

$F = \mathbb{R}$ or \mathbb{C} Given DRG $\Gamma = (X, R)$ dim $0 \leq d$

Assume $\{E_i\}_{i=0}^d$ is \mathbb{Q} -poly

Fix $x \in X$, write $T = T(x)$ etc

For time being assume Γ is bipartite

Let W denote an unred T -module with endpoint

dual endpoint t , dim $d \geq 1$ Recall W is thin by L125.

Obs $a_i(W) = 0$ ($0 \leq i \leq d$)

LEM 127 We have

$$(i) \quad \varphi_i = \theta_t (\theta_{r_{i+1}}^{\vee} - \theta_r^*)$$

$$(ii) \quad \theta_{t_{i+1}} (\theta_{r_{i+1}}^* - \theta_r^{\vee}) = \theta_t (\theta_{r_{i+1}}^{\vee} - \theta_{r_{i+1}}^*)$$

$$\text{pf (i) by L122} \quad \theta_t + \frac{\varphi_i}{\theta_r^{\vee} - \theta_{r_{i+1}}^*} = a_0(W) = 0$$

(ii) By L123 (i) and since $b_d(W) = 0$

$$c_d(W) = \frac{(\theta_{t_{i+1}} - \theta_t) (\theta_{r_{i+1}}^* - \theta_r^*) + \varphi_i}{\theta_{r_{i+1}}^{\vee} - \theta_{r_{i+1}}^*}$$

Also

$$\begin{aligned}\theta_t &= c_d(w) + \underbrace{a_d(w)}_{''_0} + \underbrace{b_d(w)}_{''_0} \\ &= c_d(w)\end{aligned}$$

Comparing the two expressions for $c_d(w)$ and using (i) above

we get the result.

□

L128 We have

$$(i) \quad c_0(\omega) = 0$$

$$(ii) \quad c_i(\omega) = \frac{\theta_{\ell} (\theta_{r+i}^* - \theta_{r+i}^*) - \theta_{\ell} (\theta_{r+i}^* - \theta_{r+i}^*)}{\theta_{r+i}^* - \theta_{r+i}^*} \quad (1 \leq i \leq d-1)$$

$$(iii) \quad c_d(\omega) = \theta_{\ell}$$

$$(iv) \quad b_0(\omega) = \theta_{\ell}$$

$$(v) \quad b_i(\omega) = \frac{\theta_{\ell} (\theta_{r+i}^* - \theta_{r+i}^*) - \theta_{\ell} (\theta_{r+i}^* - \theta_{r+i}^*)}{\theta_{r+i}^* - \theta_{r+i}^*} \quad (1 \leq i \leq d-1)$$

$$(vi) \quad b_d(\omega) = 0$$

Pf A linear equation involving $c_i(\omega)$, $b_i(\omega)$ is given in L123 (i).

We also have $c_i(\omega) + b_i(\omega) = \theta_{\ell}$

Solve this system for $c_i(\omega)$, $b_i(\omega)$ and elem

φ_i using L127 (i)

LEM 129

$$\theta_i = -\theta_{p-i} \quad (0 \leq i \leq p)$$

[We saw this earlier for the natural ordering of the eigenvalues]

Pf (For Case I) $\beta \neq \pm 2$

$$\theta_i = a + bq^{2i-p} + cq^{p-2i}$$

 $0 \leq i \leq p$

$$\theta_i^* = a^* + b^*q^{2i-p} + c^*q^{p-2i}$$

$$\beta = q^2 + q^{-2}$$

By L74(i)

$$0 = \gamma$$

$$= \theta_{i-1} - \beta \theta_i + \theta_{i+1} \quad (1 \leq i \leq p-1)$$

$$= a(2 - \beta)$$

$$\text{So } a = 0$$

Show $b = -c$

Apply L 127 (ii) to the previous module to get

$$\frac{\theta_1}{\theta_0} = \frac{\theta_{0,1}^x - \theta_1^x}{\theta_0^x - \theta_0^x} \quad (*)$$

Note $q^{2i} \neq 1$ ($0 \leq i \leq 0$) since $\{\theta_i\}_{i=0}^0$ are dist

Eval (*) to get

$$\frac{b q^{2-0} + c q^{0-2}}{b q^{-0} + c q^0} = \frac{q^{2-0} - q^{0-2}}{q^{-0} - q^0}$$

Solve to find $b = -c$

the pt for cases II, III is sum. \square

LEM 130 For our module W

(i) $2t+d = D$

(ii) $D-d$ is even

Pf (i) (For Case I) heap notation of L129

By L127 (ii)

$$\frac{\theta_{2t}}{\theta_t} = \frac{\theta_{2t+d}^* - \theta_{2t}^*}{\theta_{t+d}^* - \theta_t^*}$$

Guess

$$\frac{q^{2t+2-d} - q^{d-2t-2}}{q^{2t-d} - q^{d-2t}} = \frac{q^{2-d} - q^{d-2}}{q^{-d} - q^d}$$

this yields

$$q^{(2t+d-d)t} = 1$$

and then

$$2t+d = 0$$

using $q^{2i} \neq 1$ ($1 \leq i \leq D$)

(ii) ✓

□

LEM 131 As a T -module the iso class of W

is determined by r and d .

Pf Let W' denote an unred T -module with endst r and deam d . Show T -modules W, W' are iso.

Obs $t' = t$ by L130

Also by L128

$$c_i(W') = c_i(W), \quad b_i(W') = b_i(W) \quad (0 \leq i \leq d)$$

Let $\{w_i\}_{i=0}^d$ be a st. basis for W

Let $\{w'_i\}_{i=0}^d$ be a st. basis for W'

Define a linear trans $\sigma: W \rightarrow W'$ s.t.

$$\sigma(w_i) = w'_i \quad 0 \leq i \leq d$$

So σ is iso of vector spaces.

Matrix rep A rel $\{w_i\}_{i=0}^d$ equals the matrix rep A rel $\{w'_i\}_{i=0}^d$

$$S_0 \quad A\sigma = \sigma A$$

By const

$$A^* w_i = \theta_{ri} w_i \quad 0 \leq i \leq d$$

$$A^* w_i' = \theta_{ri} w_i' \quad 0 \leq i \leq d$$

So

$$A^* \sigma = \sigma A^*$$

A, A^* gen T so σ is T -module iso. □

For our bipartite Γ

the two classes of unred T -modules are indexed by the
endpt r and diam d

What are the possible r, d ?

Constraints:

$$(i) \quad 0 \leq d \leq D$$

$$(ii) \quad D-d \text{ even}$$

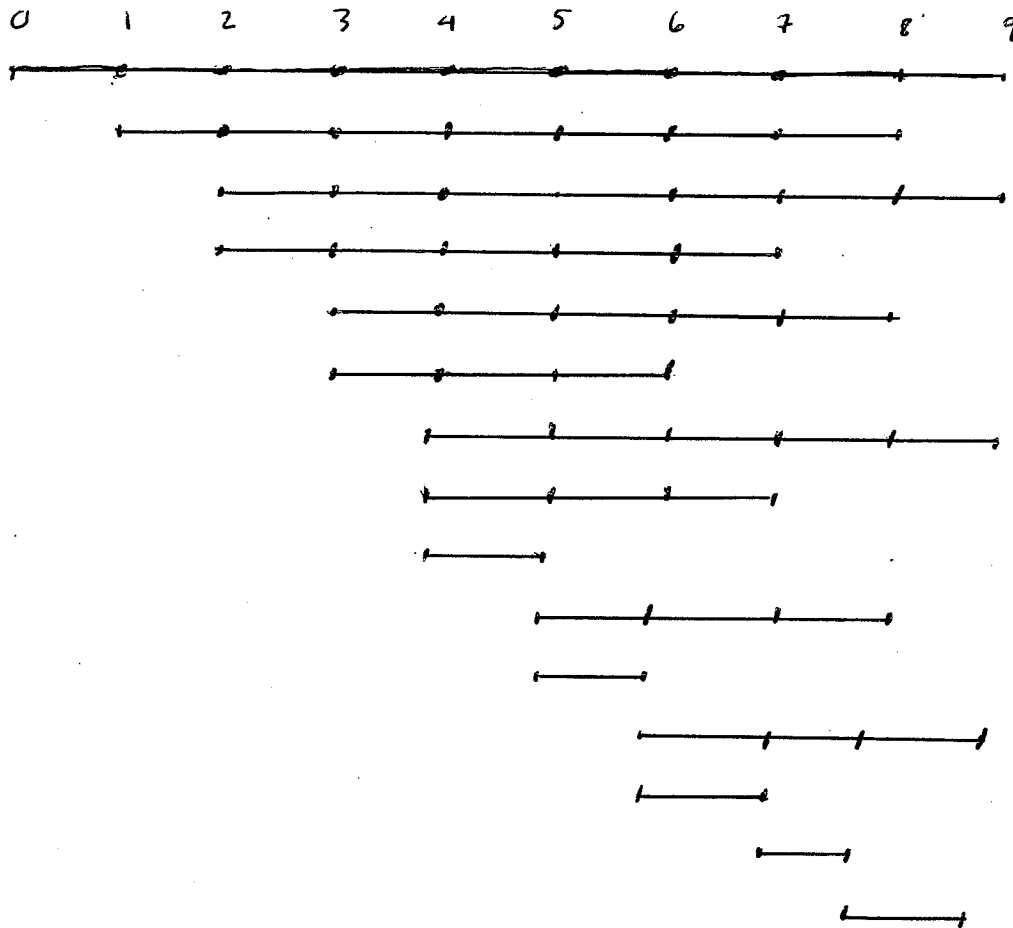
$$(iii) \quad r \leq D-d \quad (\text{by constr})$$

$$(iv) \quad \frac{1}{2}(D-d) \leq r \quad (\text{by Cor 92 (i)})$$

$$\text{Let } \Delta = \left\{ (r, d) \in \mathbb{Z}^2 \mid r, d \text{ sat (i)-(iv)} \right\}$$

Δ looks as follows (to illustrate take $D=9$)

In the diagram each (r, d) is rep by a line
segment from col r to col $r+d$



Caution: For some (rid) the corresp
module might not exist.

No longer assume Γ is bipartite

Instead assume $\{E_i\}_{i=0}^D$ is dual bip.

Then LEM 127-131 still hold if we replace

$$\theta_i \leftrightarrow \theta_i^*, \quad r \leftrightarrow t, \quad c_i(w) \leftrightarrow c_i^*(w)$$

$$b_i(w) \leftrightarrow b_i^*(w) \quad \text{etc.}$$

Indeed the proofs are purely algebraic.

Until further notice assume both

Γ is bipartite

$\{E_i\}_{i=0}^d$ is dual bipartite.

LEM 132 For each und T-module U

(i) $2r+d=0$

(ii) $2t+d=0$

(iii) $D-d$ even

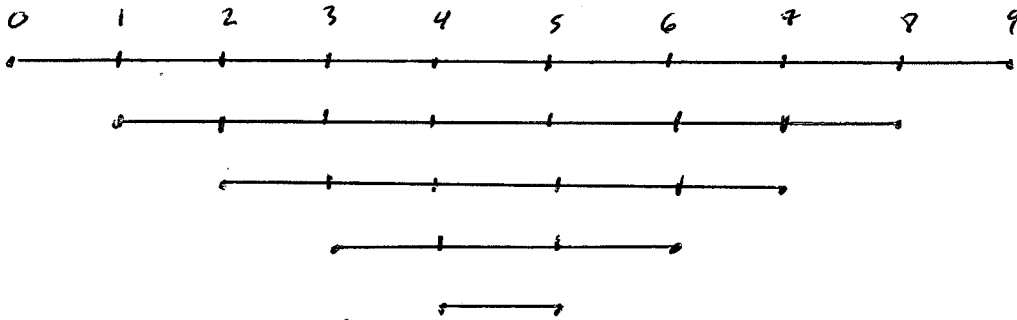
(iv) iso class of U is det by d

PF By L130 and its dual, and L131

Referring to our sets Δ from below L131

the set of (r, d) in Δ that satisfy L132 looks

as follows



LEM 133 Let W denote an r th T -module

with endpt r . Then W appears in the sth module V

with mult

$$\begin{cases} 1 & \text{if } r=0 \\ \ker - \ker & \text{if } r \geq 1 \end{cases}$$

Pf Denote the mult by μ_r .

Obs

$$\mu_0 + \mu_1 + \dots + \mu_r = \ker$$

Result follows.

□

Ex 134 $\Gamma = H(d, 2)$ hypercube

Γ bip

Γ has dual bip \mathbb{Q} -only structure with

$$\theta_i = d - 2i \quad 0 \leq i \leq d$$

$$\theta_i^* = d - 2i \quad 0 \leq i \leq d$$

Let W denote an unad T -module diam d .

By L132 $d-d$ even, and

$$r = \frac{d-d}{2}, \quad t = \frac{d-d}{2}$$

Using L128

$$c_i(W) = c, \quad 0 \leq i \leq d$$

$$b_i(W) = d - i \quad 0 \leq i \leq d$$

Using the dual of L128

$$c_i^*(W) = c \quad 0 \leq i \leq d$$

$$b_i^*(W) = d - i \quad 0 \leq i \leq d$$

By L133.

W appears in V with mult $\begin{cases} 1 & \text{if } r=0 \\ \binom{d}{r} - \binom{d}{r-1} & \text{if } r \geq 1 \end{cases}$

□

Ex 135 Assume Γ is bipartite and $\{E_i\}_{i=0}^p$ is

dual bipartite. Further assume $\beta \neq \pm 2$, and let

q be as in L80. Recall

$$\beta = q^2 + q^{-2}$$

$$e_i = e_i^\vee = (q^{p-2} + q^{2-p}) \frac{q^{p-2i} - q^{2i-p}}{q^2 - q^{-2}} \quad 0 \leq i \leq p$$

Let W denote an irreducible T -module of dimension d . By L132

$D-d$ is even and

$$r = \frac{D-d}{2}, \quad t = \frac{D+d}{2}$$

Using L128 and its dual

$$c_i(w) = c_i^\vee(w) = \frac{q^{p-2} + q^{2-p}}{q^{d-2i} + q^{2i-d}} \frac{q^{2i} - q^{-2i}}{q^2 - q^{-2}}$$

$$b_i(w) = b_i^\vee(w) = \frac{q^{p-2} + q^{2-p}}{q^{d-2i} + q^{2i-d}} \frac{q^{2d-2i} - q^{2i-2d}}{q^2 - q^{-2}}$$

for $0 \leq i \leq d$.



Lec 27 Monday March 30

No. Lec 27-1

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$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{\bar{E}_i\}_{i=0}^D$ is \mathbb{Q} -poly

Fix $x \in X$ write $T = T(x)$ etc

Fix a then used T -module W with endpts r , dual endpts t ,
diam $d \geq 1$

In L128 we found

$$c_i(W), b_i(W)$$

when Γ is bip. What if Γ not bip?

Similar formulae can be obtained by similar methods

but it is slightly complicated, so I will skip the details

and just state the results.

Thm 136 Let the T -module W be as above and let

$\varphi_i = \varphi_i(W)$ be as in L121. Then

$$c_0(W) = 0$$

$$c_i(W) = \frac{\varphi_i f_i^- + g_i^-}{(\theta_{r+i}^* - \theta_{r+i}^*)(\theta_{r+i}^* - \theta_{r+i}^*)} \quad (1 \leq i \leq d-1)$$

$$c_d(W) = \frac{\varphi_i + (\theta_{r+i} - \theta_r)(\theta_{r+i}^* - \theta_r^*)}{\theta_{r+i}^* - \theta_r^*}$$

$$b_0(W) = \frac{\varphi_i}{\theta_{r+i}^* - \theta_r^*}$$

$$b_i(W) = \frac{\varphi_i f_i^+ + g_i^+}{(\theta_{r+i}^* - \theta_{r+i}^*)(\theta_{r+i}^* - \theta_{r+i}^*)} \quad (1 \leq i \leq d-1)$$

$$b_d(W) = 0$$

where

$$f_i^\pm = \theta_{r+i}^* - \theta_{r+i \mp 1}^* - \frac{(\theta_{r+i}^* - \theta_r^*)(\theta_{r+i}^* - \theta_{r+i \mp 1}^*)}{\theta_{r+i}^* - \theta_r^*}$$

$$g_i^\pm = (\theta_{r+i}^* - \theta_r^*) \left((\theta_{r+i \mp 2} - \theta_{r+i \mp 1})(\theta_{r+i}^* - \theta_{r+i}^*) - (\theta_{r+i} - \theta_r)(\theta_{r+i \mp 1}^* - \theta_{r+i}^*) \right)$$

if $d \geq 2$ To get $c_i^*(W)$, $b_i^*(W)$ interchange $r \leftrightarrow t$ and $\theta_j \leftrightarrow \theta_j^*$

(05150) \square

COR 137 With ref to Th 136

the isomorphism class of the T -module W is determined by the 4-tuple

(r, t, d, ψ)

PF this 4-tuple determines $c: (W)$, $b: (W)$, $a: (W)$ etc. \square

Earlier we saw that A, A^* satisfy the tridiagonal relations TD1, TD2. When we restrict the actions of A, A^* to a finitely generated T -module, these relations take a nicer form often called the Askey-Wilson relations.

We now explain the details.

In what follows $\beta, \gamma, \gamma^*, \delta, \delta^*$ are from Th 57.

Our next goal is to show:

Thm 138 Let W denote a finitely generated T -module.

Then \exists scalars in \mathbb{F} :

$$\omega = \omega(W), \quad \gamma = \gamma(W), \quad \gamma^* = \gamma^*(W)$$

such that on W

$$A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^* = \gamma^* A^2 + \omega A + \gamma I \quad \text{AW1}$$

$$A^* A^2 - \beta A^* A A^* + A A^*^2 - \gamma^*(A A^* + A^* A) - \delta^* A = \gamma A^*^2 + \omega A^* + \gamma^* I \quad \text{AW2}$$

AW1, AW2 are called the Askey-Wilson relations

We will prove Thm 138 after a few lemmas.

Recall the polynomials

$$P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

$$P^*(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma^*(\lambda + \mu) - \delta^*$$

LEM 139 Let W denote a then used T -module
with dual endst t and diam d

Then for all $\omega, \gamma \in F$

$$A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^* = \gamma^* A^2 + \omega A + \gamma I \text{ on } W \quad (*)$$

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$$a_i^*(W) P(\theta_{t+i}, \theta_{t+i}) = \gamma^* \theta_{t+i}^2 + \omega \theta_{t+i} + \gamma \text{ (osied)} \quad (**)$$

pf let L, R denote LHS, RHS of $(*)$

Obs

$$L = \sum_{i=0}^d \sum_{j=0}^d E_{t+i} L E_{t+j}$$

$$R = \sum_{i=0}^d \sum_{j=0}^d E_{t+i} R E_{t+j}$$

Obs for $0 \leq i, j \leq d$

$$E_{t+i} \mathcal{L} E_{t+j} = P(\theta_{t+i}, \theta_{t+j}) E_{t+i} A^* E_{t+j}$$

$$E_{t+i} \mathcal{R} E_{t+j} = \mathcal{D}_{ij} \left(\gamma^* \theta_{t+i}^2 + \omega \theta_{t+i} + \eta \right) E_{t+i}$$

First assume $*$ so

$$\mathcal{L} = \mathcal{R} \text{ on } W$$

For $0 \leq i \leq d$

$$E_{t+i} \mathcal{L} E_{t+i} \stackrel{\text{on } W}{=} E_{t+i} \mathcal{R} E_{t+i}$$

$$P(\theta_{t+i}, \theta_{t+i}) \underbrace{E_{t+i} A^* E_{t+i}}_{a_i^*(W) E_{t+i}} = \left(\gamma^* \theta_{t+i}^2 + \omega \theta_{t+i} + \eta \right) E_{t+i}$$

By constr $E_{t+i} W \neq 0$ so

$$a_i^*(W) P(\theta_{t+i}, \theta_{t+i}) = \gamma^* \theta_{t+i}^2 + \omega \theta_{t+i} + \eta$$

We have $(*)$.

Next assume $(**)$ show $\mathcal{L} = \mathcal{R}$ on W

Suf to show

$$E_{t+i} \mathcal{L} E_{t+j} \stackrel{\text{on } W}{=} E_{t+i} \mathcal{R} E_{t+j} \quad 0 \leq i, j \leq d$$

For $0 \leq i \leq d$

$$E_{t+i} \mathcal{L} E_{t+i} = a_i^*(w) P(\theta_{t+i}, \theta_{t+i}) E_{t+i}$$

$$E_{t+i} \mathcal{R} E_{t+i} = (\gamma^2 \theta_{t+i}^2 + \omega \theta_{t+i} + \eta) E_{t+i}$$

So

$$E_{t+i} \mathcal{L} E_{t+i} = E_{t+i} \mathcal{R} E_{t+i}$$

For $0 \leq i, j \leq d$ with $|i-j| = 1$

Recall

$$P(\theta_{t+i}, \theta_{t+j}) = 0$$

by Prop 64 (iv) so

$$E_{t+i} \mathcal{L} E_{t+j} = 0$$

Also

$$E_{t+i} \mathcal{R} E_{t+j} = 0 \quad \text{since } i \neq j.$$

so

$$E_{t+i} \mathcal{L} E_{t+j} = E_{t+i} \mathcal{R} E_{t+j}$$

For $0 \leq i, j \leq d$ s.t. $|i-j| > 1$,

$$E_{t+i} A^j E_{t+j} = 0$$

so

$$E_{ti} l E_{tj} = 0$$

Also

$$E_{ti} R E_{tj} = 0$$

since $i \neq j$.

We have shown

$$E_{ti} l E_{tj} \stackrel{on W}{=} E_{ti} R E_{tj} \quad \text{for } i, j \leq d$$

so $l = R$ on W .

□

Pf of Thm 138:

Set $d = \dim W$.

Assume $d \geq 1$ else triv.

As before def

$$\mathcal{L} = A^2 A^* - \beta A A^* A + \gamma A^* A^2 - \delta (A A^* + A^* A) - \epsilon A^*$$

By TD1

$$0 = [\mathcal{L}, A]$$

Aside on lin alg: for any linear trans $\sigma: W \rightarrow W$

that is diagonalizable and has all eigenpaces dim 1

any linear trans $W \rightarrow W$ that commutes with σ is a poly in σ (ex)

By this aside $\exists f \in F[\lambda]$ s.t.

$$\mathcal{L} \stackrel{on W}{=} f(A)$$

Minimal poly of A on W has degree $d+1$ so WLOG

$$\deg f \leq d.$$

Let $h = \text{degree of } f$.

claim $h \leq 2$

Suppose $h > 2$

Obs

$$\underbrace{E_{r+h}^* \mathcal{L} E_r^*}_{= 0 \text{ by form of } \mathcal{L}} \stackrel{\text{on } W}{=} \underbrace{E_{r+h}^* f(A) E_r^*}_{\propto \underbrace{E_{r+h}^* A^h E_0^*}_{\text{Leading coeff of } f}} \quad d = \text{Leading coeff of } f$$

$$\underbrace{E_{r+h}^* A E_{r+h-1}^* A E_{r+h-2}^* A \dots E_1^* A E_0^*}_{\neq 0 \text{ on } W}$$

cont.

clarified.

Write

$$\mathcal{L} = \varepsilon A^2 + \omega A + \gamma I \quad \varepsilon, \omega, \gamma \in \mathbb{F}$$

For moment let $d=1$. Then I, A, A^2 lin dep on W so ε

can be chosen so $\varepsilon = \gamma^*$.

Next assume $d \geq 2$ show $\varepsilon = \gamma^*$. Obs

$$\underbrace{E_{r+2}^* \mathcal{L} E_r^*}_{\text{on } W} = \underbrace{E_{r+2}^* (\varepsilon A^2 + \omega A + \gamma I) E_0^*}_{\varepsilon E_{r+2}^* A^2 E_r^*}$$

$$E_{r+2}^* A^2 E_0^* \underbrace{(\theta_r^* - \beta \theta_{r-1}^* + \theta_{r-2}^*)}_{\gamma^*}$$

$E_{r+2}^* A^2 E_r^* \neq 0$ on W since W is then, so $\gamma^* = \varepsilon$

We have obtained AW1. Interchanging the roles of A, A^*

in the argument so far, we find $\exists w^*, \gamma^* \in F$ s.t.

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^*A \stackrel{\text{on } W}{=} \gamma A^2 + w^*A^* + \gamma^*I \quad (\text{AW2}')$$

Show $w = w^*$. Find the commutator of AW1 with A^* . Get

$$\begin{aligned} A^2A^{*2} - \beta A A^* A A^* + \beta A^* A A^* A - A^{*2}A^2 - \gamma(AA^{*2} - A^{*2}A) \\ \stackrel{\text{on } W}{=} \gamma^*(A^2A^* - A^*A^2) + w(AA^* - A^*A) \end{aligned}$$

Now find the commutator of AW2' with A . Get

$$\begin{aligned} A^{*2}A^2 - \beta A^*AA^*A + \beta AA^*AA^* - A^2A^{*2} - \gamma^*(A^*A^2 - A^2A^*) \\ \stackrel{\text{on } W}{=} \gamma(A^{*2}A - AA^{*2}) + w^*(A^*A - AA^*) \end{aligned}$$

Adding the above eqs get

$$0 \stackrel{\text{on } W}{=} (w - w^*)(AA^* - A^*A)$$

Obs $AA^* \neq A^*A$ on W since $d \geq 1$, so

$$w = w^*$$

□

We now consider how to find w, γ, γ^*

Notation: Recall

$$\theta_{i-1} - \beta \theta_i + \theta_{i+1} = \gamma \quad (1 \leq i \leq d-1)$$

$$\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* = \gamma^* \quad (1 \leq i \leq d-1)$$

Define $\theta_{-1}, \theta_{d+1}, \theta_{-1}^*, \theta_{d+1}^*$ so that the above eqs hold for $i=0$ and $i=d$.

Thm 140 Let W denote a Thm unid T -module with endpt r , dual endpt t , diam d . Then the scalars w, γ, γ^* from Th 138 satisfy:

$$(i) \quad w = a_i^*(w) (\theta_{tri} - \theta_{tri+1}) + a_{i+1}^*(w) (\theta_{tri+1} - \theta_{tri+2}) - \gamma^* (\theta_{tts} + \theta_{tts+1}) \quad (1 \leq i \leq d)$$

$$(ii) \quad w = a_i(w) (\theta_{rri}^* - \theta_{rri+1}^*) + a_{i+1}(w) (\theta_{rri+1}^* - \theta_{rri+2}^*) - \gamma (\theta_{rri}^* + \theta_{rri+1}^*) \quad (1 \leq i \leq d)$$

$$(iii) \quad \gamma = a_i^*(w) (\theta_{tri} - \theta_{tri+1}) (\theta_{tri} - \theta_{tri+1}) - \gamma^* \theta_{tts}^2 - w \theta_{tts} \quad (0 \leq i \leq d)$$

$$(iv) \quad \gamma^* = a_i(w) \left(\theta_{ri}^* - \theta_{ri-1}^* \middle| \left(\theta_{ri}^* - \theta_{ri+1}^* \right) \right) - \gamma \theta_{ri}^{*2} - w \theta_{ri}^*$$

0 ≤ i ≤ d

Pf We start with (iii)

(iii) Let i be given, claim

$$P(\theta_2, \theta_1) = (\theta_2 - \theta_{1+}) \middle| (\theta_2 - \theta_{1+}) \quad 0 \leq j \leq d$$

To see this for $1 \leq j \leq d$ elim θ_{j+1} using

$$\theta_{j+1} - \beta \theta_j + \theta_{j+2} = \delta$$

and eval the result using

$$P(\theta_2, \theta_1) = (2 - \beta) \theta_2^2 - 2\delta \theta_2 - \delta$$

and

$$P(\theta_{2-1}, \theta_1) = 0$$

To see the claim for $0 \leq j \leq d-1$, elim θ_{j+1} using

$$\theta_{j+1} - \beta \theta_j + \theta_{j+2} = \delta$$

and proceed in a similar manner, denproved.

Now use the claim and L159.

(Pv) Sim. (i) Subtract (iii) (at i) from (iii) (at $i-1$)

(iii) Sim.



$F = \mathbb{R} \text{ or } \mathbb{C}$ Given DRG $\Gamma = (X, R)$ $\dim D \geq 2$

Assume $\{E_i\}_{i=0}^D$ is \mathcal{Q} -poly

Fix $x \in X$, write $T = T(x)$ etc.

Def 141 For $0 \leq i \leq D$, Γ called i -min

(with respect to x) whenever each used T -module with inputs at node i is min. (So Γ is 0-min)

Γ called min (with respect to x) whenever every used T -module is min.

In what follows we fix scalars

$$p, r, r^*, s, s^*$$

from Th 57

LEM 142 Assume $F = \mathbb{C}$ and Γ is then rel x.

Then \exists central elements

$$\Omega, Z, Z^*$$

in T such that

$$(i) \quad A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* A^* A) - \delta A^* = \gamma^* A^2 + \Omega A + Z$$

$$(ii) \quad A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^* (A^* A^* A A^*) - \delta^* A = \gamma A^{*2} + \Omega A^* + Z^*$$

\therefore Decompose the st. module V into the homogeneous components

$$V = \sum_{\phi \in \Psi} V_{\phi} \quad (\text{orthog ds of } T\text{-modules})$$

Recall $\Psi =$ set of iso classes of irred T -modules

For $\phi \in \Psi$ we define Ω, Z, Z^* on V_{ϕ} .

Let W denote an irred T -module in V_{ϕ} .

Define Ω, Z, Z^* on V_{ϕ} s.t

$$(\Omega - \omega I | V_{\phi} = 0, \quad (Z - \gamma I | V_{\phi} = 0, \quad (Z^* - \gamma^* I | V_{\phi} = 0$$

where $\omega = \omega(W)$, $\gamma = \gamma(W)$, $\gamma^* = \gamma^*(W)$ are from Th 138

By construction Ω, Z, Z^* sat (31), (32) and commute with each element of T .

Show Ω, Z, Z^* are in T (sketch):

The Standard module V is direct sum of irred T -modules, so T is semi-simple. We give a basis for the center of T .

$\forall \phi \in \Psi$ define $e_\phi \in \text{Mat}_X(\mathbb{C})$ s.t

$$(e_\phi - I) | V_\phi = 0,$$

$$e_\phi | V_\psi = 0 \quad \forall \psi \neq \phi \quad (\forall \phi \in \Psi).$$

So e_ϕ is the projection $V \rightarrow V_\phi$.

By the Wedderburn theory

$$\{e_\phi \mid \phi \in \Psi\}$$

is a basis for the center of T . Each of Ω, Z, Z^* is

a linear combination of this basis and hence in T . \square

Note Referring to L142,

Given Ω we can get Z, Z^* using (i), (ii)

Open Problem 143 Find the combinatorial meaning of Ω

In a moment we will give a conjecture about Ω

First some motivation.

Since Ω is central

$$E_i^* \Omega E_j^* = 0 \quad \forall i \neq j \quad (0 \leq i, j \leq D)$$

So

$$\Omega = \sum_{i=0}^D E_i^* \Omega E_i^*$$

So $\forall y, z \in X$

$$\Omega_{yz} \neq 0 \rightarrow \partial(y, z) = \partial(z, y)$$

Conjecture 144 $\forall y, z \in X$

$$\Omega_{yz} \neq 0 \rightarrow \partial(y, z) \leq 1$$

Project 145 Compute Ω, Z, Z^* for

some examples of Q -poly DRG's. See if Cong 144

holds and try to find the combinatorial meaning.

We now put the equations of L142 in cyclic form.

For simplicity we assume $\beta \neq \mp 2$.

Thm 146 Assume $\mathbb{F} = \mathbb{C}$ and $\beta \neq \mp 2$ and Γ is Hermitian.

Pick $0 \neq t^\varepsilon \in \mathbb{F}$ and $r^\varepsilon \in \mathbb{F}$. Then $\exists A^\varepsilon \in T$ and

central $\Phi, \Phi^*, \Phi^\varepsilon \in T$ such that

$$q A A^* - q^{-1} A^* A + r^* A + r A^* + t^\varepsilon A^\varepsilon = \Phi^\varepsilon,$$

$$q A^* A^\varepsilon - q^{-1} A^\varepsilon A^* + r^\varepsilon A^* + r^* A^\varepsilon + t A = \Phi$$

$$q A^\varepsilon A - q^{-1} A A^\varepsilon + r A^\varepsilon + r^\varepsilon A + t^* A^* = \Phi^*$$

where

$$\beta = q^2 + q^{-2}$$

$$\gamma = r(q - q^{-1}),$$

$$\gamma^* = r^*(q - q^{-1})$$

$$\delta = r^2 - t^* t^\varepsilon$$

$$\delta^* = r^{*2} - t t^\varepsilon$$

$$\Omega = (r r^* - r^\varepsilon t^\varepsilon) I - \Phi^\varepsilon (q - q^{-1})$$

$$Z = \Phi^* t^\varepsilon - \Phi^\varepsilon r$$

$$Z^* = \Phi t^\varepsilon - \Phi^\varepsilon r^*$$

Pf Using $\beta, \gamma, \gamma^*, \delta, \delta^*, \Omega, Z, Z^*$

and the last 8 equations, define in order

$q, r, r^*, t, t^*, \Phi^E, \Phi, \Phi^*$

Now define A^E using the 1st eq.

It remains to verify the 2nd and 3rd eqs. In these eqs

we elim A^E using eq 1 and find the result matches the

equations in L142. Result follows. \square

Ex 147 Assume $\mathbb{F} = \mathbb{C}$ and $\beta \neq \pm 2$ and Γ bipartite.

Using th 74,

$$\gamma = 0$$

$$\Omega = 0$$

$$Z^* = 0$$

So by L142

$$Z = A^2 A^* - \beta A A^* A + \gamma A^* A^2 - \delta A^* - \gamma^* A^2$$

Now in th 146

$$r = 0$$

$$\Phi^E = - \frac{r^E t^E}{q - q^{-1}} I$$

$$\Phi = - \frac{r^* r^E}{q - q^{-1}} I$$

$$\Phi^* = \frac{1}{t^E} Z$$

$$A^E = \frac{-1}{t^E} \left(q A A^* - \gamma A^* A + \frac{\gamma^* A}{q - q^{-1}} \right) - \frac{r^E}{q - q^{-1}} I$$

Since $r^\varepsilon, t^\varepsilon$ are free choose

$$r^\varepsilon = 0, \quad t^\varepsilon = 1$$

so

$$\Phi^\varepsilon = 0$$

$$\Phi = 0$$

$$\Phi^* = -Z$$

$$A^\varepsilon = \eta A A^\varepsilon - \eta^{-1} A^* A + \frac{\delta^\varepsilon}{\eta - \eta^{-1}} A$$

Problem 148 Assume $F = \mathbb{C}$ and $\beta \neq \mathbb{F}2$ and

Γ is then rel \times .

Find combinatorial meaning of A^ε , \mathbb{F} , \mathbb{F}^\times , \mathbb{F}^ε

Algebraically A^ε should resemble A , A^\times

What are the eigenvalues of A^ε ?

Is A^ε an ε -ary adj matrix in sense of Def 55?

Try Γ bip as a special case.

Problem 149 Find analogs of M146 that

apply to $\beta = \mathbb{F}2$.



Lecture 29 Friday April 3

 $\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$ Given DRG $\Gamma = (k, R)$ diam ≥ 2

Next goal:

Given the intersection numbers of Γ

find an easy way to determine if

 Γ has a \mathcal{Q} -poly structure.Motivation. Assume $\{e_i\}_{i=0}^D$ is a \mathcal{Q} -poly orderingof the prim idempotents of Γ .

Applying Thm 140 to the primary module we find

the intersection number a_i satisfies

$$a_i (e_i^* - e_{i-1}^*) (e_i^* - e_{i+1}^*) = \gamma e_i^{*2} + \omega e_i^* + \gamma^*$$

$0 \leq i \leq D$

for some $\omega, \gamma^* \in \mathbb{F}$, where e_i^*, e_{i+1}^* are defined

as above Thm 140.

Next thm due to Arlene Pascasio

Thm 150 Given a nonzero prim idempotent

$$E = |X|^{-1} \sum_{i=0}^p \theta_i^* A_i$$

of P . Then E is Q -poly iff

(i) $\theta_i^* \neq \theta_0^* \quad 1 \leq i \leq p$

(ii) $\exists \beta, \gamma^* \in \mathbb{F}$ s.t

$$\gamma^* = \theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* \quad 1 \leq i \leq p-1 \quad (*)$$

(iii) $\exists \gamma, \omega, \zeta^* \in \mathbb{F}$ s.t

$$a_i (\theta_i^* - \theta_{i-1}^*) (\theta_i^* - \theta_{i+1}^*) = \gamma \theta_i^{*2} + \omega \theta_i^* + \zeta^* \quad 0 \leq i \leq p$$

where $\theta_{-1}^*, \theta_{p+1}^*$ are such that (*) holds for $i=0, i=p$

Pf We saw earlier that E Q -poly implies (i)-(iii)-

Next assume (i)-(iii)

Fix $x \in X$, write $T = T(x)$ etc.

Write $E = E_1, \quad A^* = A_1^*$

For the time being let $\{E_i\}_{i=2}^p$ be any ordering of the remaining nontrivial predegenerations of Γ . Let

$e_i := \text{kernel of } E_i$ for $0 \leq i \leq p$.

Define a graph Δ_E with vertex set

$$\{0, 1, \dots, p\}$$

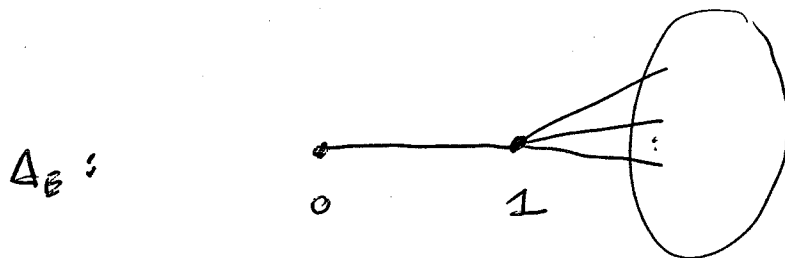
Vertices i, j are adj in Δ_E iff

$$i \neq j \text{ and } q_{ij}^1 \neq 0 \quad (0 \leq i, j \leq p)$$

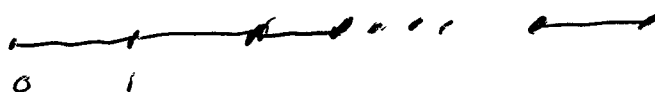
Recall

$$q_{01}^1 = \delta_{11} \quad (0 \leq i \leq p)$$

So vertex 0 is adjacent to vertex 1 and no other vertices



We will show Δ_E is a path



claim Δ_E is connected

pf let $S \subseteq \{0, 1, \dots, D\}$ be the connected

component of Δ_E that contains 0, 1.

Define

$$U = \sum_{i \in S} E_i V \quad V = \text{st. module}$$

Obs

$$E_i V \subseteq U$$

recall For $0 \leq i \leq D$

$$E_i V \circ E_j V = \sum_{\substack{0 \leq h \leq D \\ q_{ih} \neq 0}} E_h V$$

by Th 92, So

$$E_i V \circ U \subseteq U$$

But $E_i V$ generates V in the functional algebra $V, 0$ by Cor 17

So

$$U = V$$

So

$$S = \{0, 1, \dots, D\}$$

cl proved.

Applying L63 to $\{\theta_i^x\}_{i=1}^{D+1}$ $\exists \delta^x \in \mathbb{F}$ s.t.

$$\theta_{i-1}^{x2} - \beta \theta_{i-1}^x \theta_i^x + \theta_i^{x2} - \gamma^x (\theta_{i-1}^x + \theta_i^x) = \delta^x \quad 0 \leq i \leq D+1$$

Using this we check

$$(\theta_i^x - \theta_{i-1}^x) \left(\theta_i^x - \theta_{i-1}^x \right) = (2 - \beta) \theta_i^{x2} - 2\gamma^x \theta_i^x - \delta^x \quad 0 \leq i \leq D$$

claim 2 On the primary T-module W

$$\begin{aligned} A^2 A - \beta A^* A A^* + A A^{*2} - \gamma^* (A A^* + A^* A) - \delta^* A \\ = \gamma A^{*2} + \omega A^* + \zeta I. \end{aligned}$$

Pfcd (Sim to L139) let $\ell =$ LHS in above eq. On W

$$\begin{aligned} \ell &= \left(\sum_{i=0}^D E_i^* \right) \ell \left(\sum_{j=0}^D E_j^* \right) \\ &= \sum_{i=0}^D \sum_{j=0}^D E_i^* A E_j^* \left(\theta_i^{*2} - \beta \theta_i^* \theta_j^* + \theta_j^{*2} - \gamma^* (\theta_i^* + \theta_j^*) - \delta^* \right) \\ &= \sum_{i=0}^D E_i^* A E_i^* \left(\theta_i^* - \theta_{i+1}^* \right) \left(\theta_i^* - \theta_{i-1}^* \right) \\ &= \sum_{i=0}^D E_i^* a_i \left(\theta_i^* - \theta_{i+1}^* \right) \left(\theta_i^* - \theta_{i-1}^* \right) \\ &= \sum_{i=0}^D E_i^* \left(\gamma \theta_i^{*2} + \omega \theta_i^* + \zeta^* \right) \\ &= \gamma A^{*2} + \omega A^* + \zeta^* I \quad \checkmark \end{aligned}$$

claim proved.

claim 3 Given vertices i, j in Δ_E at $d(i, j) = 2$. Assume

there exists a unique vertex h in Δ_E that is adjacent

both i, j . Then

$$\gamma = \theta_i - \beta\theta_h + \theta_j$$

pf cl 3 In the equation of cl 2 multi each term on left

by E_i and right by E_j , and simplify. To help in this simplification

note

$$\begin{aligned} E_i A^{x^2} E_j &= E_i A^x \left(\sum_{r=0}^p E_r \right) A^x E_j \\ &= E_i A^x E_h A^x E_j \end{aligned}$$

and

$$\begin{aligned} E_i A^x A A^x E_j &= E_i A^x \left(\sum_{r=0}^p \theta_r E_r \right) A^x E_j \\ &= \theta_h E_i A^x E_h A^x E_j \end{aligned}$$

By above comments

$$0 \stackrel{w}{=} E_i A^x E_h A^x E_j (\theta_i - \beta\theta_h + \theta_j - \delta)$$

But

$$W = E_1 A^* E_n A^* E_2$$

so

$$r = \theta_1 - \beta \theta_n + \theta_2$$

claim proved.

We can now easily show Δ_E is a path.

Since Δ_E is connected, and since vertex 0 is adj only

to vertex 1, it suff to show each vertex in Δ_E is adj

at most 2 vertices in Δ_E . Suppose \exists vertex i in Δ_E

that is adj at least 3 vertices in Δ_E . Of all such vertices

pick i such that $d(0, i)$ is minimal. WLOG the

vertices of Δ_E are labelled st $d(0, i) = i$, and that

$0, 1, \dots, i$ is a path in Δ_E



By constr $i \geq 1$

By assumption \exists distinct vertices j, j' in $\Delta_E \setminus \{0, 1, \dots, i\}$

that are both adj i

Obs $d(i-1, j) = 2$, and i is unique vertex in Δ_E

adj both $i-1, j$. So by cl 3

$$y = \theta_{i-1} - \beta \theta_i + \theta_j.$$

Replacing j by j' in above argument

$$y = \theta_{i-1} - \beta \theta_i + \theta_{j'}$$

So $\theta_j = \theta_{j'}$ cont.

Therefore Δ_E is a path

□

Note 151 Referring to Thm 150, assume E is Q -poly
and let $\{\theta_i\}_{i=0}^D$ be the eigenvalue ordering for the corresp

Q -poly structure. These eigenvalues are found as follows:

$$\theta_0 = k \text{ the valency}$$

$\theta_1 =$ eigenvalue of E , found using

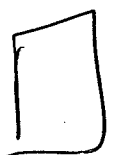
$$c_i \theta_{i+1}^x + a_i \theta_i^x + b_i \theta_{i-1}^x = \theta_i \theta_i^x \quad (0 \leq i \leq D)$$

For $1 \leq i \leq D-1$, θ_{i+1} is found recursively using

$$\theta_{i+1} - \beta \theta_i + \theta_{i-1} = \gamma$$

where β, γ is from Th 150.

Project Redo Th 150, Note 151 at level of
Leonard systems



$F = \mathbb{R} \text{ or } \mathbb{C}$ Given DRG $\Gamma = (X, R)$ dim $D \geq 2$

We now define what it means for Γ to have

classical parameters

Notation $\forall b \in \mathbb{Z}$ units

$$\begin{aligned} \begin{bmatrix} i \\ 1 \end{bmatrix} &= 1 + b + b^2 + \dots + b^{i-1} \\ &= \begin{cases} \frac{b^i - 1}{b - 1} & \text{if } b \neq 1 \\ i & \text{if } b = 1 \end{cases} \end{aligned}$$

DEF 152 Γ has classical parameters

(D, b, d, σ) whenever the intersection numbers

$$c_i = \begin{bmatrix} i \\ i \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ i \end{bmatrix} \right) \quad 0 \leq i \leq D$$

$$b_i = \left(\begin{bmatrix} 0 \\ i \end{bmatrix} - \begin{bmatrix} i \\ i \end{bmatrix} \right) \left(\sigma - \alpha \begin{bmatrix} i \\ i \end{bmatrix} \right) \quad 0 \leq i \leq D$$

Note in this case $b \neq 0, b \neq -1$

Th 153 (Brouwer, Cohen, Neumann)

Assume Γ has classical parameters (d, b, α, σ) , then

(i) $\theta = \frac{b_1}{b} - 1$ is an eigenvalue of Γ with $\theta \neq k$

(ii) Let $E = |X|^{-1} \sum_{i=0}^d \theta_i^* A_i$ be the associated primitive idempotent then

$$\theta_i^* / \theta_0^* = 1 + \left(\frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ i \end{bmatrix} b^{i-1} \quad 0 \leq i \leq d$$

(iii) E is Q -poly

Pf (i), (ii) To show that θ is an eigenvalue of Γ suffices

to check

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \quad 0 \leq i \leq d$$

$$\text{where } a_i = k - c_i - b_i \quad k = b_0$$

this check is routine.

Show $\theta \neq k$: Suppose $\theta = k$. Then

$$\frac{b_1}{b} - 1 = k$$

so

$$b > 0$$