Lecture 34 Wednesday April 15

3. Some Algebra related to Q-poly DRGs

We would like to investigate the structure of an irred T-module for a Q-poly DRG. (not nec thin)

To do this efficiently it is helpful to introduce the notion of a <u>tridiagonal pair</u> of linear transformations.

We will use our results on TD pairs to get an action of the quantum gp $U_q(\widehat{sl_2})$ on the st. module of certain DRGs.

Until further notice $\mathbb{F}$ is <u>any field</u>

Def 1 Let V denote a non0, finite dim'l vector space over $\mathbb{F}$. A <u>tridiagonal pair</u> (TD pair) on V is an ordered pair of linear transformations

$$A : V \to V, \qquad A^*: V \to V$$

such that

(i) Each of A, A^* is diagonalizable on V

(ii) $\exists$ an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A s.t.

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \le i \le d$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) $\exists$ an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* s.t.

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad 0 \le i \le \delta$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$

(iv) there does not exist a subspace W of V s.t.

$$A W \subseteq W, \quad A^* W \subseteq W, \qquad W \ne 0, \qquad W \ne V.$$

"V is irreducible as a module for A, A^*"

Note 2. Given a TD pair A, A^* on V. Then

A^*, A is a TD pair on V. Call these TD pairs <u>duals</u>

EX 3 (Frac) Given a DRG $\Gamma = (X, R)$ diam D

Assume $\{E_i\}_{i=0}^{D}$ is a Q-poly ordering of the prim

idempotents of Γ. Fix $x \in X$ write $T = T(x)$ etc.

Let W denote an irred T-module. Then

the pair A, A^* acts on W as a TD pair

Pf. Let r = endpt of W
t = dual endpt of W
δ = diameter
d = dual diameter

Ref to Def 1 take V to be W

(i) clear

(ii) take $V_i = E_{t+i} W$ $\quad 0 \le i \le d$

(iii) take $V_i^* = E^*_{r+i} W$ $\quad 0 \le i \le \delta$

(iv) Since W is irred as T-module, and since T is gen by A, A^*. $\square$

Referring to Def 1, it will turn out $d = \delta$.

EX 4. A Leonard pair is the same thing as a TD pair for which the V_i, V_i^* all have dim 1.

We now give an example of a TD pair related to quantum gps

Until further notice

assume $\mathbb{F}$ is alg closed

Fix $0 \neq q \in \mathbb{F}$ s.t. q not a root of 1

Recall

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{Z}$$

Let A_q denote the associative $\mathbb{F}$-algebra with 1, defined by generators x, y and relations

$$x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = 0$$

$$y^3x - [3]_q y^2xy + [3]_q yxy^2 - xy^3 = 0$$

"cubic q-Serre relations"

A_q called "positive part of $U_q\hat{sl}_2$"

Recall q-Serre relations are special case of the TD relations.

Ex 5 let V denote a finite-dim'l irred A_q-module on which neither of x, y is nilpotent. Then x, y act on V as a TD pair.

Pf $\forall\, \theta \in \mathbb{F}$ def

$$V(\theta) = \theta\text{-eigenspace of } x \text{ on } V$$
$$= \{ v \in V \mid x.v = \theta v \}$$

Possibly $V(\theta) = 0$

We show that $\forall\, \theta \in \mathbb{F}$

$$y\, V(\theta) \subseteq V(q^{-2}\theta) + V(\theta) + V(q^{2}\theta) \qquad (*)$$

To see $(*)$ pick $v \in V(\theta)$ so $xv = \theta v$. Obs

$$0 = \left(x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 \right) v$$
$$= x^3 y v - [3]_q \theta x^2 y v + [3]_q \theta^2 x y v - \theta^3 y v$$
$$= (x - q^2 I)(x - I)(x - q^{-2} I) y v$$

We assume q not a root of 1 so

$$q^2, 1, q^{-2} \quad \text{must distinct}$$

so

$$yv \in V(q^{-2}\theta) + V(\theta) + V(q^2\theta)$$

Since $\mathbb{F}$ is alg closed and x is not nil on V

x has at least one non0 eigval θ on V.

Consider sequence

$$\theta, q^{-2}\theta, q^{-4}\theta, \ldots$$

these scalars must distinct since q not a root of 1.

So they are not all eigvals of x on V

So $\exists$ eigval α of x on V s.t $q^{-2}\alpha$ is not an eigval of x on V.

Consider sequence

$$\alpha, \alpha q^2, \alpha q^4, \ldots$$

$\exists$ nonneg integer d s.t αq^{2i} is eigval of x on V

pro $0 \le i \le d$ but not $i = d+1$.

Set

$$V_i = V(\alpha q^{2i}) \qquad 0 \le i \le d$$

Obs

$$V_0 + V_1 + \cdots + V_d$$

is nonzero and x-inv by constr. By $(*)$

$$y V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d)$$

where $V_{-1} = 0$, $V_{d+1} = 0$. So $V_0 + \cdots + V_d$ is y-inv.

Now

$$V = \sum_{i=0}^{d} V_i$$

by irreducibility. Now x is diagonalizable on V

and Def 1 (ii) holds.

Int the roles of x, y we find y is diag on V, and

Def 1 (iii) holds.

Def 1 (iv) is from irred of V. Result follows. $\square$

Until further notice

$\mathbb{F}$ alg closed, char 0

View $sl_2(\mathbb{F})$ as the Lie algebra of all

2×2 trace 0 matrices over $\mathbb{F}$, with Lie bracket

$$[r, s] = rs - sr.$$

Given $x \in sl_2(\mathbb{F})$, char poly of x has form

$$\lambda^2 - \alpha \qquad \alpha \in \mathbb{F}$$

(Since x has trace 0)

Call x <u>nilpotent</u> if $\alpha = 0$ and <u>semi simple</u> if $\alpha \neq 0$

Given $x, y \in sl_2(\mathbb{F})$ obs x, y generate

$sl_2(\mathbb{F})$ iff $x, y, [x, y]$ are lin indep (since $sl_2(\mathbb{F})$

has dim 3 as v.s / $\mathbb{F}$)

Ex 6. Given semi-simple $x, y \in sl_2(\mathbb{F})$

that generate $sl_2(\mathbb{F})$. Let V denote a f.d.

irred $sl_2(\mathbb{F})$-module. Then x, y act on V as a TD pair.
(In fact LP)

pf ex. $\square$

Recall Onsager algebra $\mathcal{O}$ is the Lie algebra over $\mathbb{F}$

defined by gens x, y and rels

$$[x, [x, [x, y]]] = 4[x, y]$$

$$[y, [y, [y, x]]] = 4[y, x]$$

Ex 7. Let V denote a f.d. irred $\mathcal{O}$-module.

Then x, y act on V as a TD pair

pf Similar to pf of Ex 5. $\square$

Assume $\mathbb{F}$ arb

Given TD pair A, A^* on V

An ordering of the eigenspaces of A (resp A^*) is called standard whenever it satisfies Def 1 (ii) (resp. Def 1 (iii))

Obs if $\{V_i\}_{i=0}^d$ is a st. ordering of the eigenspaces of A then so is $\{V_{d-i}\}_{i=0}^d$, and no other ordering is standard.

Sim for A^*.

An ordering of the primitive idempotents or eigenvalues of A (resp A^*) is called standard if the corresp ordering of the eigenspaces is standard.

Def 8. Let V denote a non 0 f.d. vector space / $\mathbb{F}$

By a <u>tridiagonal system</u> (TD system) on V we mean a sequence

$$\Phi = (A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta})$$

s.t

(i) A, A^* is a TD pair on V

(ii) $\{E_i\}_{i=0}^{d}$ is a st. ordering of the prim idempotents of A

(iii) $\{E_i^*\}_{i=0}^{\delta}$ " " A^*

Say Φ is <u>over</u> $\mathbb{F}$. Call V the <u>underlying vector space</u>

—

Ref to Def 8, obs

$$E_i A^* E_j = 0 \quad \text{if } |i-j| > 1 \qquad (0 \le i,j \le d)$$

$$E_i^* A E_j^* = 0 \quad \text{if } |i-j| > 1 \qquad (0 \le i,j \le \delta)$$

Next goal: explain the relations of a TD system.

Given a TD system Φ on V as in Def 8

Then the following are TD systems on V:

$$\Phi^{*} := \left(A^{*};\ \{E_i^{*}\}_{i=0}^{\delta};\ A,\ \{E_i\}_{i=0}^{d} \right) \qquad \text{"dual"}$$

$$\Phi^{\downarrow} := \left(A;\ \{E_i\}_{i=0}^{d};\ A^{*};\ \{E_{\delta-i}^{*}\}_{i=0}^{\delta} \right) \qquad \text{"1st inv"}$$

$$\Phi^{\Downarrow} := \left(A;\ \{E_{d-i}\}_{i=0}^{d};\ A^{*};\ \{E_i^{*}\}_{i=0}^{\delta} \right) \qquad \text{"2nd inv"}$$

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all TD systems

$$*^2 = \downarrow^2 = \Downarrow^2 = 1$$

$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow \Downarrow = \Downarrow \downarrow$$

the group generated by symbols $*, \downarrow, \Downarrow$ subject to the relations alone is the Octahedral gp D_4. this is the group of symmetries of a square and has 8 elements.

Obs $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all TD systems.

Two TD systems will be called relatives whenever they are in the same orbit of this D_4-action.

DEF 9. Given TD system

$$\Phi = \left(A, \{E_i\}_{i=0}^{d}, A^*, \{E_i^*\}_{i=0}^{\delta} \right)$$

For $0 \le i \le d$ let θ_i denote the eigenvalue of A for E_i

For $0 \le i \le \delta$ .. θ_i^* - - - A^* for E_i^*

Call $\{\theta_i\}_{i=0}^{d}$ the eigenvalue sequence of Φ

Call $\{\theta_i^*\}_{i=0}^{\delta}$ the dual eigenvalue sequence of Φ

Notation 10 Given TD system Φ Given $g \in D_4$

For any object f assoc with Φ, f^g will denote the corresp object for $\Phi^{g^{-1}}$. So

$$\theta_i(\Phi^*) = \theta_i^*(\Phi) \qquad \text{etc.}$$

Next goal: split decomp.

Notation. Let V be nonO, f.d v.s $/\mathbb{F}$.

d = nonneg integer.

A <u>decomposition of V of length d</u> is a sequence of subspaces

$\{U_i\}_{i=0}^{d}$ s.t $\quad U_i \neq 0 \quad 0 \le i \le d$ and

$$V = \sum_{i=0}^{d} U_i \qquad \text{(dir sum)}.$$

We set $U_{-1} = 0$, $U_{d+1} = 0$.

Given a TD system

$$\Phi = (A, \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta})$$

on V we now show $d = \delta$. Also show $\exists$

unique dec $\{U_i\}_{i=0}^{d}$ of V s.t

$$(A - \theta_i I) U_i \subseteq U_{i+1} \qquad 0 \le i \le d$$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1} \qquad 0 \le i \le d$$

where $\{\theta_i\}_{i=0}^{d}$ (resp $\{\theta_i^*\}_{i=0}^{d}$) is the eigval seq (dual eigval seq) of Φ.

$\square$

Lecture 35 Friday April 17

Field $\mathbb{F}$ arb

V denotes non0 f.d. v.s $/\mathbb{F}$

$\Phi = (A, \{E_i\}_{i=0}^d ; A^*; \{E_i^*\}_{i=0}^\delta)$ is TD system on V.

with eigval seq $\{\theta_i\}_{i=0}^d$ and dual eigval seq $\{\theta_i^*\}_{i=0}^\delta$

DEF 11 For all integers i, j define

$$V_{ij} = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_jV + E_{j+1}V + \cdots + E_dV)$$

Interpret sum on left to be 0 if $i<0$ and V if $i>\delta$

Interpret sum on right to be 0 if $j>d$ and V if $j<0$

LEM 12 We have

(i) $V_{i0} = E_0^*V + \cdots + E_i^*V \qquad (0 \le i \le \delta)$

(ii) $V_{\delta j} = E_jV + \cdots + E_dV \qquad (0 \le j \le d)$

Pf clear

LEM 13 for $0 \le i \le \delta$ and $0 \le j \le d$,

(i) $(A - \theta_j I) V_{ij} \subseteq V_{i+1, j+1}$

(ii) $A V_{ij} \subseteq V_{ij} + V_{i+1, j+1}$

(iii) $(A^* - \theta_i^* I) V_{ij} \subseteq V_{i-1, j-1}$

(iv) $A^* V_{ij} \subseteq V_{ij} + V_{i-1, j-1}$

Pf (i) We have

$$(A - \theta_j I)(E_0^* V + \cdots + E_i^* V) \subseteq E_0^* V + \cdots + E_{i+1}^* V$$

and

$$(A - \theta_j I)(E_j V + \cdots + E_d V) = E_{j+1} V + \cdots + E_d V$$

(ii) By (i)

(iii), (iv) Sim. $\square$

LEM 14. We have $d=\delta$. Moreover

$$V_{ij}=0 \quad \text{if } i<j \qquad (0\le i,j\le d) \qquad (*)$$

Pf Switching A, A^* if nec. WLOG $\delta \le d$

First show $(*)$. To do this show

$$V_{0r}+V_{1,r+1}+\cdots+V_{d-r,d} \qquad (**)$$

is 0 for $0<r\le d$

Let r be given. Let W be sum in $**$. By L13 (iii), (iv)

$$AW\subseteq W, \qquad A^*W\subseteq W$$

So $W=0$ or $W=V$ by irred of V.

Show $W=0$. By Def 11 each term of $**$ is contained in

$$E_rV+\cdots+E_dV$$

so

$$W\subseteq E_rV+\cdots+E_dV$$

We assume $0<r$ so W is properly contained in V. So $W=0$

We have shown $(**)$ is 0 for $0<r\le d$ so $**$ holds.

Show $d=\delta$ Suppose $d \neq \delta$, so $\delta < d$

Set $i=\delta$, $j=d$ in $(*)$ to get

$$V_{\delta d} = 0$$

But by LEM 12

$$V_{\delta d} = E_d V$$

cont. So $d = \delta$. $\square$

Thm 15 For any subspaces $\{U_i\}_{i=0}^d$ of V TFAE:

(i) $U_i = (E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV) \quad 0 \le i \le d$

(ii) $\{U_i\}_{i=0}^d$ is a decomp of V and

$$(A - \theta_i I) U_i \subseteq U_{i+1}, \quad (A^* - \theta_i^* I) U_i \subseteq U_{i-1}$$

for $0 \le i \le d$

(iii) For $0 \le i \le d$ both

$$U_i + \cdots + U_d = E_iV + \cdots + E_dV \qquad (1)$$

$$U_0 + \cdots + U_i = E_0^*V + \cdots + E_i^*V \qquad (2)$$

pf (i)→(ii) To get the inclusions set $i=j$ in L13

and note $U_i = V_{ii}$

Show $\{U_i\}_{i=0}^d$ is dec of V

show $V = U_0 + U_1 + \cdots + U_d$

Define $W = U_0 + \cdots + U_d$. By inclusions $AW \subseteq W$, $A^*W \subseteq W$

so $W = 0$ or $W = V$. Also W contains U_0 and $U_0 = E_0^*V \neq 0$

$\varepsilon_0 W \neq 0$ So $W = V$.

Show sum $V = U_0 + \cdots + U_d$ is direct.

Suf to show

$$U_i \cap (U_0 + \cdots + U_{i-1}) = 0 \qquad 1 \leq i \leq d.$$

Let i be given. For $0 \leq j \leq i-1$

$$U_j \subseteq E_0^* V + \cdots E_j^* V \quad \subseteq E_0^* V + \cdots + E_{i-1}^* V$$

so

$$U_0 + \cdots + U_{i-1} \subseteq E_0^* V + \cdots + E_{i-1}^* V$$

Also

$$U_i \subseteq E_i V + \cdots + E_d V$$

So

$$U_i \cap (U_0 + \cdots + U_{i-1}) \subseteq (E_i V + \cdots + E_d V) \cap (E_0^* V + \cdots + E_{i-1}^* V)$$

$$= V_{i-1, i}$$

$$= 0$$

by L14.

Show $U_i \neq 0 \quad 0 \leq i \leq d$:

We have $U_0 = E_0^* V \neq 0$
$U_d = E_d V \neq 0$

Suppose $\exists\, i$ $(1 \le i \le d-1)$ s.t. $U_i = 0$.

Then $U_0 + \cdots + U_{i-1}$ is a non 0 subspace of V that is

inv under A, A^*, contradicting the irred of V.

We have shown $\{U_i\}_{i=0}^{d}$ is a dec of V.

(ii) → (iii) Show (1).

Abbr

$$W = U_i + \cdots + U_d$$

$$Z = E_i V + \cdots + E_d V$$

show $Z \subseteq W$: Def $X = \prod_{h=0}^{i-1} (A - \theta_h I)$

Obs

$$Z = XV$$

Also Using the inclusions in (ii)

$$XV \subseteq W$$

So $Z \subseteq W$ ✓

show $W \subseteq Z$: Def $Y = \prod_{h=i}^{d} (A - \theta_h I)$

Obs $Z = \{ v \in V \mid Yv = 0 \}$

By the inclusions in (iii)

$$Y U_j = 0 \quad \text{for} \quad i \le j \le d$$

so

$$Y W = 0$$

So

$$W \subseteq Z$$

We have shown (1) and (2) is sim.

(iii) $\to$ (i) First show sum $U_0 + \cdots + U_d$ is direct.

To do this show

$$(U_0 + \cdots + U_{i-1}) \cap U_i = 0 \qquad 1 \le i \le d$$

let i be given. Obs

$$(U_0 + \cdots + U_{i-1}) \cap U_i \subseteq (E_0^* V + \cdots E_{i-1}^* V) \cap (E_i V + \cdots + E_d V)$$

$$= V_{i-1,i}$$
$$= 0$$

So $U_0 + \cdots + U_d$ is direct. Now

$$U_i = (U_0 + \cdots + U_i) \cap (U_i + \cdots + U_d)$$
$$= (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V)$$

Lec 36 Monday April 20

$\mathbb{F}$ arb field

V is non 0 fd vs over $\mathbb{F}$

$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$

is TD system on V with eigenval seq $\{\theta_i\}_{i=0}^d$ and

dual eigenval seq $\{\theta_i^*\}_{i=0}^d$.

DEF 16 By the Φ-split decomposition of V

we mean the sequence $\{U_i\}_{i=0}^d$ from Th 15.

Next goal: For the split dec $\{U_i\}_{i=0}^d$

show

- E_i^*V, U_i, E_iV have same dim $(=\rho_i)$
- $\rho_i = \rho_{d-i}$ for $0 \le i \le d$
- $\rho_{i-1} \le \rho_i$ for $1 \le i \le d/2$

So $\{\rho_i\}_{i=0}^d$ is "symmetric" and "unimodal"

DEF 17 For $0 \le i \le d$ def lin trans

$$F_i : V \to V$$

by

$$(F_i - I)\, U_i = 0$$

$$F_i U_j = 0 \quad \text{if } i \neq j \qquad (0 \le j \le d)$$

So F_i is the projection onto U_i

Define $F_{-1} = 0$, $F_{d+1} = 0$.

Obs

$$F_i F_j = \delta_{ij} F_i \qquad 0 \le i, j \le d$$

$$I = \sum_{i=0}^{d} F_i$$

$$U_i = F_i V \qquad 0 \le i \le d$$

LEM 18 For $0 \le i < j \le d$

(i) $E_i F_j = 0$

(ii) $F_i E_j = 0$

(iii) $E_j^* F_i = 0$

(iv) $F_j E_i^* = 0$

pf (i)

$$
\begin{aligned}
E_i F_j V &= E_i U_j \\
&\subseteq E_i (U_j + \cdots + U_d) \\
&= E_i (E_j V + \cdots + E_d V) \\
&= 0
\end{aligned}
$$

(ii)

$$
\begin{aligned}
F_i E_j V &\subseteq F_i (E_j V + \cdots + E_d V) \\
&= F_i (U_j + \cdots + U_d) \\
&= 0
\end{aligned}
$$

(iii), (iv) Sim $\square$

LEM 19 For $0 \le i \le d$

(i) $F_i E_i F_i = F_i$

(ii) $E_i F_i E_i = E_i$

(iii) $F_i E_i^* F_i = F_i$

(iv) $E_i^* F_i E_i^* = E_i^*$

Pf (i)

$$F_i = F_i^2$$

$$= F_i (E_0 + \cdots + E_d) F_i$$

[For $0 \le j \le d$ $F_i E_j = 0$ if $j > i$ and $E_j F_i = 0$ if $j < i$]

$$= F_i E_i F_i$$

(ii) – (iv) Sim $\square$

LEM 20 For $0 \le i \le d$

(i) the lin trans

$U_i \to E_i V$ $\qquad$ $E_i V \to U_i$

$v \to E_i v$ $\qquad$ $v \to F_i v$

are bijections, and moreover they are inverses

(ii) the lin trans

$U_i \to E_i^* V$ $\qquad$ $E_i^* V \to U_i$

$v \to E_i^* v$ $\qquad$ $v \to F_i v$

are bijections, and moreover they are inverses

pf (i) they are inverses by L19 (i), (ii)

It follows they are bijections.

(ii) Sim $\square$

COR 21 For $0 \le i \le d$ the dimensions of

$$E_iV, \quad U_i, \quad E_i^*V \quad \text{are equal}$$

Denoting this dim by ρ_i we have

$$\rho_i = \rho_{d-i}.$$

pf. By L20 the dimensions of

$$E_iV, \quad U_i, \quad E_i^*V \quad \text{are equal.}$$

call it ρ_i

To show $\rho_i = \rho_{d-i}$ suf to show

$$\dim E_i^*V = \dim E_{d-i}^*V \qquad (*)$$

We just showed

$$\dim E_iV = \dim E_i^*V$$

Apply this result to $\Phi^{\Downarrow}$ to get $(*)$ $\square$

DEF 22 Set

$$R = A - \sum_{h=0}^{d} \theta_h F_h$$

$$L = A^* - \sum_{h=0}^{d} \theta_h^* F_h$$

We call R (resp. L) the raising (resp. lowering) map

LEM 23 For $0 \le i \le d$ the following hold on U_i

$$R = A - \theta_i I, \qquad L = A^* - \theta_i^* I$$

Pf Since F_h is proj onto U_h for $0 \le h \le d$. $\square$

COR 24 For $0 \le i \le d$

(i) $R U_i \subseteq U_{i+1}$

(ii) $L U_i \subseteq U_{i-1}$

pf Combine Th 15(ii) and L 23 $\square$

LEM 25 For $0 \le i \le j \le d$ the lin trans

$$U_i \longrightarrow U_j$$
$$v \longrightarrow R^{j-i} v$$

is an injection if $i+j \le d$, a bij if $i+j=d$, and a surjection if $i+j \ge d$. The lin trans

$$U_j \longrightarrow U_i$$
$$v \longrightarrow L^{j-i} v$$

is an injection if $i+j \ge d$, a bij if $i+j=d$, and a surjection if $i+j \le d$.

(Caution: above maps are not inverses, even if $i+j=d$)

pf Concerning R

Case $i+j \le d$: Given $v \in U_i$ s.t $R^{j-i} v = 0$ show $v=0$

Obs
$$0 = R^{j-i} v$$
$$= (A-\theta_{j-1} I) \cdots (A-\theta_{i+1} I)(A-\theta_i I) v$$

so
$$v \in E_i V + \cdots + E_{j-1} V$$
$$\subseteq E_0 V + \cdots + E_{j-1} V$$

Also

$$v \in U_i$$

$$\subseteq U_0 + U_1 + \cdots + U_i$$

$$= E_0^* V + \cdots + E_i^* V$$

So

$$v \in \underbrace{(E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_{j-1} V)}_{= 0 \text{ by L14 applied to } \Phi^{\Downarrow}}$$

So

$$v = 0 \checkmark$$

Case $i+j=d$ $\quad U_i, U_j$ have same dim so above inj is bij.

Case $i+j \geq d$ $\quad$ Given $w \in U_j$ find $v \in U_i$ s.t $R^{j-i} v = w$

Consider map

$$U_{d-j} \to U_j$$
$$u \to R^{d-2j} u$$

By above comments this is bij. So $\exists\, u \in U_{d-j}$ s.t

$$R^{d-2j} u = w$$

Def $\quad v = R^{i+j-d} u.$

then $v \in U_i$ and $R^{j-i} v = w$, $\quad$ Pf for L sim $\quad \square$

COR 26 We have

$$\rho_{i-1} \le \rho_i \qquad 1 \le i \le d/2$$

Pf the map

$$U_{i-1} \to U_i$$
$$v \to Rv$$

is an injection by L25 so

$$\underset{\rho_{i-1}}{\underset{\parallel}{\dim U_{i-1}}} \le \underset{\rho_i}{\underset{\parallel}{\dim U_i}}$$

$\square$

Next goal: the tetrahedron diagram

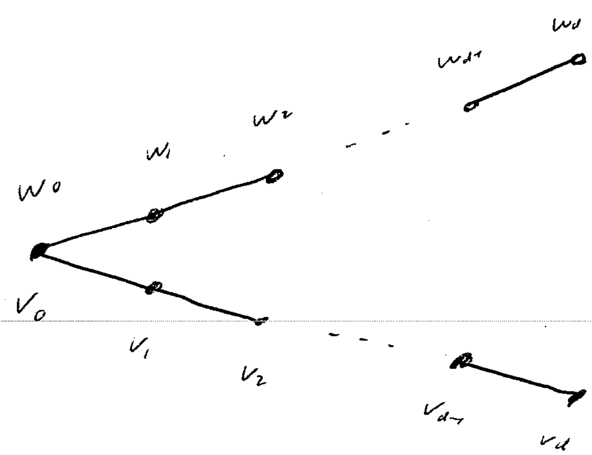
Notation Given decomp of V of length d

$$\{V_i\}_{i=0}^{d}$$

Represent by dotted line segment

Given two decomps of V of length d:

$$\{V_i\}_{i=0}^{d} \qquad \{W_i\}_{i=0}^{d}$$



means

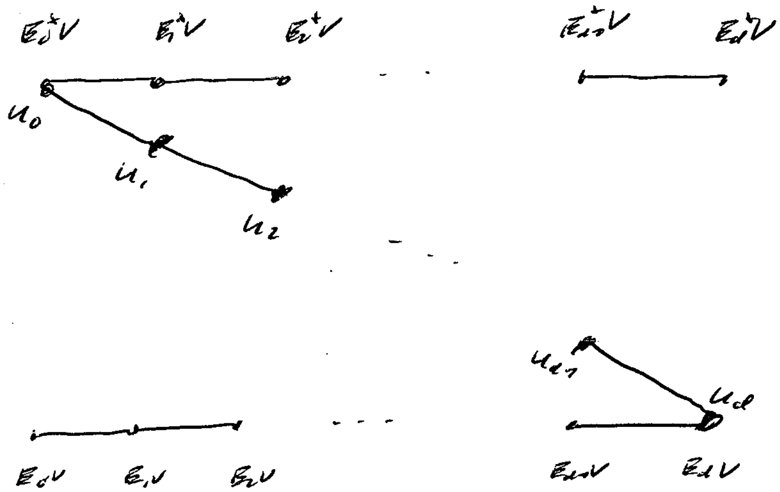
$$\sum_{h=0}^{i} V_h = \sum_{h=0}^{i} W_h \qquad 0 \le i \le d$$

Recall the split dec satisfies

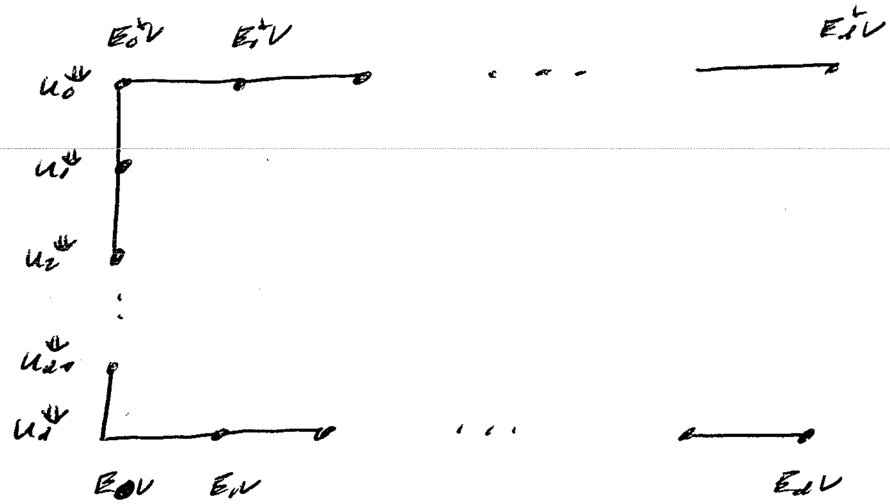
$$U_0+U_1+\cdots+U_i = E_0^*V+\cdots+E_i^*V \qquad 0\le i\le d$$

$$U_i+U_{i+1}+\cdots+U_d = E_iV+\cdots+E_dV \qquad \text{"}$$

Corresp diagram is

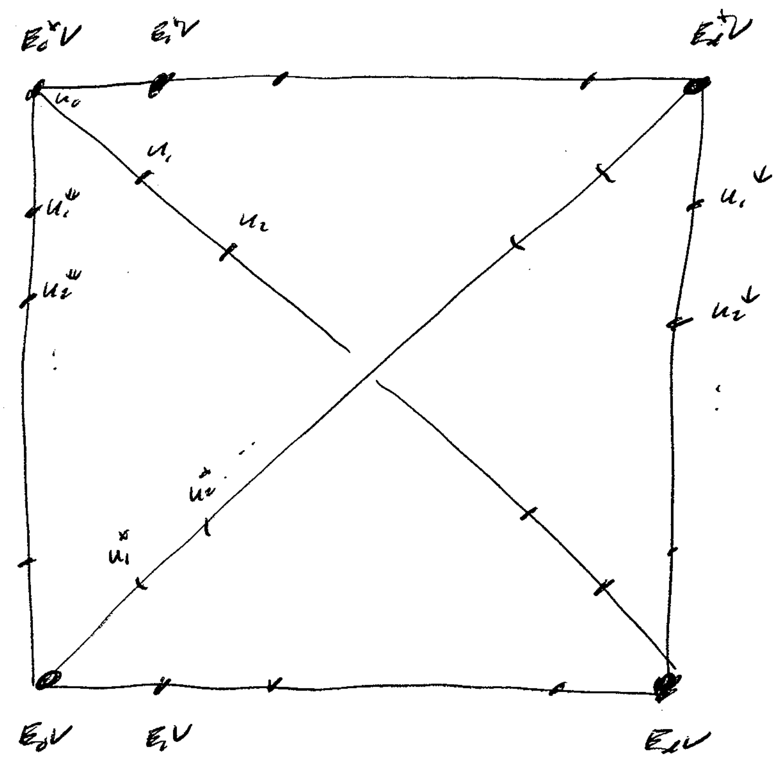


Applying this to $\Phi^\Downarrow$ get



Other relatives of Φ give similar diagrams

Altogether get



Lecture 37 Wednesday April 22

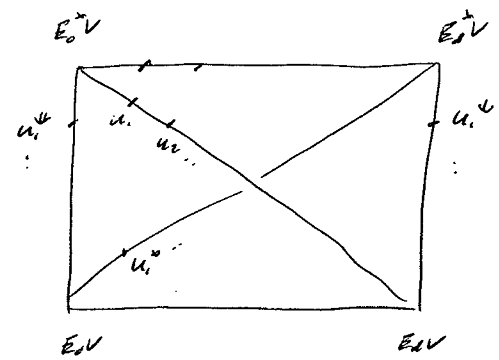
Field $\mathbb{F}$ is arb

$0 \neq V =$ f.d v.s / $\mathbb{F}$

$\Phi = (A, \{E_i\}_{i=0}^d \; A^* \; \{E_i^*\}_{i=0}^d)$ TD system on V shape $\{\rho_i\}_{i=0}^d$

Split dec $\{U_i\}_{i=0}^d$

Last lecture we got



We now describe this picture further.

Notation. Let $\{s_i\}_{i=0}^d$ be sequence of pos

integers whose sum is dim of V.

A <u>flag</u> on V of shape $\{s_i\}_{i=0}^d$ is a nested sequence

of subspaces

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_d$$

such that

$$\dim V_i = s_0 + s_1 + \cdots + s_i \qquad 0 \le i \le d$$

So $V_d = V$. Call V_i the ith component of the flag.

This construction yields a flag on V of shape $\{s_i\}_{i=0}^d$

Let $\{W_i\}_{i=0}^d$ be a decomp of V with $s_i = \dim W_i \quad 0 \le i \le d$

def
$$V_i = W_0 + \cdots + W_i \qquad 0 \le i \le d$$

then $\{V_i\}_{i=0}^d$ is flag on V shape $\{s_i\}_{i=0}^d$

Given 2 flags on V: $\{V_i\}_{i=0}^d$ and $\{V_i'\}_{i=0}^d$

Call these <u>opposite</u> whenever $\exists$ decomp $\{W_i\}_{i=0}^d$ of V

s.t.

$$V_i = W_0 + \cdots + W_i \qquad 0 \le i \le d$$

$$V_i' = W_d + \cdots + W_{d-i} \qquad 0 \le i \le d$$

In this case

$$V_i \cap V_j' = 0 \quad \text{if } i+j < d \qquad (0 \le i, j \le d)$$

$$W_i = V_i \cap V_{d-i}' \qquad 0 \le i \le d.$$

So $\{W_i\}_{i=0}^d$ is determined by the given flags. Call this the <u>associated</u> decomp.

DEF 27 Ref to our TD system Φ

we now define 4 flags on V, denoted

$$[0],\ [D],\ [0^*],\ [D^*]$$

Each has shape $\{\rho_i\}_{i=0}^{d}$

flag	ith component
$[0]$	$E_0V + \cdots + E_iV$
$[D]$	$E_dV + \cdots + E_{d-i}V$
$[0^*]$	$E_0^*V + \cdots + E_i^*V$
$[D^*]$	$E_d^*V + \cdots + E_{d-i}^*V$

Obs $[0], [D]$ are opp and $[0^*], [D^*]$ are opp.

LEM 28. The four flags in Def 27 are mutually opposite.

Pf. Show $[0^*], [D]$ opp.

take split dec $\{U_i\}_{i=0}^{d}$ fn Φ.

Fn $0 \leq i \leq d$

ith comp of $[\delta^*]$ $= E_d^* V + \cdots + E_i^* V$

$= U_0 + \cdots + U_i$

ith comp of $[\delta]$ $= E_d V + \cdots + E_{d-i} V$

$= U_d + \cdots + U_{d-i}$

Rest of pf is sim. $\square$

In our tetrahedron picture there are 6 decomps of V

we now give them more convenient names.

Given an ordered pair of distinct flags in Def 27

$[\alpha], [\beta]$ denote by $[\alpha, \beta]$ the associated decomp.

Note that $[\beta, \alpha]$ is the "inversion" of $[\alpha, \beta]$ ie $[\alpha, \beta]$

written in reverse order.

We have

decomp	ith subspace of decomp
$[0 \quad 0]$	E_iV
$[0^* \quad 0^*]$	E_i^*V
$[0^* \quad 0]$	$(E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV)$
$[0^* \quad 0]$	$(E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_{d-i}V)$
$[0^* \quad 0]$	$(E_d^*V + \cdots + E_{d-i}^*V) \cap (E_0V + \cdots + E_{d-i}V)$
$[0^* \quad 0]$	$(E_d^*V + \cdots + E_{d-i}^*V) \cap (E_iV + \cdots + E_dV)$

We now summarize the action of A, A^* on our 6 decomps

LEM 29. Let $\{W_i\}_{i=0}^d$ be any one of the 6 decomps of V given in the above table. Then for $0 \le i \le d$ the action of A and A^* on W_i is:

Name	A action	A^* action
$[0\ D]$	$(A-\theta_i I)W_i = 0$	$A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$
$[0^*\ D^*]$	$AW_i \subseteq W_{i-1} + W_i + W_{i+1}$	$(A^* - \theta_i^* I)W_i = 0$
$[0^*\ D]$	$(A-\theta_i I)W_i \subseteq W_{i+1}$	$(A^*-\theta^*_i I)W_i \subseteq W_{i-1}$
$[0^*\ 0]$	$(A-\theta_{d-i} I)W_i \subseteq W_{i+1}$	$(A^*-\theta^*_i I)W_i \subseteq W_{i-1}$
$[D^*\ 0]$	$(A-\theta_{d-i} I)W_i \subseteq W_{i+1}$	$(A^*-\theta^*_{d-i} I)W_i \subseteq W_{i-1}$
$[0^*\ D]$	$(A-\theta_i I)W_i \subseteq W_{i+1}$	$(A^*-\theta^*_{d-i} I)W_i \subseteq W_{i-1}$

Pf Rows $[0\ D]$ and $[0^*\ D^*]$ restate def of TDS.

Row $[0^* D]$ is from Th 15 (ii)

Remaining rows are Th 15 (ii) applied to relatives of Φ. $\square$

Consider our 6 decomps of V, from the tetrahedron picture.

The decomps $[0,D]$ and $[0^*,D^*]$ are eigenspace decomps for A, A^* resp. Tempting to view remaining 4 decomps also as eigenspace decomps

To make progress here we assume until further notice

$$0 \neq q \in \mathbb{F} \qquad q^2 \neq 1.$$

$$\theta_i = q^{2i-d} \qquad 0 \le i \le d$$

$$\theta_i^* = q^{d-2i} \qquad 0 \le i \le d.$$

(obs $q^{2i} \neq 1$ $\quad 1 \le i \le d$ since eigenvals must dist)

In this case TD relations become q-Serre rels

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0$$

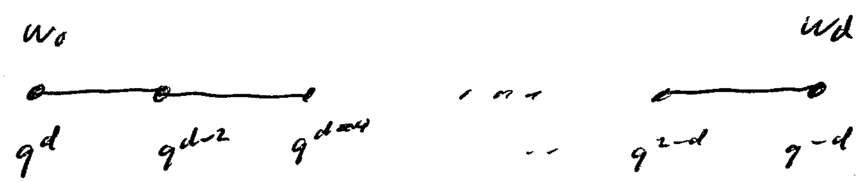
$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0$$

So A, A^* comes from irred A_q-module.

Notation Given a dec $\{W_i\}_{i=0}^{d}$ of V

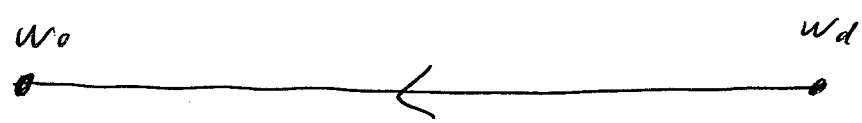
Consider the lin trans $T: V \to V$ s.t. for $0 \le i \le d$

W_i is eigenspace for T with eigval q^{d-2i}

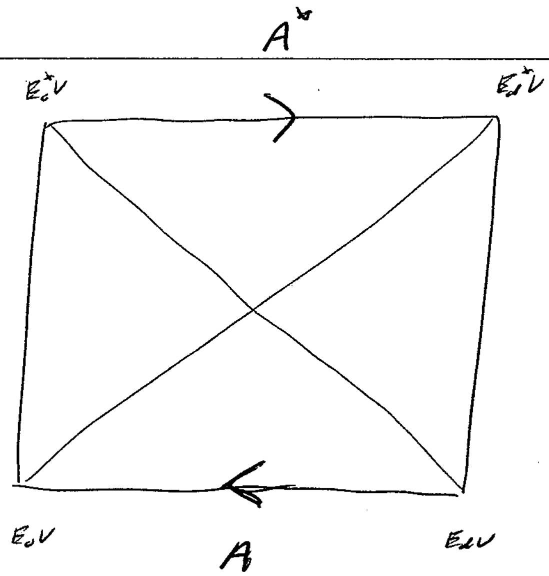


We often represent this lin trans by directed arc

Obs the inverse of T is rep by



Ex



DEF 30. We def lin trans B, B^*, K, K^* $V \to V$

as follows

$$\begin{tikzcd}
E_0^*V \arrow[rr, "A^*"] & & E_d^*V \arrow[dd, "B^*"] \arrow[ddll, "K^*"', near start] \\
& & \\
E_0V \arrow[uu, "B"] & & E_dV \arrow[ll, "A"] \arrow[uull, "K", near start]
\end{tikzcd}$$

So for example, for $0 \le i \le d$

$$(E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV)$$

is eigenspace for K with eigval q^{2i-d}.

Next goal: Find relations sat by

$$A, A^*, B, B^*, K, K^*, K^{-1}, K^{*-1}$$

We will use the following handy facts.

For any lin trans $Y: V \to V$ $\forall \theta \in \mathbb{F}$ def

$$V_Y(\theta) = \{ v \in V \mid Yv = \theta v \}$$

LEM 31 Given any lin trans $Y: V \to V$ and $Z: V \to V$

Given $0 \neq \theta \in \mathbb{F}$ TFAE

(i) $$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I \quad \text{on } V_Y(\theta)$$

"q-Weyl equation"

(ii) $$(Z - \theta^{-1} I)\, V_Y(\theta) \subseteq V_Y(q^{-2}\theta)$$

Pf $\forall$ $v \in V_Y(\theta)$ we have

$$\left(qYZ - q^{-1}ZY - (q - q^{-1}) I \right) v$$

$$= q\left(Y - q^{-2}\theta I \right)\left(Z - \theta^{-1} I \right) v$$

LEM 32 Given any lin trans $Y : V \to V$ $Z : V \to V$

Given $0 \neq \theta \in \mathbb{F}$ TFAE

(i) $\dfrac{qYZ - q^{-1}ZY}{q - q^{-1}} = I$ on $V_Z(\theta)$

(ii) $(Y - \theta^{-1} I) V_Z(\theta) \subseteq V_Z(q^2\theta)$

Pf Replace (Y, Z, q) by (Z, Y, q^{-1}) in L31. $\square$

LEM 33 Given lin trans $Y : V \to V$ and $Z : V \to V$

(i) $Y^3 Z - [3]_q Y^2 Z Y + [3]_q Y Z Y^2 - Z Y^3 = 0$ on $V_Y(\theta)$

(ii) $Z V_Y(\theta) \subseteq V_Y(q^2\theta) + V_Y(\theta) + V_Y(q^{-2}\theta)$

Pf ex

LEM 34 We have

(i) $$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I$$

(ii) $$\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = I$$

(iii) $$\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} = I$$

(iv) $$\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} = I$$

pf In each case, combine L30/L31 and L29. □

LEM 35 We have

(i) $$\frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} = I$$

(ii) $$\frac{qKA^* - q^{-1}A^*K}{q - q^{-1}} = I$$

(iii) $$\frac{qAK^{\flat} - q^{-1}K^{\flat}A}{q - q^{-1}} = I$$

(iv) $$\frac{qA^*K^{*-1} - q^{-1}K^{*-1}A^*}{q - q^{-1}} = I$$

pf In each case, combine L30/L31 and L29. □

We now give the action of B, B^* on the 6 decomps

LEM 36 Let $\{W_i\}_{i=0}^{d}$ be any one of the 6 decomps of V from ⊠ picture. For $0 \le i \le d$ the action of B, B^* on W_i is:

Name	B-action	B^*-action
$[0\ \ 0]$	$(B - q^{d-2i} I) W_i \subseteq W_{i+1}$	$(B^* - q^{d-2i} I) W_i \subseteq W_{i+1}$
$[0^*\ 0^*]$	$(B - q^{2i-d} I) W_i \subseteq W_{i-1}$	$(B^* - q^{2i-d} I) W_i \subseteq W_{i+1}$
$[0^*\ 0]$	$(B - q^{2i-d} I) W_i \subseteq W_{i+1}$	$(B^* - q^{d-2i} I) W_i \subseteq W_{i+1}$
$[0^*\ 0]$	$(B - q^{2i-d} I) W_i = 0$	$B^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$
$[0^*\ 0]$	$(B - q^{2i-d} I) W_i \subseteq W_{i+1}$	$(B^* - q^{d-2i} I) W_i \subseteq W_{i-1}$
$[0^*\ 0]$	$B W_i \subseteq W_{i-1} + W_i + W_{i+1}$	$(B^* - q^{d-2i} I) W_i = 0$

pf $[0, 0]$ to get B action use L 34 (i) and L 31

to get B^* action use L 34 (iv) and L 32.

$[0^*, 0^*]$ sim

$[0^*, 0]$ ..

$[0^*, 0]$ ..

[0*.0] : $(B - q^{2i-d} I) W_i = 0$ by def of B

Find $B^* W_i$:

View

$$W_i = (W_0 + \cdots + W_i) \cap (W_i + \cdots + W_d)$$

$$
\begin{aligned}
B^* W_i &\subseteq B^*(W_0 + \cdots + W_i) \\
&= B^*(E_0^* V + \cdots + E_i^* V) \\
&\subseteq E_0^* V + \cdots + E_{i+1}^* V && \text{by row } [0^* \; 0^*] \\
&= W_0 + \cdots + W_{i+1}
\end{aligned}
$$

$$
\begin{aligned}
B^* W_i &\subseteq B^*(W_i + \cdots + W_d) \\
&= B^*(E_0 V + \cdots + E_{d-i} V) \\
&\subseteq E_0 V + \cdots + E_{d-i+1} V && \text{by row } [0 \; 0] \\
&= W_{i-1} + \cdots + W_d
\end{aligned}
$$

So

$$
\begin{aligned}
B^* W_i &\subseteq (W_0 + \cdots + W_{i+1}) \cap (W_{i-1} + \cdots + W_d) \\
&= W_{i-1} + W_i + W_{i+1}
\end{aligned}
$$

[0* 0] sim. $\square$

LEM 37 We have

(i) $$\frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} = I$$

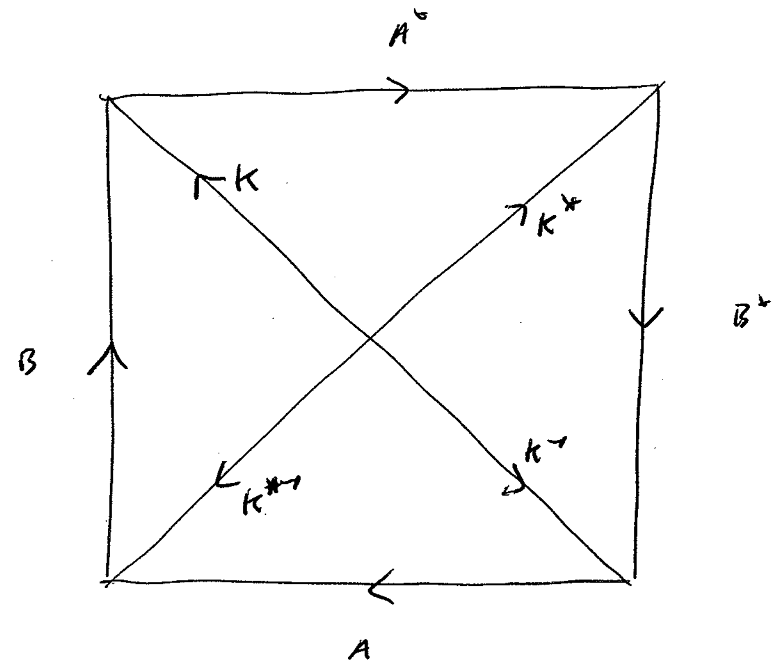
(ii) $$\frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} = I$$

(iii) $$\frac{qK^{*-1}B - q^{-1}BK^{*-1}}{q - q^{-1}} = I$$

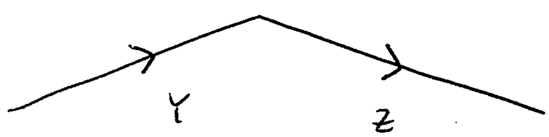
(iv) $$\frac{qK^*B^* - q^{-1}B^*K^*}{q - q^{-1}} = I$$

Pf In each case, combine L36 and L31/L32

The q-Weyl rels in summary



For each config



we have

$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I$$

$\square$

No.

Date 4/24/09

$\mathbb{F}$ arb

$0 \neq V = $ f.d v.s$/\mathbb{F}$

$\Phi = (A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d)$ TDS on V

assume

$$\theta_i = q^{2i-d} \qquad \theta_i^* = q^{d-2i} \qquad 0 \le i \le d$$

$$0 \neq q \in \mathbb{F} \qquad q^{2i} \neq 1 \qquad 1 \le i \le d$$

$$\begin{tikzcd}
E_0^*V \arrow[r, "A^*"] \arrow[dr, "K^{-1}" near end] & E_d^*V \arrow[d, "B^*"] \arrow[dl, "K^{*-1}" near end] \\
E_dV \arrow[u, "B"] \arrow[ur, "K^*"' near end] & E_0V \arrow[l, "A"] \arrow[ul, "K"' near end]
\end{tikzcd}$$

$$q^d \quad q^{d-2} \quad \cdots \quad q^{2-d} \quad q^{-d}$$

eigenvalues

Lec 38 Friday Ap 24

A few more relations

LEM 38 We have

(i) $B^3 B^* - [3]_q B^2 B^* B + [3]_q B B^* B^2 - B^* B^3 = 0$,

(ii) $B^{*3} B - [3]_q B^{*2} B B^* + [3]_q B^* B B^{*2} - B B^{*3} = 0$.

Pf Combine L33 with L36, rows $[0^* 0]$ and $[0^* 0]$. $\square$

We now interpret our results so far in terms of an algebra $\boxtimes_q$ "q-tetrahedron algebra"

let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic gp of order 4.

DEF 39. let $\boxtimes_q$ denote the assoc $\mathbb{F}$-alg with 1 that has generators

$$\{x_{ij} \mid i,j \in \mathbb{Z}_4, \quad i-j=1 \text{ or } i-j=2\}$$

and the following relations

(i) For $i,j \in \mathbb{Z}_4$ s.t. $i-j=2$

$$x_{ij}x_{ji} = 1$$

(ii) For $i,j,k \in \mathbb{Z}_4$ s.t. the pair $(j-i, k-j)$ is one of

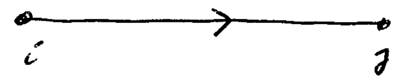
$(1,1)$, $(1,2)$, $(2,1)$

(q-Weyl)
$$\frac{q x_{ij} x_{jk} - q^{-1} x_{jk} x_{ij}}{q-q^{-1}} = 1$$

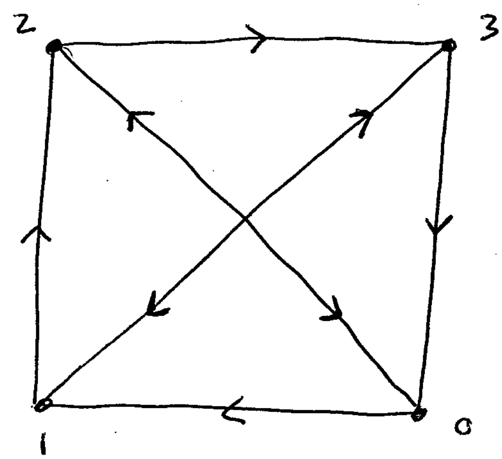
(iii) For $i,j,k,l \in \mathbb{Z}_4$ s.t. $j-i=k-j=l-k=1$.

(q-Serre)
$$x_{ij}^3 x_{kl} - [3]_q x_{ij}^2 x_{kl} x_{ij} + [3]_q x_{ij} x_{kl} x_{ij}^2 - x_{kl} x_{ij}^3 = 0$$

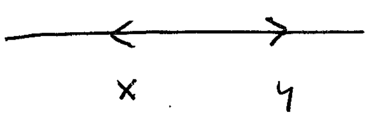
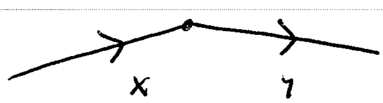
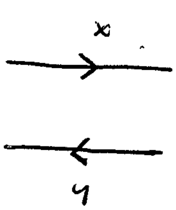
Diagram for $\boxtimes_q$

Represent generator x_{ij} by 

Generators for $\boxtimes_q$:



We read off the relations from the diagram as follows

picture	meaning
	$xy = yx = 1$
	$\dfrac{qxy - q^{-1}yx}{q - q^{-1}} = 1$
	$x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = 0$

the following is now immediate

thm 40 Ref to our TD system Φ, $\exists$ unique

$\boxtimes_q$-module structure on V s.t the generators act as follows

gen	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}	x_{20}	x_{31}
action	A	B	A^*	B^*	K	K^*	K^{-1}	K^{*-1}

this $\boxtimes_q$-module str is irred. $\square$

Next goal: How is $\boxtimes_q$ related to $U_q \hat{sl_2}$

Start with $U_q(sl_2)$

DEF 41. Let $U_q(sl_2)$ denote the assoc $\mathbb{F}$-algebra with 1 with gens e, f, k, k^{-1} and relations

$$kk^{-1} = k^{-1}k = 1,$$

$$kek^{-1} = q^2 e$$

$$kfk^{-1} = q^{-2} f$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

Note the above def gives the "Chevalley presentation" of $U_q(sl_2)$

Sometimes gens are written

$$e^+ \equiv e$$

$$e^- \equiv f$$

To get a feel for $U_q(sl_2)$ it is helpful to work out the f.d. irred modules. I will give the ans and leave the pf as an ex.

LEM 42 Assume $\mathbb{F}$ alg closed, q not a root of unity.

Up to iso the f.d irred $U_q(sl_2)$-modules are

$L(d,\varepsilon)$ $\qquad d = 0,1,2\ldots$ $\qquad \varepsilon \in \{1,-1\}$

$L(d,\varepsilon)$ has a basis $\{v_i\}_{i=0}^{d}$ s.t

$$Kv_i = \varepsilon q^{d-2i} v_i \qquad 0 \le i \le d$$

$$fv_i = [i+1]_q v_{i+1} \qquad 0 \le i \le d-1, \qquad fv_d = 0$$

$$ev_i = \varepsilon [d-i+1]_q v_{i-1} \qquad 1 \le i \le d, \qquad ev_0 = 0$$

(If char $k=2$, view $\{1,-1\}$ as having single element)

pf ex.

Def 43 Ref to L42, call ε the <u>type</u> of the module.

We now give another presentation of $U_q(sl_2)$ said to be <u>equitable</u>

LEM 44 The algebra $U_q(sl_2)$ is iso to the $\mathbb{F}$-algebra with gens x, x^{-1}, y, z and rels

$$x x^{-1} = x^{-1} x = 1$$

$$\frac{q x y - q^{-1} y x}{q - q^{-1}} = 1$$

$$\frac{q y z - q^{-1} z y}{q - q^{-1}} = 1$$

$$\frac{q z x - q^{-1} x z}{q - q^{-1}} = 1$$

An iso with the presentation in Def 41 is given by

$$x^{\pm 1} \longrightarrow K^{\pm 1}$$

$$y \longrightarrow K^{-1} + f$$

$$z \longrightarrow K^{-1} - K^{-1} e\, q (q - q^{-1})^2$$

The inverse of this iso is given by

$$k^{\pm 1} \longrightarrow x^{\pm 1}$$

$$f \longrightarrow y - x^{-1}$$

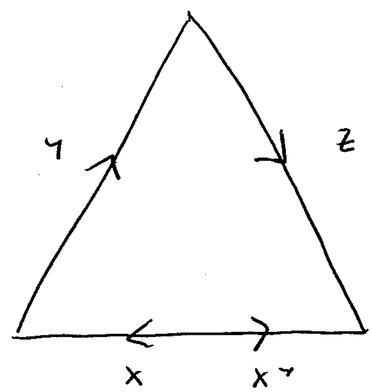
$$e \longrightarrow (1 - xz)\, q^{-1} (q - q^{-1})^{-2}$$

pf. One checks each map above is hom of $\mathbb{F}$-algebras and that the maps are inverses. Therefore each map is iso of $\mathbb{F}$-algebras. $\square$

DEF 45. We call $x^{\pm 1}, y, z$ the <u>equitable generators</u> for $U_q(sl_2)$

Diagram for $U_q(sl_2)$

Represent each equitable generator by directed arc.



Read the relations off the diagram using the conventions for $\boxtimes_q$

LEM 46 For $i \in \mathbb{Z}_4$ $\exists$ $\mathbb{F}$-alg hom $U_q(sl_2) \to \boxtimes_q$ that sends

$$x \to x_{i,i+2}$$
$$x^{-1} \to x_{i+2,i}$$
$$y \to x_{i+2,i+3}$$
$$z \to x_{i+3,i}$$

"map the triangle onto one of the faces of the tetrahedron"

Open Problem 47. Show that the map in L46 is injective

We now define $U_q(\hat{sl}_2)$. Roughly speaking this algebra is generated by 2 copies of $U_q(sl_2)$ glued together in a certain way.

DEF 48 Let $U_q(\hat{sl_2})$ denote the assoc $\mathbb{F}$-alg with 1, with gens e_i^+, e_i^-, k_i, k_i^{-1} $(i=0,1)$ and rels:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_0 k_1 = k_1 k_0$$

$$k_i e_i^{\pm} k_i^{-1} = q^{\pm 2} e_i$$

$$k_i e_j^{\pm} k_i^{-1} = q^{\mp 2} e_j^{\pm} \qquad i \neq j$$

$$[e_i^+, e_i^-] = \frac{k_i - k_i^{-1}}{q - q^{-1}} \qquad [r,s] = rs - sr$$

$$[e_0^{\pm}, e_1^{\mp}] = 0$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \quad i \neq j \ (q\text{-Serre})$$

"Chevalley presentation"

$\square$

No. Lec 40 -1

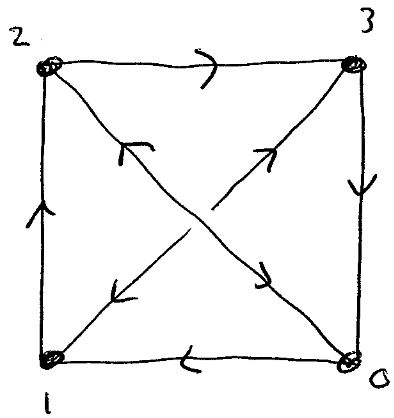
Lecture 40 Wednesday April 29

Date 4/29/09

fixed $\mathbb{F}$ arb

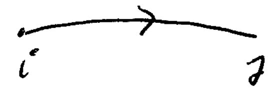
$0 \neq q \in \mathbb{F}$ $\qquad q^2 \neq 1$

$\mathbb{F}$- Algebra $\boxtimes_q$



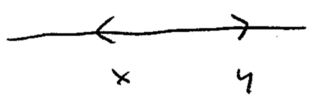
gen x_{ij} represented by

directed arc

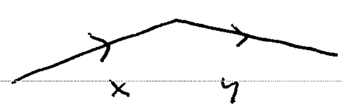


picture

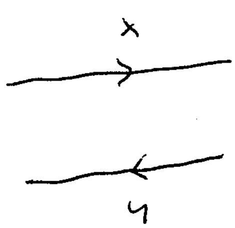
meaning



$$xy = yx = 1$$



$$\frac{q\,xy - q^{-1}yx}{q - q^{-1}} = 1$$



$$x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = 0$$

We now give the equitable presentation for $U_q(\widehat{sl_2})$

LEM 49 the $\mathbb{F}$-algebra $U_q \widehat{sl_2}$ is iso to the $\mathbb{F}$-algebra with generators $x_i^{\pm 1}, y_i, z_i$ $(i = 0, 1)$ and rels:

$$x_i x_i^{-1} = x_i^{-1} x_i = 1$$

$x_0 x_1$ central

$$\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} = 1$$

$$\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} = 1$$

$$\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} = 1$$

$$\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} = x_0^{-1} x_1^{-1} \qquad i \neq j$$

$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0 \qquad i \neq j$$

$$z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0 \qquad i \neq j$$

An iso with the presentation in Def 48 is given by

$$x_i^{\pm 1} \to k_i^{\pm 1}$$

$$y_i \to k_i^{-1} + e_i^-$$

$$z_i \to k_i^{-1} - k_i^{-1} e_i^+ q(q-q^{-1})^2$$

The inverse of this iso is given by

$$k_i^{\pm 1} \to x_i^{\pm 1}$$

$$e_i^- \to y_i - x_i^{-1}$$

$$e_i^+ \to (1 - x_i z_i) q^{-1} (q-q^{-1})^{-2}$$

Pf Check each map is hom of $\mathbb{F}$-algebras, and that the maps are inverses. It follows each map is iso of $\mathbb{F}$-algebras. $\square$

Let $U_q(L(sl_2))$ denote the quotient of $U_q(\hat{sl}_2)$ by the 2-sided ideal gen by $X_0 X_1 - 1$ (in the equit pres). $U_q(L(sl_2))$ is called the

<u>$U_q sl_2$ loop algebra</u>.

The following result shows that $U_q(L(sl_2))$ is closely related to $\boxtimes_q$.

LEM 50 The $\mathbb{F}$-algebra $U_q(L(sl_2))$ is iso to the $\mathbb{F}$-algebra with gens x_i, y_i, z_i $(i=0,1)$ and rels:

$$x_0 x_1 = x_1 x_0 = 1$$

$$\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} = 1$$

$$\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} = 1$$

$$\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} = 1$$

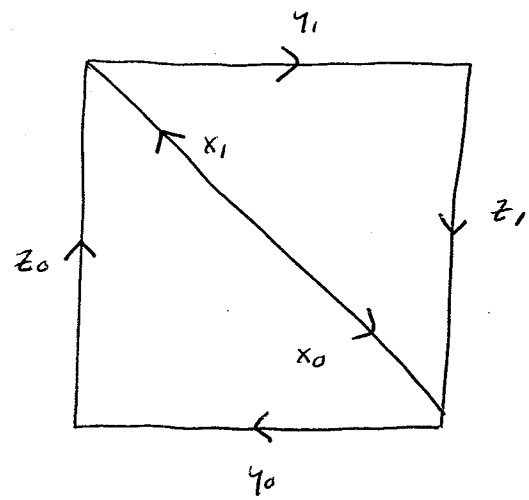
$$\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} = 1 \qquad i \neq j$$

$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0 \qquad i \neq j$$

$$z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0 \qquad i \neq j$$

pf ex.

Diagram for $U_q(L(sl_2))$



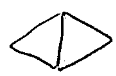
Same conventions as for $\boxtimes_q$

The following is now immediate.

Th 51 $\forall i \in \mathbb{Z}_4$ $\exists$ $\mathbb{F}$-alg hom $U_q(L(sl_2)) \to \boxtimes_q$ that sends

$$X_1 \to x_{i,i+2} \qquad y_1 \to x_{i+2,i+3} \qquad z_1 \to x_{i+3,i}$$

$$X_0 \to x_{i+2,i} \qquad y_0 \to x_{i,i+1} \qquad z_0 \to x_{i+1,i+2}$$

"map the diamond  onto a pair of adj faces in the tetrahedron"

Open Problem 52. Show the map in Th 51 is injective

Note 53 Composing the canonical hom $U_q \widehat{sl_2} \to U_q(L(sl_2))$ with the hom in Th 51 we get an algebra hom $U_q \widehat{sl_2} \to \boxtimes_q$

Let V denote ~~an~~ $\boxtimes_q$-module. Pulling back the $\boxtimes_q$-module structure to $U_q\widehat{sl_2}$ via the above hom, V becomes a $U_q \widehat{sl_2}$-module.

In particular, the $\boxtimes_q$-modules from Th 40 support a $U_q\widehat{sl_2}$-module structure.

Conjecture 54 Given nonzero

$$b, b^*, c, c^* \in \mathbb{F}$$

and consider these elements in $\boxtimes_q$:

$$A = b\, x_{01} + c\, x_{02}$$

$$A^* = b^*\, x_{23} + c^*\, x_{30}$$

then A, A^* satisfy the TD relations

$$0 = \left[A,\ A^2A^* - (q^2+q^{-2})AA^*A + A^*A^2 + bc(q^2-q^{-2})^2A^*\right],$$

$$0 = \left[A^*,\ A^{*2}A - (q^2+q^{-2})A^*AA^* + AA^{*2} + b^*c^*(q^2-q^{-2})^2A\right].$$

Conj 55 Given $0 \neq V$ = f.d v.s $/\mathbb{F}$

Given TD system

$$(A, \{E_i\}_{i=0}^d, A^* \{E_i^*\}_{i=0}^d)$$

on V. Assume the eigvals / dual eigvals have form

$$\theta_i = a + b q^{2i-d} + c q^{d-2i} \qquad 0 \leq i \leq d$$

$$\theta_i^* = a^* + b^* q^{2i-d} + c^* q^{d-2i} \qquad 0 \leq i \leq d$$

where $0 \neq q \in \mathbb{F}$ $\quad a, b, c, a^*, b^*, c^* \in \mathbb{F}$.

$bb^*cc^* \neq 0, \qquad q^2 \neq 1, \qquad q^2 \neq -1$

"q-Racah type"

Then $\exists$ $\boxtimes_q$-module structure on V s.t.

A acts as $\quad aI + b x_{01} + c x_{12}$

A^* acts as $\quad a^* I + b^* x_{23} + c^* x_{30}$

$\square$

Lecture 41 Friday May 1

Back to DRGs

Next goal: For a certain type of DRG we get an action of $\boxtimes_q$ on the standard module.

<u>Assumptions</u> $\mathbb{F} = \mathbb{C}$

Given DRG $\Gamma = (X, R)$ diam $D \geq 2$, st. module V.

Fix $b \in \mathbb{Z}$ $b \neq 0, 1$.

Assume Γ has classical parameters (D, b, α, σ)

where $\alpha = b-1$. So the intersection numbers are

$$c_i = b^{i-1} \frac{b^i - 1}{b-1} \qquad 0 \leq i \leq D$$

$$b_i = (\sigma + 1 - b^i) \frac{b^D - b^i}{b-1} \qquad 0 \leq i \leq D$$

Obs $c_2 = b(b+1)$ so $b \neq -1$

We fix $q \in \mathbb{C}$ s.t.

$$q^2 = b$$

Note q is nonzero and not a root of 1.

Recall Γ has Q-poly str as in Th 153.

Let $\{\theta_i\}_{i=0}^{D}$, $\{\theta_i^*\}_{i=0}^{D}$ be the corresp eigvals / dual eigvals

LEM 56 We have

$$\theta_i = \alpha_0 + \alpha_1 q^{D-2i} \qquad 0 \le i \le D$$

where

$$\alpha_0 = -\frac{q^{2D}+\sigma}{q^2-1}$$

$$\alpha_1 = \frac{(\sigma+1)q^D}{q^2-1}$$

Moreover $\alpha_1 \neq 0$

Pf. By L155

$$\theta_i = \frac{b_i}{b^i} - \frac{b^i - 1}{b-1} \qquad 0 \le i \le D$$

Use this to get α_0, α_1

Note $\alpha_1 \neq 0$ since $\theta_0 \neq \theta_1$ $\square$

LEM 57. We have

$$\frac{\theta_i^*}{\theta_0^*} = \frac{\theta_i}{\theta_0} \qquad 0 \le i \le D.$$

Pf. By Th 153

$$\frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta_1}{k} - 1\right) \frac{b^i - 1}{b - 1} \quad b^{1-i}$$

$k = \theta_0$

Compare this with LEM 56 $\square$

Note: It turns out $\theta_i = \theta_i^*$ $0 \le i \le D$

Recall distance matrices A_i $(0 \le i \le D)$

Fix $x \in X$ write $T = T(x)$ etc.

Recall dual distance matrices A_i^* $(0 \le i \le D)$

Earlier we abbreviated

$$A = A_1 \qquad A^* = A_1^*$$

For this unit we use a different convention.

A_1, A_1^* will denote the adj matrix and dual adj matrix

We define $A \in Mat_X(\mathbb{C})$ s.t

$$A_1 = \alpha_0 I + \alpha_1 A$$

So for $0 \le i \le D$

$E_i V$ is an eigenspace for A with eigval q^{D-2i}

We def $A^* \in Mat_X(\mathbb{C})$ s.t

$$A_1^* = \alpha_0' I + \alpha_1' A^*$$

with α_0', α_1' chosen so that $\theta_i^* = \alpha_0' + \alpha_1' q^{D-2i}$ $0 \le i \le D$

So for $0 \le i \le D$

$E_i^* V$ is an eigenspace of A^* with eigval q^{D-2i}

LEM 58 For $0 \le i, j \le D$ s.t $|i-j| > 1$,

(i) $E_i^* A E_j^* = 0$

(ii) $E_i A^* E_j = 0$

pf (i) Since $p^1_{ij} = 0$ we have

$$E_i^* A_1 E_j^* = 0$$

Also

$$E_i^* E_j^* = 0$$

and

$$A_1 = \alpha_0 I + \alpha_1 A,$$

$\alpha_1 \neq 0$

(ii) Sim. □

LEM 59. A, A^* satisfy the q-Serre relations

(i) $A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0$.

(ii) $A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0$.

Pf Recall A_1, A_1^* satisfy the relations TD1, TD2.

After we adjust A_1, A_1^* to get A, A^* the TD1, TD2 become

the q-Serre relations.

We can see this directly as follows.

Concerning (i). Let $C =$ LHS. Show $C = 0$

Abbrev $\sigma_i = q^{D-2i}$

$$C = ICI$$

$$= \left(\sum_{i=0} E_i\right) C \left(\sum_{j=0}^{D} E_j\right)$$

For $0 \le i, j \le D$

$$E_iCE_j = E_iA^*E_j\left(\sigma_i^3 - [3]_q\sigma_i^2\sigma_j + [3]_q\sigma_i\sigma_j^2 - \sigma_j^3\right)$$

$$= \underbrace{E_iA^*E_j}_{0 \text{ if } |i-j|>1}\underbrace{(\sigma_i - \sigma_jq^2)(\sigma_i - \sigma_j)}_{0 \text{ if } |i-j| \le 1}(\sigma_i - \sigma_jq^{-2})$$

$= 0$. So $C = 0$

the eq (iii) is sim proved. $\square$

Recall the split dec for Γ

$$V_{ij} = (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_j V)$$

$\tilde{V}_{ij}$ = orthog complement of

$V_{i-1,j} + V_{i,j-1}$ in V_{ij}

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij} \qquad (ds)$$

We are going to do something a bit more general.

Def 60 For $-1 \le i, j \le D$ define

$$V_{ij}^{\downarrow\downarrow} = (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_j V)$$

$$V_{ij}^{\uparrow\downarrow} = (E_D^* V + \cdots + E_{D-i}^* V) \cap (E_0 V + \cdots + E_j V)$$

$$V_{ij}^{\downarrow\uparrow} = (E_0^* V + \cdots + E_i^* V) \cap (E_D V + \cdots + E_{D-j} V)$$

$$V_{ij}^{\uparrow\uparrow} = (E_D V + \cdots + E_{D-i} V) \cap (E_D V + \cdots + E_{D-j} V)$$

Ref to Def 60, for $\eta, \mu \in \{\downarrow, \uparrow\}$

and for $0 \le i, j \le D$ we have

$$V_{i,j-1}^{\eta,\mu} + V_{i-1,j}^{\eta,\mu} \subseteq V_{ij}^{\eta,\mu} \qquad (*)$$

Let $\tilde{V}_{ij}^{\eta,\mu}$ denote the orthog complement of LHS in RHS.

LEM 61 For $\eta, \mu \in \{\downarrow, \uparrow\}$,

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij}^{\eta,\mu} \qquad (ds)$$

Pf. For $\eta = \downarrow$, $\mu = \downarrow$ this is Cor 165.

For the other values of η, μ the proof is very similar. $\square$

DEF 62 We def $B, B^*, K, K^*, \Phi, \Psi$

to be the unique matrices in $\mathrm{Mat}_X(\mathbb{C})$ that satisfy

the following table for $0 \le i, j \le d$

the matrix	is 0 on
$B - q^{i-j} I$	$\tilde{V}_{ij}^{\downarrow\uparrow}$
$B^* - q^{j-i} I$	$\tilde{V}_{ij}^{\uparrow\downarrow}$
$K - q^{i-j} I$	$\tilde{V}_{ij}^{\downarrow\downarrow}$
$K^* - q^{i-j} I$	$\tilde{V}_{ij}^{\uparrow\uparrow}$
$\Phi - q^{i+j-d} I$	$\tilde{V}_{ij}^{\downarrow\downarrow}$
$\Psi - q^{i+j-d} I$	$\tilde{V}_{ij}^{\downarrow\uparrow}$

We are going to show

Thm 63 Under our assumption $\exists$ a $\boxtimes_q$ module structure on V s.t the generators x_{ij} act as follows

generator	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}
action on V	$A\Phi\Psi^{-1}$	$B\Phi^{-1}$	$A^*\Phi\Psi$	$B^*\Phi^{-1}$	$K\Psi^{-1}$	$K^*\Psi$

First we need some lemmas.

Comment. Let W denote an irred T-module [for any Q-poly DRG] then the

diameter of W = dual diameter of W

Pf. By L14 in this chapter and since A, A^* act on W as a TD pair. $\square$

LEM 64 Let W denote an irred T-module with endpt r, dual endpt t, diam d.

Then $\exists$ unique $\boxtimes_q$-module structure on W s.t. the gens x_{01}, x_{23} act as Aq^{d-D+2t}, A^*q^{d-D+2r} resp.

This $\boxtimes_q$-module is irred.

Pf. Put

$$a = Aq^{d-D+2t}$$

$$a^* = A^*q^{d-D+2r}$$

$$e_i = E_{d+t-i} \qquad 0 \le i \le d$$

$$e_i^* = E^*_{r+i} \qquad 0 \le i \le d$$

Obs a, a^* acts on W as a TD pair diam d

For $0 \le i \le d$

e_iW is the eigenspace of a on W with eigval q^{2i-d}

e_i^*W " " a^* " " q^{d-2i}

Result follows in view of Th 40 $\square$

Lecture 42 Monday May 4

$\mathbb{F} = \mathbb{C}$

Given DRG $\Gamma = (X, R)$ diam $D \geq 2$ st module V

classical params (D, b, α, σ) $\alpha = b-1$

Fix $q^2 = b$

Fix $x \in X$ write $T = T(x)$

A_1, A_1^* are adj / dual adj matrices

$A_1 = \alpha_0 I + \alpha_1 A,$ $\qquad A_1^* = \alpha_0' I + \alpha_1' A^*$

with α_i, α_i' chosen so that A, A^* have eigvals

$$q^{D-2i} \qquad 0 \leq i \leq D$$

LEM 65 Let W denote an irred T-module with endpt r, dual endpt t, diam d.

Consider the $\boxtimes_q$-module str on W from L 64.

For each generator x_{rs} of $\boxtimes_q$ and $0 \le i \le d$,

the eigenspace for x_{rs} on W associated with ~~e~~ eigval q^{d-2i}

is given in the table below

r	s	eigenspace of x_{rs} for eigval q^{d-2i}
0	1	$E_{t+i}W$
1	2	$(E^*_r W + \cdots + E^*_{r+d-i}W) \cap (E_{t+d-i}W + \cdots + E_{t+d}W)$
2	3	$E^*_{r+i}W$
3	0	$(E^*_{r+d-i}W + \cdots E^*_{r+d}W) \cap (E_t W + \cdots E_{t+d-i}W)$
0	2	$(E^*_r W + \cdots + E^*_{r+d-i}W) \cap (E_t W + \cdots + E_{t+i}W)$
1	3	$(E^*_{r+i}W + \cdots + E^*_{r+d}W) \cap (E_{t+d-i}W + \cdots + E_{t+d}W)$

Pf Immed from Def 30 and Th 40 $\square$

LEM 66. Let W denote an irred T-module with endpt r, dual endpt t, diam d. Then for $0 \le i \le d$

Space	is contained in
$(E^*_r W + \cdots + E^*_{r+d-i}W) \cap (E_{t+d-i}W + \cdots + E_{t+d}W)$	$\tilde{V}^{\downarrow\uparrow}_{r+d-i,\, D-d-t+i}$
$(E^*_{r+d-i}W + \cdots + E^*_{r+d}W) \cap (E_t W + \cdots + E_{t+d-i}W)$	$\tilde{V}^{\uparrow\downarrow}_{D-d-r+i,\, t+d-i}$
$(E^*_r W + \cdots + E^*_{r+d-i}W) \cap (E_t W + \cdots + E_{t+i}W)$	$\tilde{V}^{\downarrow\downarrow}_{r+d-i,\, t+i}$
$(E^*_{r+i}W + \cdots + E^*_{r+d}W) \cap (E_{t+d-i}W + \cdots + E_{t+d}W)$	$\tilde{V}^{\uparrow\uparrow}_{D-r-i,\, D-d-t+i}$

Pf For each row the pf is sim. We will give careful pf for row 3. Obs

call this W_i

$$(E^*_r W + \cdots + E^*_{r+d-i}W) \cap (E_t W + \cdots + E_{t+i}W) \qquad (*)$$

$$\subseteq (E^*_0 V + \cdots + E^*_{r+d-i}V) \cap (E_0 V + \cdots + E_{t+i}V)$$

$$= V^{\downarrow\downarrow}_{r+d-i,\, t+i}$$

Show $(*)$ is orthog to

$$V^{\downarrow\downarrow}_{r+d-i-1,\, t+i} + V^{\downarrow\downarrow}_{r+d-i,\, t+i-1}$$

For $w \in (*)$ and $v \in V^{\downarrow\downarrow}_{r+d-i-1,\, t+i}$

show $\langle w, v \rangle = 0$

Let $W^{\perp}$ denote the orthog complement of W in V

So $V = W + W^{\perp}$ (ds of T-modules)

Write

$$v = w_1 + v_1 \qquad w_1 \in W, \quad v_1 \in W^{\perp}$$

Obs

$$\langle w, v_1 \rangle = 0 \qquad \text{since} \quad w \in W, \quad v_1 \in W^{\perp}$$

Show $w_1 = 0$.

Since $v \in V^{\downarrow\downarrow}_{r+d-i-1,\, t+i}$ we have

$$E^*_j v = 0 \qquad r+d-i \le j \le D$$

$$E_j v = 0 \qquad t+i+1 \le j \le D$$

Since $V = W + W^{\perp}$ is ds of T-modules

$$E^*_j w_1 = 0 \qquad r+d-i \le j \le D$$

$$E_j w_1 = 0 \qquad t+i+1 \le j \le D$$

Since $w_1 \in W$ and W has endpt r

$$E^*_j w_1 = 0 \qquad 0 \le j \le r-1$$

Similarly since W has dual endpt t.

$$E_j w_1 = 0 \qquad 0 \le j \le t-1$$

So

$$w_1 \in \left(E^*_r W + \cdots + E^*_{r+d-i-1} W\right) \cap \left(E_t W + \cdots + E_{t+i} W\right)$$

$$= W_i \cap W_{i+1}$$

The sum $W = \sum_{i=0}^{d} W_i$ is direct so

$W_i \cap W_{i+1} = 0$ so $w_1 = 0$.

So $v = v_1$ so $\langle w, v \rangle = 0$

this shows $(*)$ is orthog $V^{\downarrow\downarrow}_{r+d-i-1,\, t+i}$

Similar arg shows $(*)$ is orthog $V^{\downarrow\downarrow}_{r+d-i,\, t+i-1}$

therefore $(*)$ is in

$$\tilde{V}^{\downarrow\downarrow}_{r+d-i,\, t+i}$$

$\square$

LEM 67 Let W denote an irred T-module with endpt r, dual endpt t, diam d.

Consider the $\boxtimes_q$-module structure on W from L 64.

In the table below, each row contains a matrix in $\mathrm{Mat}_X(\mathbb{C})$ and an element of $\boxtimes_q$. The actions of these two objects on W coincide.

Matrix	el of $\boxtimes_q$
A	$x_{01}\, q^{D-d-2t}$
B	$x_{12}\, q^{d-D+r+t}$
A^*	$x_{23}\, q^{D-d-2r}$
B^*	$x_{30}\, q^{d-D+r+t}$
K	$x_{02}\, q^{r-t}$
K^*	$x_{13}\, q^{t-r}$
Φ	$1\, q^{d-D+r+t}$
Ψ	$1\, q^{r-t}$

Pf A: By constr ✓

B: For $w \in W$ show

$$\left(B - X_{12}\, q^{d-D+r+t}\right) w = 0$$

By constr X_{12} is diagonalizable on W with eigvals q^{d-2i} $(0 \le i \le d)$. WLOG w is eigvector for X_{12} with eigval q^{d-2i}

By L65 (row $r=1, s=2$) and L66 (top row)

$$w \in \tilde{V}^{\downarrow\uparrow}_{r+d-i,\, D-d-t+i}$$

Now by Def 62

$$Bw = q^{2d-D+r+t-2i}\, w$$

$$= q^{d-2i}\, q^{d-D+r+t}\, w$$

$$= X_{12}\, q^{d-D+r+t}\, w \quad ✓$$

Remaining rows similar $\square$

COR 68 let W denote an irred T-module with endpt r, dual endpt t, diam d.

Consider the $\boxtimes_q$-module str on W from L 64.

In the table below each row contains a matrix in $\mathrm{Mat}_X(\mathbb{C})$ and an element of $\boxtimes_q$; the actions of these two objects on W coincide

Matrix	el of $\boxtimes_q$
$A\Phi\Psi^{-1}$	x_{01}
$B\Phi^{-1}$	x_{12}
$A^*\Phi\Psi$	x_{23}
$B^*\Phi^{-1}$	x_{30}
$K\Psi^{-1}$	x_{02}
$K^*\Psi$	x_{13}

Pf $A\Phi\Psi^{-1}$ acts on W as

$$x_{01}\ q^{D-d-2t} \qquad q^{d-D+r+t} \qquad q^{t-r}$$

which is x_{01}

Other rows sim. $\square$

We are now ready to prove Th 63.

Pf of Th 63 V decomposes into a direct sum of irred T-modules. Each irred T-module in this sum supports a $\boxtimes_q$-module structure as in L 64. Combining these $\boxtimes_q$-modules we get a $\boxtimes_q$-module structure on V. Remains to show this $\boxtimes_q$-module str satisfies the table in Th 63. This is the case by Cor 68. $\square$

Note 69. In thm 63 we displayed a $\boxtimes_q$-module structure on V. In Note 53 we mentioned 4 algebra homs $U_q\hat{sl}_2 \to \boxtimes_q$.
Using these homs to pull back the $\boxtimes_q$-action we get four $U_q\hat{sl}_2$-module structures on V. $\square$

We now consider how the $\boxtimes_q$-action is related to T.

LEM 70 We have

(i) Each of $A, A^*, B, B^*, K, K^*, \Phi, \Psi$ is in T

(ii) Φ, Ψ are in the center $Z(T)$

pf (i) By constr $A, A^* \in T$. For the other elements the pf is just like the pf of LEM 142 in ch 2.

(ii) By L67 Φ, Ψ act as a scalar mult of identity on each irred T-module. $\square$

Our action of $\boxtimes_q$ on V induces a $\mathbb{C}$-algebra hom $\varphi : \boxtimes_q \to \mathrm{Mat}_X(\mathbb{C})$

thm 71.

(i) The image $\varphi(\boxtimes_q)$ is contained in T.

(ii) T is generated by $\varphi(\boxtimes_q)$ together with Φ, Ψ.

pf (i) Combine th 63 and L 70 (i)

(ii) T is gen by A, A^* and by th 63

A, A^* are in the subalg of T gen by $\varphi(\boxtimes_q)$ and Φ, Ψ $\square$

Open Problem 72. What is the combinatorial meaning of matrices B, B^*, K, K^*, Φ, Ψ

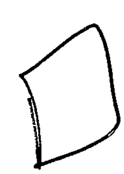
The way we defined them, not clear what are the entries. For $y, z \in X$, find the (y,z)-entry for each of these matrices.

Open Problem 73 For any Q-poly DRG what is the combinatorial meaning of the split dec

$$V = \sum_{i,j=0}^{D} \tilde{V}_{ij} \qquad (ds)$$

For $0 \le i, j \le D$ find a basis for $\tilde{V}_{ij}$. Find the matrix that represents A, A^* wrt this basis.

Start with Hypercube $H(D,2)$.



Lec 43 Wednesday May 6

Here is a problem concerning $\boxtimes_q$

To motivate, start with $U_q(sl_2)$

Recall Chev gens

e, f, K, K^{-1}

Define
$$C = ef + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}$$
"Casimir element"

LEM 74. C is in center $Z(U_q(sl_2))$

Pf Show C commutes with K:

$$[K, ef] = \underbrace{Kef}_{\underbrace{q^2eKf}_{q^2e\,q^{-2}fK}} - efK$$

$= 0$ ✓

Show C commutes with e:

$$[e, C] = \underbrace{eef - efe}_{e\frac{K-K^{-1}}{q-q^{-1}}} + \frac{q^{-1}eK - q^{-1}\overbrace{Ke}^{qeK} + qeK^{-1} - q\overbrace{K^{-1}e}^{q^{-1}eK^{-1}}}{(q-q^{-1})^2}$$

$= 0$

Sim C commutes with f ✓ $\square$

LEM 75 For the equitable gens x, y, z

$$C = qx + q^{-1}y + qz - qxyz$$
$$= qy + q^{-1}z + qx - qyzx$$
$$= qz + q^{-1}x + qy - qzxy$$

pf Use

$$k = x$$
$$f = y - x^{-1}$$
$$e = (1 - xz)\, q^{-1}(q - q^{-1})^{-2}$$
$\square$

Back to $\boxtimes_q$

$\boxtimes_q$ contains 4 copies of $U_q(sl_2)$, each

with its own Casimir element

DEF 76 For $i \in \mathbb{Z}_4$ def $C_i \in \boxtimes_q$ by

$$C_i = q x_{i,i+2} + q^{-1} x_{i+2,i+3} + q x_{i+3,i} - q x_{i,i+2} x_{i+2,i+3} x_{i+3,i}$$

By constr C_i commutes with

$x_{i,i+2}$, $x_{i+2,i+3}$, $x_{i+3,i}$

Problem 77. How are the C_i $i \in \mathbb{Z}_4$ related to each other? Find the equations in $\boxtimes_q$ involving these elements.

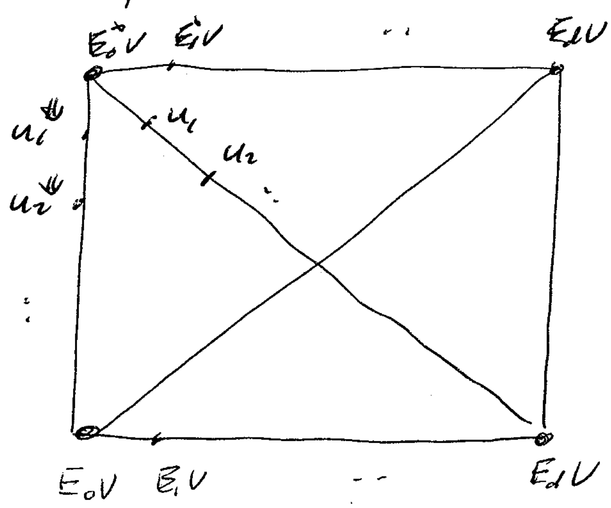
this suggests a more general problem about TD pairs

Assume $\mathbb{F}$ arb

Fix $0 \neq V =$ f.d vs/$\mathbb{F}$

Given TD system $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V

Recall $\boxtimes$ picture



Call a lin trans $X : V \to V$ <u>0-Casimir</u> whenever

(i) $X E_i V \subseteq E_i V$ $\quad 0 \le i \le d$

(ii) $X U_i \subseteq U_i$ $\quad 0 \le i \le d$

(iii) $X U_i^{\Downarrow} \subseteq U_i^{\Downarrow}$ $\quad 0 \le i \le d$

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Set of all o-Casimir elements forms a subalg of $\mathrm{End}_{\mathbb{F}}(V)$. Call it C_o.

Subalgebras C_o^*, C_d, C_d^* similarly defined.

Problem 78. How are C_o, C_o^*, C_d, C_d^* related? What equations relate the elements of these subalgebras? Are C_o, C_o^*, C_d, C_d^* commutative? If so what are their eigenspaces?

We have discussed the algebra $\boxtimes_q$

This is a q-analog of a Lie algebra $\boxtimes$

"tetrahedron algebra"

In time remaining we discuss $\boxtimes$

Start with Lie algebra sl_2. Assume

Field $\mathbb{F}$ char 0

Recall $sl_2 = sl_2(\mathbb{F})$ is the Lie algebra over $\mathbb{F}$ of all

2×2 trace 0 matrices $/\mathbb{F}$, together with Lie bracket

$$[x, y] = xy - yx$$

Basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Satisfies

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h.$$

Define

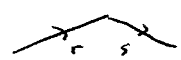
$$x = 2e - h, \qquad y = -2f - h, \qquad z = h$$

then x, y, z is basis for sl_2 and

$$[x, y] = 2x + 2y, \qquad [y, z] = 2y + 2z, \qquad [z, x] = 2z + 2x$$

Call x, y, z the <u>equitable</u> <u>basis</u>

Diag 



means $[r, s] = 2r + 2s$

Recall sl_2 - loop algebra

t = indet $\qquad \mathbb{F}[t, t^{-1}]$ = $\mathbb{F}$-alg of all Laurent polys in t

Laurent poly: $\qquad \sum_{i \in \mathbb{Z}} a_i t^i \qquad a_i \in \mathbb{F} \qquad$ f. many a_i nonzero

$\mathbb{F}$-vector space

$$sl_2 \otimes \mathbb{F}[t, t^{-1}] \qquad \otimes = \otimes_{\mathbb{F}}$$

becomes a Lie algebra with Lie bracket

$$[u \otimes f, v \otimes g] = [u, v] fg \qquad \begin{array}{l} u, v \in sl_2 \\ f, g \in \mathbb{F}[t, t^{-1}] \end{array}$$

Call this Lie alg

$L(sl_2)$: the <u>Loopalgebra</u> for sl_2

LEM 79 $L(sl_2)$ is isomorphic to the Lie algebra $/\mathbb{F}$ with gens e_i, f_i, h_i $i \in \{0,1\}$ and relations

$$h_0 + h_1 = 0$$

$$[h_i, e_j] = A_{ij} e_j \qquad A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$[h_i, f_j] = -A_{ij} f_j$$

$$[e_i, f_j] = \delta_{ij} h_j$$

$$[e_i, [e_i, [e_i, e_j]]] = 0 \quad \forall\, i \neq j$$

$$[f_i, [f_i, [f_i, f_j]]] = 0 \quad \forall\, i \neq j$$

An iso is given by

$$e_1 \to e \otimes 1 \qquad e_0 \to f \otimes t$$

$$f_1 \to f \otimes 1 \qquad f_0 \to e \otimes t^{-1}$$

$$h_1 \to h \otimes 1 \qquad h_0 \to -h \otimes 1$$

pf See V. Kac ∞ dim'l Lie algebras.

We now give the equitable pres for $L(sl_2)$

LEM 80 $L(sl_2)$ is iso to Lie algebra / $\mathbb{F}$

with gens x_i, y_i, z_i $i=0,1$ and rels

$$z_0 + z_1 = 0$$

$$[x_i, y_i] = 2x_i + 2y_i$$

$$[y_i, z_i] = 2y_i + 2z_i$$

$$[z_i, x_i] = 2z_i + 2x_i$$

$$[y_i, x_j] = 2y_i + 2x_j \qquad i \neq j$$

$$[x_i[x_i[x_i, x_j]]] = 4[x_i, x_j] \qquad i \neq j.$$

"Dolan Grady"

$$[y_i[y_i[y_i, y_j]]] = 4[y_i, y_j] \qquad i \neq j.$$

An iso with the presentation in L79 is

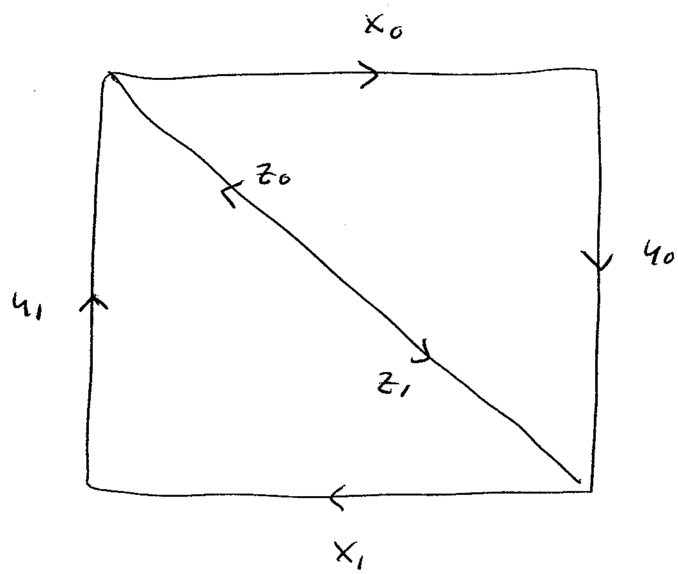
$$x_i \to 2e_i - h_i$$
$$y_i \to -2f_i - h_i$$
$$z_i \to h_i$$

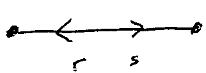
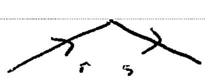
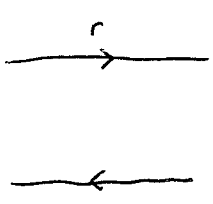
The inverse of this iso is given by

$$e_i \to \frac{x_i + z_i}{2}, \qquad f_i \to -\frac{y_i + z_i}{2}, \qquad h_i \to z_i$$

Pf. One checks each map iso hom of Lie algebras and that maps are inverses. $\square$

Diagram



picture	meaning
	$r+s=0$
	$[r,s] = 2r+2s$
	$[r,[r,[r,s]]] = 4[r,s]$

We add the missing diagonal to get $\boxtimes$

Def. 81. Let $\boxtimes$ denote the Lie algebra over $\mathbb{F}$ defined

by gens $\{ x_{ij} \mid i,j \in \mathbb{I},\ i \neq j \}$ $\qquad \mathbb{I} = \{0,1,2,3\}$

and rels

(i) For dist $i,j \in \mathbb{I}$

$$x_{ij} + x_{ji} = 0$$

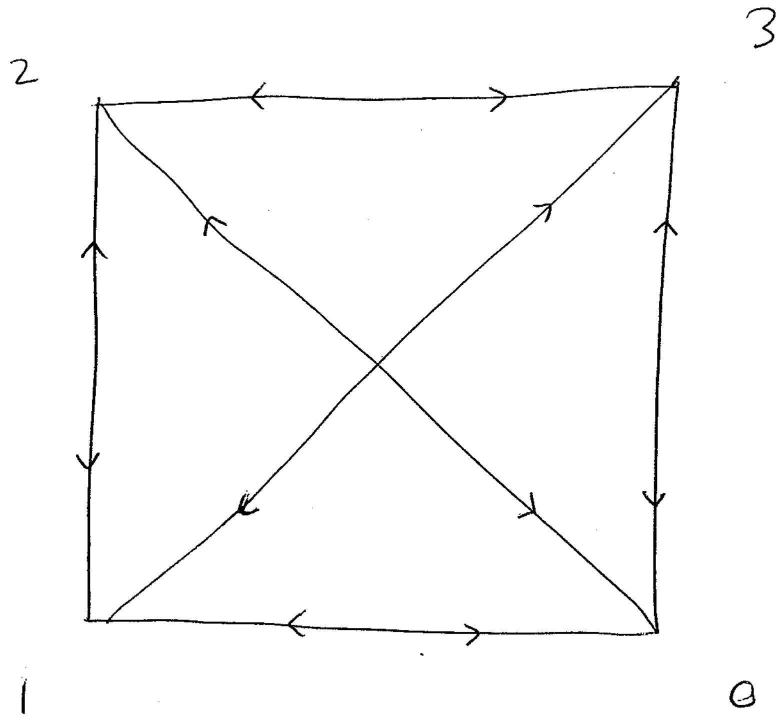
(ii) For mut dist $h,i,j \in \mathbb{I}$

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}$$

(iii) For mut dist $h,i,j,k \in \mathbb{I}$

$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}]$$

$\boxtimes$:



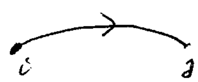
X_{ij} rep by 

Diagram has same interp as for $L(sl_2)$.

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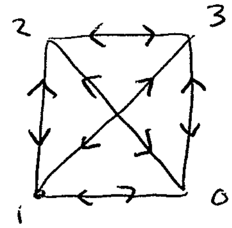
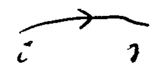
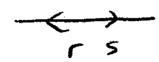
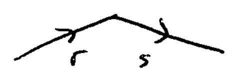
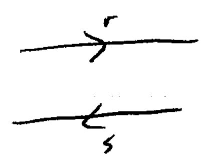
$\mathbb{F}$ char 0

Lie algebra $\boxtimes$ has presentation



Gen X_{ij} rep by 

picture	meaning
	$r + s = 0$
	$[r,s] = 2r+2s$
	$[r,[r,[r,s]]] = 4[r,s]$

It turns out $\boxed{\times}$ isiso to "3-point sl_2 loop algebra"

Define

$$\mathbb{F}[t, t^{-1}, (t-1)^{-1}] = \mathbb{F}\text{-algebra of all}$$

Laurent polys in $t, t-1$

ie all rational functions $\dfrac{f(t)}{t^r(t-1)^s}$ $\quad f \in \mathbb{F}[t]$

$0 \le r, s < \infty$

$sl_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$ becomes Lie alg with

$$[u \otimes f, \ v \otimes g] = [u, v] \otimes fg \qquad u, v \in sl_2$$

$$f, g \in \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$$

This is <u>3-pt sl_2 loop algebra</u>

One can check that there ~~map~~ exists an aut

$$\mathbb{F}[t, t^{-1}, (t-1)^{-1}] \to \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$$

$$t \to 1 - T^{-1}$$

One finds $\quad T'' = (1-T)^{-1}$

$T''' = T$

so ' has order 3

Thm 82 $\exists$ iso of Lie algebras

$$\boxtimes \to sl_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$$

that sends

gen	image
X_{12}	$x \otimes 1$
X_{23}	$y \otimes 1$
X_{31}	$z \otimes 1$
X_{03}	$y \otimes t + z \otimes (t-1)$
X_{01}	$z \otimes t' + x \otimes (t'-1)$
X_{02}	$x \otimes t'' + y \otimes (t''-1)$

Here x, y, z is equit basis for sl_2.

Pf. see Brian Hartwig + T: The tetrahedron algebra, the Onsager algebra and the sl_2 loop algebra $\square$

Thm 82 has the following corollaries.

Cor 83 For mut dist $h, i, j \in I$ $\exists$ Lie alg

injection

$$sl_2 \longrightarrow \boxtimes$$

that sends

$$x \longrightarrow x_{hi},$$

$$y \longrightarrow x_{ij}$$

$$z \longrightarrow x_{jh}.$$

"Send the triangle into a face of the tetrahedron"

where x, y, z is equiv basis for sl_2.

Pf ex.

COR 84 For mut dist $h, i, j, k \in \mathbb{I}$

$\exists$ Lie algebra injection

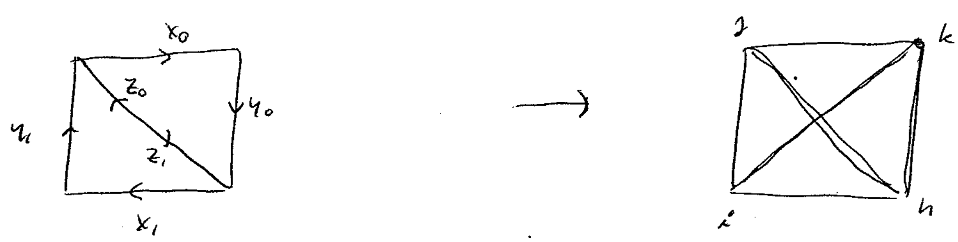
$$L(sl_2) \longrightarrow \boxtimes$$

that sends

$$X_1 \to X_{hi} \qquad y_1 \to X_{ij} \qquad Z_1 \to X_{jh}$$

$$X_0 \to X_{jk} \qquad Y_0 \to X_{kh} \qquad Z_0 \to X_{hj}$$

pf "Send the diamond into adj faces in tetrahedron"



Recall the Onsager algebra $\mathcal{O}$ is the Lie algebra

def by gens x, y and relations

$$[x,[x,[x,y]]] = 4[x,y]$$

$$[y,[y,[y,x]]] = 4[y,x]$$

COR 85. For mut dist $h,i,j,k \in \mathbb{I}$ $\exists$ Lie alg

injection

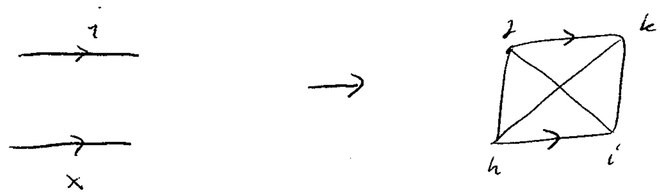
$$\mathcal{O} \longrightarrow \boxtimes$$

that sends

$$x \longrightarrow x_{hi}$$

$$y \longrightarrow x_{jk}$$

Pf "send x, y to a pair of opp edges in the tetrahedron"



Define

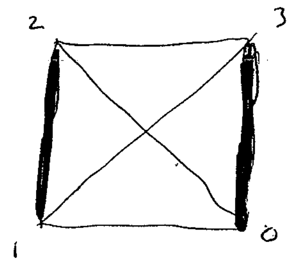
Ω = subalg of $\boxtimes$ gen by X_{12}, X_{03}

Ω' = " " " X_{23}, X_{01}

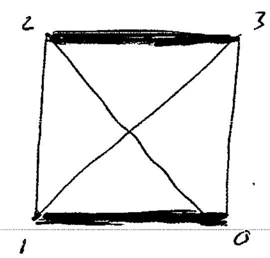
Ω'' = " " " X_{31}, X_{02}

(By Cor 85 each of $\Omega, \Omega', \Omega''$ is iso to Onsager algebra)

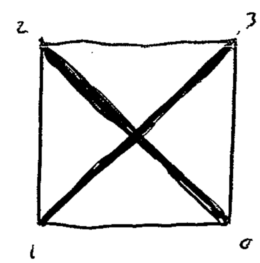
Ω



Ω'



Ω''



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COR 86 (B. Hartwig) We have

$$\boxtimes = \Omega + \Omega' + \Omega'' \qquad \text{(ds of } \mathbb{F}\text{-vector spaces)}$$

We now show how $\boxtimes$ is related to a certain kind of TD pair.

Fix $0 \neq V$ = f.d v.s / $\mathbb{F}$

Given TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V

Assume

$$\theta_i = d - 2i \qquad 0 \le i \le d$$

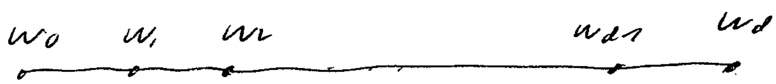
$$\theta_i^* = 2i - d \qquad 0 \le i \le d$$

As we saw before, in this case A, A^* satisfy Dolan-Grady rels

$$[A, [A, [A, A^*]]] = 4[A, A^*]$$

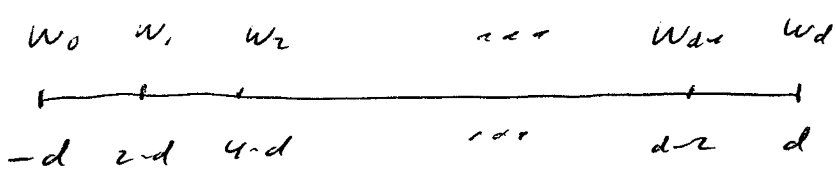
$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A]$$

<u>Notation</u> Given any dec $\{W_i\}_{i=0}^d$ of V



Consider lin trans $T: V \to V$ s.t for $0 \le i \le d$

W_i is the eigenspace for T with eigenval $2i-d$



We often represent T by a directed arc

$$w_0 \longrightarrow w_d$$

Obs $-T$ is rep by

$$w_0 \longleftarrow w_d$$

Ex

$$\begin{tikzcd} E_0^*V \arrow[r, "A^*"] \arrow[d, dash] \arrow[dr, dash] & E_d^*V \arrow[d, dash] \arrow[dl, dash] \\ E_0V & E_dV \arrow[l, "A"'] \end{tikzcd}$$

DEF 87 We def lin trans $B, B^*, h, h^*: V \to V$ as follows

$$
\begin{tikzcd}
E_0^*V \arrow[r, "A^*"] & E_d^*V \arrow[d, "B^*"] \\
E_0V \arrow[u, "B"] \arrow[ur, "h^*"] & E_dV \arrow[l, "A"] \arrow[ul, "h"]
\end{tikzcd}
$$

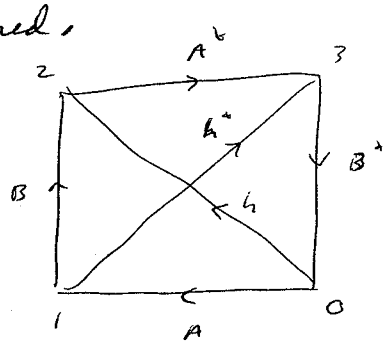
The lin trans A, A^*, B, B^*, h, h^* satisfy many equations. These equations are described in the following thm.

thm 88 (B. Hartwig) Ref to our TD system Φ

$\exists$ unique $\boxtimes$-module str on V s.t the gens x_{ij} act as follows

gen	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}
action	A	B	A^*	B^*	h	h^*

this $\boxtimes$-module str is lined.



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THE END