \text{"2nd inv"}$$

Viewing $*$, \downarrow , \Downarrow as permutations on the set of all

TD systems

$$*^2 = \downarrow^2 = \Downarrow^2 = 1$$

$$\Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow \Downarrow = \Downarrow \downarrow$$

the group generated by symbols $*$, \downarrow , \Downarrow subject to

the relations above is the dihedral gp D_4 . This is the

group of symmetries of a square, and has 8 elements.

Obs $\ast, \downarrow, \Downarrow$ induce an action of D_4 on the set of all TD systems.

Two TD systems will be called relatives whenever they are in the same orbit of this D_4 -action.

DEF 9. Given TD system

$$\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^{\delta})$$

For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A for E_i

For $0 \leq i \leq \delta$... θ_i^* ... A^* for E_i^*

Call $\{\theta_i\}_{i=0}^d$ the eigenvalue sequence of Φ

Call $\{\theta_i^*\}_{i=0}^{\delta}$ the dual eigenvalue sequence of Φ

Notation 10. Given TD system Φ Given $g \in D_4$

For any object f assoc with Φ , f^g will denote

the corresp object for Φ^g , So

$$\theta_i(\Phi^g) = \theta_i^*(\Phi) \quad \text{etc.}$$

Next goal: split decomp.

Notation. Let V be nmo, f.d. v.s. \mathbb{F} .

$d = \dim V$ integer.

A decomposition of V of length d is a sequence of subspaces

$$\{U_i\}_{i=0}^d \text{ s.t. } U_i \neq 0 \quad 0 \leq i \leq d \quad \text{and}$$

$$V = \sum_{i=0}^d U_i \quad (\text{dir sum})$$

We set $U_{-1} = 0$, $U_{d+1} = 0$.

Given a TD system

$$\mathbb{F} = (A, \{E_i\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^d)$$

in V we now show $d = \delta$. Also show \exists

unique dec $\{U_i\}_{i=0}^d$ of V s.t.

$$(A - \theta_i I) U_i \subseteq U_{i+1} \quad 0 \leq i \leq d$$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1} \quad 0 \leq i \leq d$$

where $\{\theta_i\}_{i=0}^d$ (resp $\{\theta_i^*\}_{i=0}^d$) is the equal

eq (dual equal eq) of \mathbb{F} .



Field \mathbb{F} arb V denotes nonzero f.d. v.s. / \mathbb{F} $\Phi = (A, \{E_i\}_{i=0}^d; A^*, \{E_i^*\}_{i=0}^{\delta})$ is TD system on V .with equal req $\{E_i\}_{i=0}^d$ and dual equal req $\{E_i^*\}_{i=0}^{\delta}$ DEF 11 For all integers i, j define

$$V_{ij} = (E_0^*V + E_1^*V + \dots + E_i^*V) \cap (E_jV + E_{j+1}V + \dots + E_dV)$$

Interpret sum on left to be 0 if $i < 0$ and V if $i \geq \delta$ Interpret sum on right to be 0 if $j > d$ and V if $j \leq 0$

LEM 12 We have

$$(i) \quad V_{i0} = E_0^*V + \dots + E_i^*V \quad (0 \leq i \leq \delta)$$

$$(ii) \quad V_{\delta j} = E_jV + \dots + E_dV \quad (0 \leq j \leq d)$$

pf clear

LEM 13 for $0 \leq i \leq \delta$ and $0 \leq j \leq d$,

$$(i) \quad (A - \theta_j I) V_{ij} \subseteq V_{i+1, j+1}$$

$$(ii) \quad A V_{ij} \subseteq V_{ij} + V_{i+1, j+1}$$

$$(iii) \quad (A^* - \theta_j^* I) V_{ij} \subseteq V_{i+1, j+1}$$

$$(iv) \quad A^* V_{ij} \subseteq V_{ij} + V_{i+1, j+1}$$

Pf (i) We have

$$(A - \theta_j I) (E_0^* V + \dots + E_i^* V) \subseteq E_0^* V + \dots + E_{i+1}^* V$$

and

$$(A - \theta_j I) (E_j V + \dots + E_d V) = E_{j+1} V + \dots + E_d V$$

(ii) By (i)

(iii), (iv) Sum. □

LEM 14. We have $d = \delta$. Moreover

$$V_{ij} = 0 \quad \text{if } i < j \quad (\text{ordered}) \quad (*)$$

Pf Switching A, A^* it nec. wlog $\delta \leq d$

First show $(*)$. To do this show

$$V_{0r} + V_{1,r+1} + \dots + V_{d-r,d} \quad (**)$$

is 0 for $0 \leq r \leq d$

Let r be given, let W be sum in $**$. By L13 (iii), (iv)

$$AW \subseteq W, \quad A^*W \subseteq W$$

So $W = 0$ or $W = V$ by used of V .

Show $W = 0$. By det II each term of $**$ is contained in

$$E_r V + \dots + E_d V$$

so

$$W \subseteq E_r V + \dots + E_d V$$

We assume $0 < r$ so W is properly contained in V . So $W = 0$

We have shown $(**)$ is 0 for $0 \leq r \leq d$ so $**$ holds.

Show $d = \delta$ Suppose $d \neq \delta$, so $\delta < d$

Set $i = \delta$, $j = d$ (*) to get

$$V_{\delta d} = 0$$

But by LEM 12

$$V_{\delta d} = E_d V$$

cont. So $d = \delta$.

□

Thm 15 For any subspaces $\{U_i\}_{i=0}^d$ of V TFAE:

$$(i) \quad U_i = (E_0^*V + \dots + E_i^*V) \cap (E_iV + \dots + E_dV) \quad 0 \leq i \leq d$$

(ii) $\{U_i\}_{i=0}^d$ is a decp of V and

$$(A - \theta_i I)U_i \subseteq U_{i+1}, \quad (A^* - \theta_i^* I)U_i \subseteq U_{i-1}$$

proved

(iii) For $0 \leq i \leq d$ both

$$U_{i+1} + \dots + U_d = E_iV + \dots + E_dV \quad (1)$$

$$U_0 + \dots + U_i = E_0^*V + \dots + E_i^*V \quad (2)$$

pf (i) \rightarrow (ii) To get the inclusions set $i=j$ in L13

and note $U_i = V_i$

Show $\{U_i\}_{i=0}^d$ is dec of V

show $V = U_0 + U_1 + \dots + U_d$

Define $W = U_0 + \dots + U_d$. By inclusions $AW \subseteq W$, $A^*W \subseteq W$

so $W = 0$ or $W = V$. Also W contains U_0 and $U_0 = E_0^*V \neq 0$

so $W \neq 0$ so $W = V$.

Show sum $V = u_0 + \dots + u_d$ is direct.

Suf to show

$$u_i \cap (u_0 + \dots + u_{i-1}) = 0 \quad 1 \leq i \leq d.$$

Let i be given. For $0 \leq j \leq i-1$

$$u_j \subseteq E_0^* V + \dots + E_j^* V \subseteq E_0^* V + \dots + E_{i-1}^* V$$

so

$$u_0 + \dots + u_{i-1} \subseteq E_0^* V + \dots + E_{i-1}^* V$$

Also

$$u_i \subseteq E_i V + \dots + E_d V$$

So

$$\begin{aligned} u_i \cap (u_0 + \dots + u_{i-1}) &\subseteq (E_i V + \dots + E_d V) \cap (E_0^* V + \dots + E_{i-1}^* V) \\ &= V_{i-1}, \mathcal{L} \\ &= 0 \end{aligned}$$

by L14.

Show $u_i \neq 0$ $0 \leq i \leq d$:

$$\text{We have } u_0 = E_0^* V \neq 0$$

$$u_d = E_d V \neq 0$$

Suppose $\exists i$ ($1 \leq i \leq d$) s.t. $u_i = 0$.

Then $u_0 + \dots + u_i$ is a non-0 subspace of V that is inv under A, A^* , contradicting the inv of V .

We have shown $\{u_i\}_{i=0}^d$ is a dec of V .

(iii) \rightarrow (i) show (1)

Abbr

$$W = u_0 + \dots + u_d$$

$$Z = E_0 V + \dots + E_d V$$

show $Z \subseteq W$: Def $X = \prod_{h=0}^{i-1} (A - \theta_h I)$

obs

$$Z = XV$$

Also Using the inclusions in (i)

$$XV \subseteq W$$

So $Z \subseteq W$

show $W \subseteq Z$: Def $Y = \prod_{h=i}^{d-1} (A - \theta_h I)$

obs

$$Z = \{v \in V \mid Yv = 0\}$$

By the inclusion (1)

$$Y U_i = 0 \quad \text{for } i \leq d$$

so

$$Y W = 0$$

so

$$W \subseteq Z$$

We have shown (1) and (2) is sim.

(iii) \rightarrow (i) First show sum $U_0 + \dots + U_d$ is direct.

To do this show

$$(U_0 + \dots + U_{i-1}) \cap U_i = 0 \quad \text{for } i \leq d$$

Let i be given. obs

$$\begin{aligned} (U_0 + \dots + U_{i-1}) \cap U_i &\subseteq (E_0^x V + \dots + E_{i-1}^x V) \cap (E_i V + \dots + E_d V) \\ &= V_{i,i} \\ &= 0 \end{aligned}$$

So $U_0 + \dots + U_d$ is direct. Now

$$\begin{aligned} U_i &= (U_0 + \dots + U_i) \cap (U_i + \dots + U_d) \\ &= (E_0^x V + \dots + E_i^x V) \cap (E_i V + \dots + E_d V) \end{aligned}$$



\mathbb{F} arb field

V is non 0 f.d vs over \mathbb{F}

$$\Phi = (A; \{E_i\}_{i=0}^d; A^* ; \{E_i^*\}_{i=0}^d)$$

is TD system on V with equal rep $\{\theta_i\}_{i=0}^d$ and

dual equal rep $\{\theta_i^*\}_{i=0}^d$.

DEF 16 By the Φ -split decomposition of V

we mean the rep $\{U_i\}_{i=0}^d$ from Th 15.

Next goal: For the split dec $\{U_i\}_{i=0}^d$

show

- $E_i^* V, U_i, E_i V$ have same dim ($= p_i$)
- $p_i = p_{d-i}$ for $0 \leq i \leq d$
- $p_{i-1} \leq p_i$ for $1 \leq i \leq d/2$

So $\{p_i\}_{i=0}^d$ is "symmetric" and "unimodal"

DEF 17 For $0 \leq i \leq d$ def lin trans

$$F_i : V \rightarrow V$$

by

$$(F_i - I) u_i = 0$$

$$F_i u_j = 0 \quad \text{if } i \neq j \quad (0 \leq i, j \leq d)$$

So F_i is the projection onto U_i

Define $F_{-1} = 0, F_{d+1} = 0.$

Obs

$$F_i F_j = \delta_{ij} F_i \quad 0 \leq i, j \leq d$$

$$I = \sum_{i=0}^d F_i$$

$$U_i = F_i V \quad 0 \leq i \leq d$$

LEM 18 F_α $0 \leq i < j \leq d$

$$(i) \quad E_i F_j = 0$$

$$(ii) \quad F_i E_j = 0$$

$$(iii) \quad E_j^* F_i = 0$$

$$(iv) \quad F_j E_i^* = 0$$

pf (i)

$$\begin{aligned} E_i F_j V &= E_i U_j \\ &\leq E_i (U_j + \dots + U_d) \\ &= E_i (E_j V + \dots + E_d V) \\ &= 0 \end{aligned}$$

$$\begin{aligned} (ii) \quad F_i E_j V &\leq F_i (E_j V + \dots + E_d V) \\ &= F_i (U_j + \dots + U_d) \\ &= 0 \end{aligned}$$

(iii), (iv) Sim

□

LEM 19 F_n orthogonal

(i) $F_i E_i F_i = F_i$

(ii) $E_i F_i E_i = E_i$

(iii) $F_i E_i^* F_i = F_i$

(iv) $E_i^* F_i E_i^* = E_i^*$

Pf (i)

$$F_i = F_i^2$$

$$= F_i (E_0 + \dots + E_n) F_i$$

$$\left[\begin{array}{l} F_n \text{ orthogonal} \quad F_i E_j = 0 \text{ if } j > i \text{ and } E_j F_i^* = 0 \\ \text{if } j < i \end{array} \right]$$

$$= F_i E_i F_i$$

(ii) - (iv) Sim

□

LEM 20 For $\mathcal{O} \subseteq \mathcal{E}$

(i) the \mathcal{L} maps

$$U_i \longrightarrow E_i V$$

$$E_i V \longrightarrow U_i$$

$$v \longrightarrow E_i v$$

$$v \longrightarrow F_i v$$

are bijections, and moreover they are inverses

(ii) the \mathcal{L} maps

$$U_i \longrightarrow E_i^* V$$

$$E_i^* V \longrightarrow U_i$$

$$v \longrightarrow E_i^* v$$

$$v \longrightarrow F_i v$$

are bijections, and moreover they are inverses

Pf (i) they are inverses by L19 (i), (ii)

It follows they are bijections.

(ii) Sim

□

COR 21 For fixed the dimensions of

$E_i V$, U_i , $E_i^* V$ are equal

Denoting this dim by p_i we have

$$p_i = p_{d-i}$$

Pf. By L20 the dimensions of

$E_i V$, U_i , $E_i^* V$ are equal.

call it p_i

To show $p_i = p_{d-i}$ suf to show

$$\dim E_i^* V = \dim E_{d-i} V \quad (*)$$

We just showed

$$\dim E_i V = \dim E_i^* V$$

Apply this result to \mathbb{F}^d to get $(*)$ □

DEF 22 Set

$$R = A - \sum_{h=0}^d \theta_h F_h$$

$$L = A^* - \sum_{h=0}^d \theta_h^* F_h$$

We call R (resp. L) the raising (resp. lowering) map

LEM 23 For $0 \leq i \leq d$ the following hold on U_i

$$R = A - \theta_i I,$$

$$L = A^* - \theta_i^* I$$

Pf. Since F_h is proj onto U_h for $0 \leq h \leq d$.

□

COR 24 F_n ordered

(i) $R u_i \leq u_{i+1}$

(ii) $L u_i \leq u_{i+1}$

Pf Combine Th 15 (ii) and L 23

□

LEM 25 For $0 \leq i \leq j \leq d$ the lin trans

$$U_i \rightarrow U_j$$

$$v \rightarrow R^{j-i} v$$

is an injection if $i+j \leq d$, a bij if $i+j = d$, and a

surjection if $i+j \geq d$. The lin trans

$$U_j \rightarrow U_i$$

$$v \rightarrow L^{j-i} v$$

is an injection if $i+j \geq d$, a bij if $i+j = d$, and a

surjection if $i+j \leq d$.

(Caution: above maps are not inverses, even if $i+j = d$)

Pf Concerning R

Case $i+j \leq d$: Given $v \in U_i$ s.t. $R^{j-i} v = 0$ show $v = 0$

Obs

$$0 = R^{j-i} v$$

$$= (A - \theta_j I) \cdots (A - \theta_i I) (A - \theta_i I) v$$

so

$$v \in E_i V + \cdots + E_{j-1} V$$

$$\subseteq E_0 V + \cdots + E_{j-1} V$$

Also

$$\begin{aligned} v &\in U_i \\ &\subseteq U_0 + U_1 + \dots + U_i \\ &= E_0^* V + \dots + E_i^* V \end{aligned}$$

So

$$v \in \underbrace{(E_0^* V + \dots + E_i^* V)}_{=0 \text{ by L14 applied to } \mathbb{F}^d} \cap (E_0 V + \dots + E_{i-1} V)$$

So

$$v = 0^d$$

Case $i \neq d$ U_i, U_j have same dim so above is b.c.g.

Case $i \neq d$ Given $w \in U_j$ find $v \in U_i$ s.t. $R^{j-i} v = w$

Consider map

$$\begin{aligned} U_{d-j} &\rightarrow U_j \\ u &\rightarrow R^{d-j} u \end{aligned}$$

By above comments this is b.c.g. So $\exists u \in U_{d-j}$ s.t.

$$R^{d-j} u = w$$

Def $v = R^{i-d} u,$

then $v \in U_i$ and $R^{j-i} v = w,$ P.F. for L 511 \square

COR 26 We have

$$p_{i-1} \leq p_i \quad 1 \leq i \leq d/2$$

Pf the map

$$U_{i-1} \rightarrow U_i$$

$$v \rightarrow Rv$$

is an isomorphism by L25 so

$$\begin{array}{ccc} \dim U_{i-1} & = & \dim U_i \\ \text{"} & & \text{"} \\ p_{i-1} & & p_i \end{array}$$

□

Next goal: the tetrahedron diagram

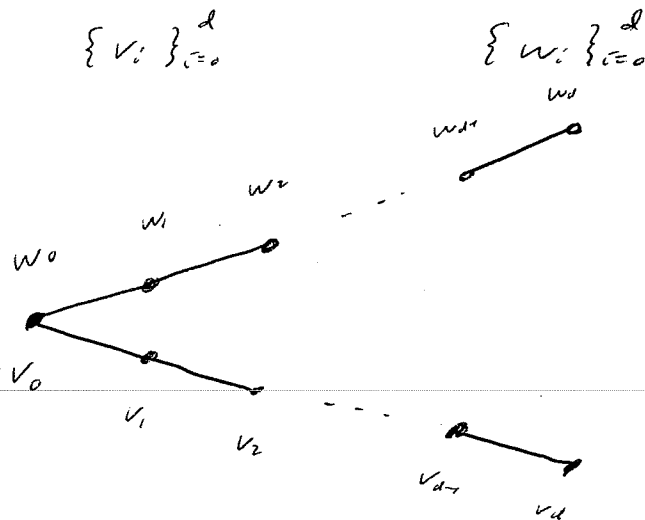
Notation Given decomp of V of length d

$$\{v_i\}_{i=0}^d$$

Represent by dotted line segment



Given two decomp of V of length d :



means

$$\sum_{h=0}^d v_h = \sum_{h=0}^d w_h$$

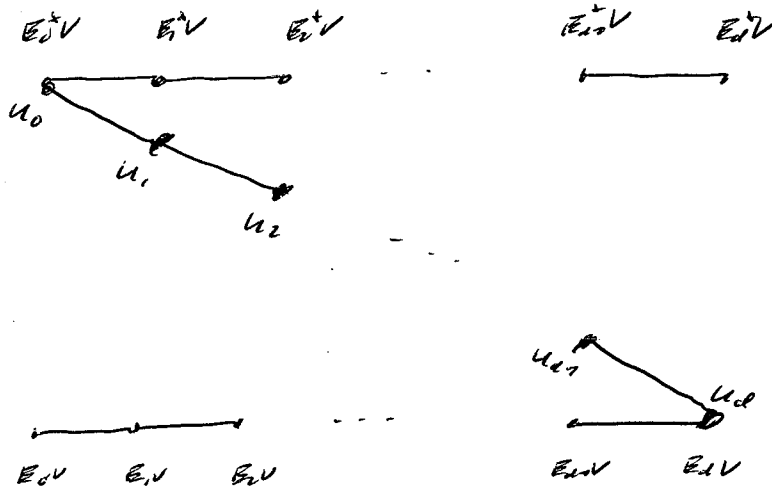
$$0 \leq i \leq d$$

Recall the split dec satisfies

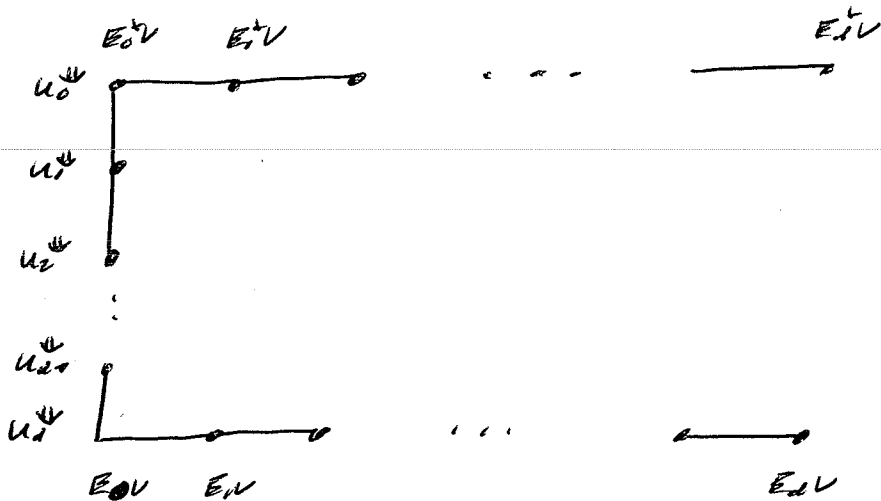
$$u_0 + u_1 + \dots + u_i = E_0^+ V + \dots + E_i^+ V \quad \text{o e e e d}$$

$$u_1 + u_2 + \dots + u_d = E_1^- V + \dots + E_d^- V$$

Corresp. diagram is

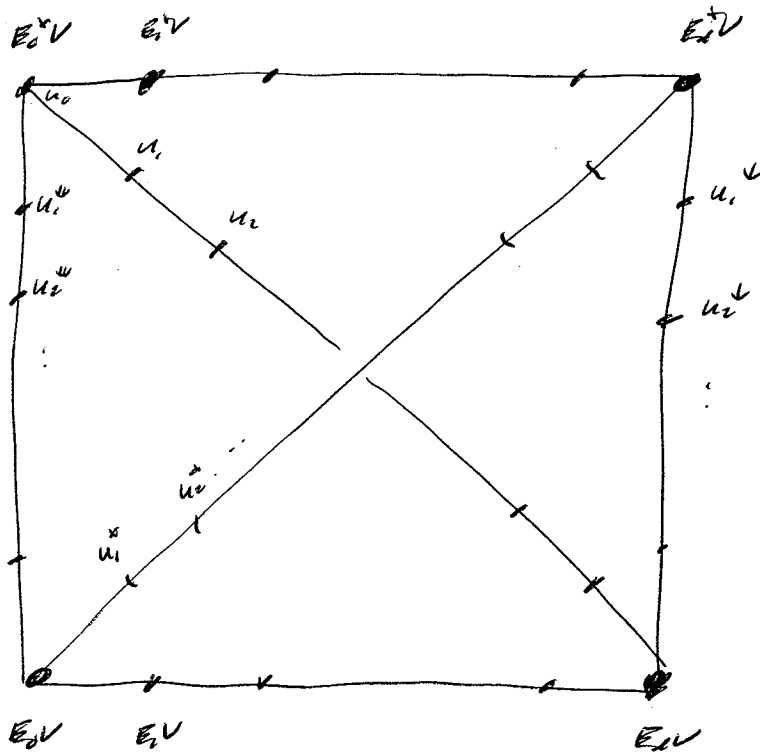


Applying this to Φ^{\downarrow} get



Other relations of Φ give similar diagrams

Altogether gut



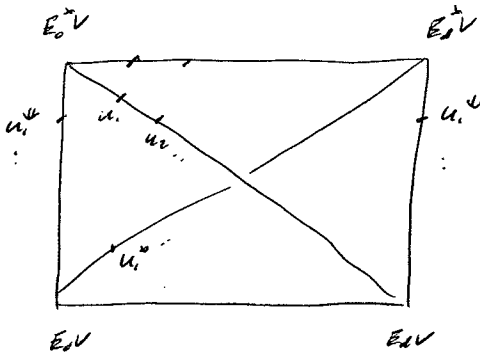
Field \mathbb{F} is arb

$0 \neq V = \text{f.d. } V \text{ is } \mathbb{F}$

$\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ TD system on V shape $\{p_i\}_{i=0}^d$

Split dec $\{u_i\}_{i=0}^d$

Last lecture we got



We now describe this picture further.

Notation. Let $\{\Delta_i\}_{i=0}^d$ be sequence of pos

integers whose sum is $\dim V$.

A flag on V of shape $\{\Delta_i\}_{i=0}^d$ is a nested sequence of subspaces

$$V_0 \subseteq V_1 \subseteq \dots \subseteq V_d$$

such that

$$\dim V_i = s_0 + s_1 + \dots + s_i \quad 0 \leq i \leq d$$

So $V_d = V$. Call V_i the i th component of the flag.

this construction yields a flag on V of shape $\{s_i\}_{i=0}^d$

Let $\{W_i\}_{i=0}^d$ be a decomp of V with $s_i = \dim W_i \quad 0 \leq i \leq d$

$$\text{def } V_i = W_0 + \dots + W_i \quad 0 \leq i \leq d$$

then $\{V_i\}_{i=0}^d$ is flag on V shape $\{s_i\}_{i=0}^d$

Given 2 flags on V : $\{V_i\}_{i=0}^d$ and $\{V_i'\}_{i=0}^d$

Call these opposite whenever \exists decomp $\{W_i\}_{i=0}^d$ of V

set

$$V_i = W_0 + \dots + W_i \quad 0 \leq i \leq d$$

$$V_i' = W_0 + \dots + W_{d-i}$$

In this case

$$V_i \cap V_j' = 0 \quad \text{if } i+j < d \quad (0 \leq i, j \leq d)$$

$$W_i = V_i \cap V_{d-i}' \quad 0 \leq i \leq d.$$

So $\{W_i\}_{i=0}^d$ is determined by the given flags. Call this the associated ^{decomp.}

DEF 27 Ref to our TD system \mathbb{F}

we now define 4 flags on V , denoted

$$[0], [0], [0^*], [0^*]$$

Each has shape $\{p_i\}_{i=0}^d$

flag	i th component
$[0]$	$E_0V + \dots + E_dV$
$[0]$	$E_dV + \dots + E_0V$
$[0^*]$	$E_0^*V + \dots + E_d^*V$
$[0^*]$	$E_d^*V + \dots + E_0^*V$

Obs $[0], [0]$ are opp and $[0^*], [0^*]$ are opp.

LEM 28. the four flags in Def 27 are

mutually opposite.

Pf. Show $[0^*], [0]$ opp.

take split dec $\{u_i\}_{i=0}^d$ for \mathbb{F} .

For odd d

$$\begin{aligned} i\text{th comp of } [0^d] &= E_d^i V + r E_d^i V \\ &= u_0 + \dots + u_i \end{aligned}$$

$$\begin{aligned} i\text{th comp of } [0] &= E_d^i V + \dots + r E_d^i V \\ &= u_0 + \dots + u_{d-i} \end{aligned}$$

Rest of pt is sim.



In our tetrahedron picture there are 6 decomp of V

we now give them more convenient names.

Given an ordered pair of distinct flags in Def 27

$[\alpha], [\beta]$ denote by $[\alpha, \beta]$ the associated decomp.

Note that $[\beta, \alpha]$ is the "inverse" of $[\alpha, \beta]$ i.e. $[\alpha, \beta]$

written in reverse order.

We have

decomp	its subspace of decomp
$[0 \ 0]$	$E_i V$
$[0^* \ 0^*]$	$E_i^* V$
$[0^* \ 0]$	$(E_0^* V + \dots + E_i^* V) \cap (E_0 V + \dots + E_i V)$
$[0^* \ 0^*]$	$(E_0^* V + \dots + E_i^* V) \cap (E_0 V + \dots + E_{i-1} V)$
$[0^* \ 0]$	$(E_0^* V + \dots + E_{i-1}^* V) \cap (E_0 V + \dots + E_{i-1} V)$
$[0^* \ 0^*]$	$(E_0^* V + \dots + E_{i-1}^* V) \cap (E_i V + \dots + E_n V)$

We now summarize the action of A, A^* on our 6 decomp's

LEM 29. Let $\{W_i\}_{i=0}^d$ be any one of the 6 decomp's of V

given in the above table. Then for $0 \leq i \leq d$ the action of A and

A^* on W_i is:

Name	A action	A^* action
$[0 \ 0]$	$(A - \theta_i I) W_i = 0$	$A^* W_i \subseteq W_{i+1} \oplus W_i \oplus W_{i-1}$
$[0^\vee \ 0^\vee]$	$A W_i \subseteq W_{i+1} \oplus W_i \oplus W_{i-1}$	$(A^* - \theta_i^* I) W_i = 0$
$[0^* \ 0]$	$(A - \theta_i I) W_i \subseteq W_{i+1}$	$(A^* - \theta_i^* I) W_i \subseteq W_{i+1}$
$[0^* \ 0]$	$(A - \theta_{i-1} I) W_i \subseteq W_{i+1}$	$(A^* - \theta_{i-1}^* I) W_i \subseteq W_{i+1}$
$[0^* \ 0]$	$(A - \theta_{i-2} I) W_i \subseteq W_{i+1}$	$(A^* - \theta_{i-2}^* I) W_i \subseteq W_{i+1}$
$[0^\vee \ 0]$	$(A - \theta_i I) W_i \subseteq W_{i+1}$	$(A^* - \theta_i^* I) W_i \subseteq W_{i+1}$

Pf Rows $[0 \ 0]$ and $[0^* \ 0^\vee]$ result def of TDS.

Row $[0^\vee \ 0]$ is from Th 15 (ii)

Remaining rows are Th 15 (ii) applied to relatives of \mathbb{F} . \square

Consider our 6 decoms of V , from the tetrahedron picture,
 the decoms $[0, 0]$ and $[0^* 0^*]$ are eigenspace decoms
 for A, A^* resp. Tempting to view remaining 4 decoms
 also as eigenspace decoms

To make progress here we assume until further notice

$$0 \neq q \in \mathbb{F} \quad q^2 \neq 1$$

$$\theta_i = q^{2i-d} \quad 0 \leq i \leq d$$

$$\theta_i^* = q^{d-2i} \quad 0 \leq i \leq d.$$

(obs $q^{2i} \neq 1 \quad 1 \leq i \leq d$ since equals mult dist)

In this case TD relations become q -Serre rels

$$A^3 A^* - [3]_q A^2 A^* A + [3]_q A A^* A^2 - A^* A^3 = 0$$

$$A^* A^3 - [3]_q A^* A A^2 + [3]_q A^* A A^2 - A^* A^3 = 0$$

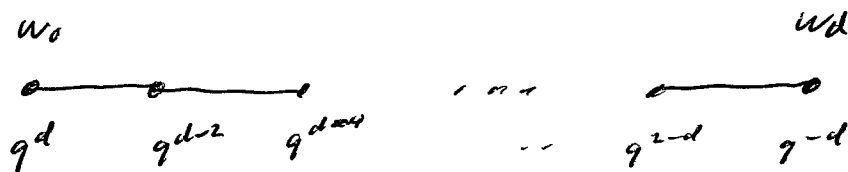
So A, A^* comes from unred A_q -module.

Notation Given a dec $\{w_i\}_{i=0}^d$ of V

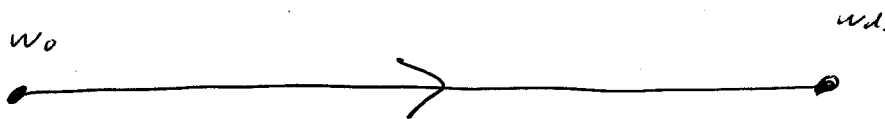


Consider the lin trans $T: V \rightarrow V$ s.t. $\mu \in \text{eig}$

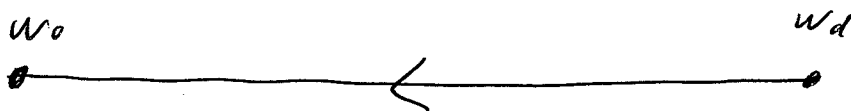
W_i is eigenspace for T with eigenval μ^{d-2i}



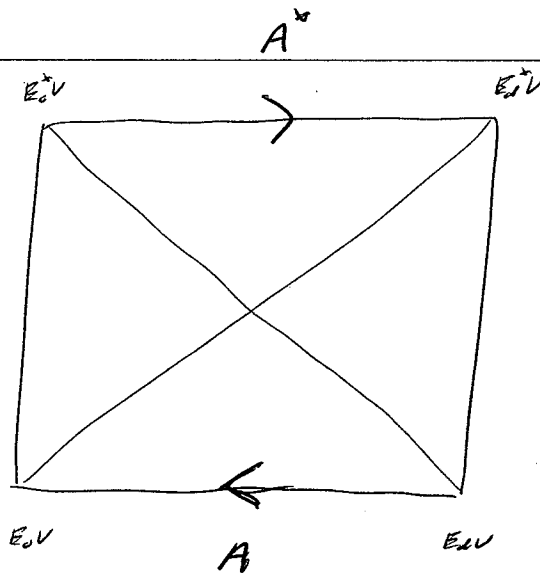
We often represent this lin trans by directed arc



Obs the inverse of T is rep by

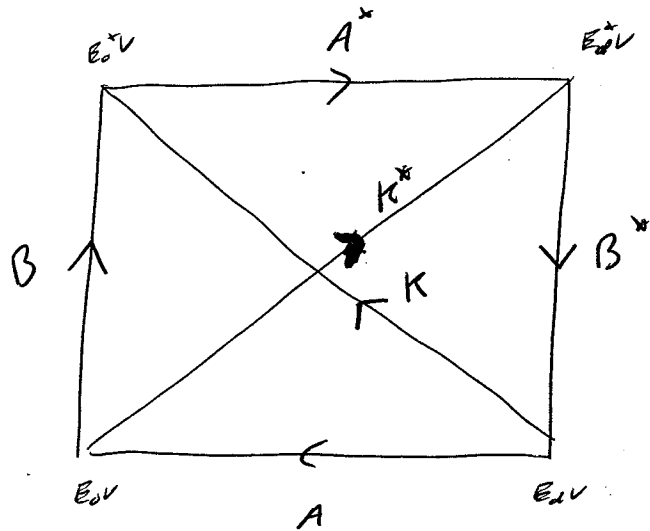


EX



DEF 30. We def lin trans $B, B^*, K, K^* \quad V \rightarrow V$

as follows



So for example, μ is called

$$(E_0^*V + \mu E_0V) \wedge (E_0V + \mu E_0^*V)$$

is eigenspace for K with equal q 2^{nd} .

Next goal: Find relations sat by

$$A, A^*, B, B^*, K, K^*, K^{-1}, K^{*-1}$$

We will use the following handy facts.

For any lin trans $Y: V \rightarrow V$ $\forall \theta \in \mathbb{F}$ def

$$V_Y(\theta) = \{v \in V \mid Yv = \theta v\}$$

LEM 31 Given any lin trans $Y: V \rightarrow V$ and $Z: V \rightarrow V$

Given $\theta \neq 0 \in \mathbb{F}$ TFAE

(i)
$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I \quad \text{on } V_Y(\theta)$$
 "q-Weyl equation"

(ii)
$$(Z - \theta^{-1}I) V_Y(\theta) \subseteq V_Y(q^{-2}\theta)$$

Pf $\forall v \in V_Y(\theta)$ we have

$$\begin{aligned} & \left(qYZ - q^{-1}ZY - (q^{-2}\theta I) \right) v \\ &= q(Y - q^{-2}\theta I)(Z - \theta^{-1}I)v \end{aligned}$$

LEM 32 Given any lin trans $Y: V \rightarrow V$ $Z: V \rightarrow V$

Given $0 \neq \theta \in F$ TFAE

$$(i) \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I \quad \text{on} \quad V_Z(\theta)$$

$$(ii) \quad (Y - \theta^{-1}I) | V_Z(\theta) \leq V_Z(q^2\theta)$$

PF Replace (Y, Z, q) by (Z, Y, q^{-1}) in L31. \square

LEM 33 Given lin trans $Y: V \rightarrow V$ and $Z: V \rightarrow V$

$$(i) \quad Y^3Z - [3]_q Y^2ZY + [3]_q YZY^2 - ZY^3 = 0 \quad \text{on} \quad V_Y(\theta)$$

$$(ii) \quad Z | V_Y(\theta) \leq V_Y(q^2\theta) + V_Y(\theta) + V_Y(q^{-2}\theta)$$

PF ex

LEM 34 We have

$$(i) \quad \frac{qAD - q^{-1}BA}{q - q^{-1}} = I$$

$$(ii) \quad \frac{qBA^{\vee} - q^{-1}A^{\vee}B}{q - q^{-1}} = I$$

$$(iii) \quad \frac{qA^{\vee}B^{\vee} - q^{-1}B^{\vee}A^{\vee}}{q - q^{-1}} = I$$

$$(iv) \quad \frac{qB^{\vee}A - q^{-1}AB^{\vee}}{q - q^{-1}} = I$$

Pf In each case, combine L30/L31 and L29. \square

LEM 35 We have

$$(i) \quad \frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} = I$$

$$(ii) \quad \frac{qKAK^{\vee} - q^{-1}A^{\vee}K}{q - q^{-1}} = I$$

$$(iii) \quad \frac{qAK^{\vee} - q^{-1}K^{\vee}A}{q - q^{-1}} = I$$

$$(iv) \quad \frac{qA^{\vee}K^{\vee\vee} - q^{-1}K^{\vee\vee}A^{\vee}}{q - q^{-1}} = I$$

Pf In each case, combine L30/L31 and L29. \square

We now give the action of B, B^* on the \mathbb{C} decomp

LEM 36 Let $\{W_i\}_{i=0}^d$ be any one of the \mathbb{C} decomp of V

from \boxtimes picture. For $0 \leq i \leq d$ the action of B, B^* on W_i is:

Name	B -action	B^* -action
$[0 \ 0]$	$(B - q^{d-2i}I)W_i \subseteq W_{i-1}$	$(B^* - q^{2i}I)W_i \subseteq W_{i+1}$
$[0^r \ 0^s]$	$(B - q^{2i-d}I)W_i \subseteq W_{i-1}$	$(B^* - q^{2i}I)W_i \subseteq W_{i+1}$
$[0^r \ 0]$	$(B - q^{2i-d}I)W_i \subseteq W_{i+1}$	$(B^* - q^{2i}I)W_i \subseteq W_{i+1}$
$[0^s \ 0]$	$(B - q^{2i-d}I)W_i = 0$	$B^*W_i \subseteq W_{i-1} + W_i + W_{i+1}$
$[0^s \ 0]$	$(B - q^{2i-d}I)W_i \subseteq W_{i+1}$	$(B^* - q^{4-2i}I)W_i \subseteq W_{i+1}$
$[0^s \ 0]$	$BW_i \subseteq W_{i-1} + W_i + W_{i+1}$	$(B^* - q^{4-2i}I)W_i = 0$

pf $[0, 0]$ to get B action use L34 (i) and L31

to get B^* action use L34 (iv) and L32.

$[0^r, 0^s]$ sim

$[0^s, 0]$...

$[0^s, 0]$..