Lecture 19 Wednesday March 4

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam D

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly ordering of the prim idempotents of Γ

Fix $x \in X$ write $T = T(x)$ etc.

Recall our goal is to prove thm 56, 57.

Notation 61

(i) Given $\beta, \gamma, \delta \in \mathbb{F}$ define a 2-variable polynomial

$$P(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

(ii) Given $\beta, \gamma^*, \delta^* \in \mathbb{F}$ define a 2-variable poly

$$P^*(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma^*(\lambda + \mu) - \delta^*$$

LEM 62 Given $\beta, \gamma, \delta \in \mathbb{F}$

$$0 = \left[A, \; A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* \right] \qquad (*)$$

iff

$$P(\theta_{i-1}, \theta_i) = 0 \quad \text{for} \quad 1 \le i \le D.$$

Pf Let C = RHS of $(*)$

$$C = (E_0 + E_1 + \cdots + E_D)\, C\, (E_0 + E_1 + \cdots + E_D)$$

$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i C E_j$$

For $0 \le i, j \le D$ use $E_i A = \theta_i E_i$ and $A E_j = \theta_j E_j$

to get

$$E_i C E_j = (\theta_i - \theta_j)\, P(\theta_i, \theta_j)\, E_i A^* E_j$$

$\Rightarrow$: For $1 \le i \le D$ show $P(\theta_{i-1}, \theta_i) = 0$.

$C = 0$ so

$$0 = E_{i-1} C E_i$$

$$= \underbrace{(\theta_{i-1} - \theta_i)}_{\neq 0}\, P(\theta_{i-1}, \theta_i)\, \underbrace{E_{i-1} A^* E_i}_{\neq 0}$$

so $\quad 0 = P(\theta_{i-1}, \theta_i)$

$\Leftarrow$: $\quad P$ is symmetric in its arguments so

$$P(\theta_i, \theta_{i-1}) = 0 \qquad 1 \le i \le D$$

To show $C = 0$ show

$$E_i C E_j = 0 \qquad 0 \le i, j \le D$$

Given i, j

If $|i-j| > 1$ then $E_i A^* E_j = 0$ so $E_i C E_j = 0$

If $|i-j| = 1$ then $P(\theta_i, \theta_j) = 0$ so $E_i C E_j = 0$

If $i = j$ then $\theta_i - \theta_j = 0$ so $E_i C E_j = 0$

In each case $E_i C E_j = 0$ so $C = 0$. $\square$

For the moment let $\{\theta_i\}_{i=0}^{D}$ be any sequence of scalars in $\mathbb{F}$. Given $\beta \in \mathbb{F}$, call this sequence

<u>β-recurrent</u> whenever

$$\theta_{i-2} - (\beta+1)\theta_{i-1} + (\beta+1)\theta_i - \theta_{i+1} = 0$$

for $2 \le i \le D-1$. Given $\beta, \gamma \in \mathbb{F}$ call $\{\theta_i\}_{i=0}^{D}$

<u>(β,γ)-recurrent</u> whenever

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma$$

for $1 \le i \le D-1$. Obs $\forall \beta \in \mathbb{F}$ TFAE

(i) $\{\theta_i\}_{i=0}^{D}$ is β-rec

(ii) $\exists \gamma \in \mathbb{F}$ s.t. $\{\theta_i\}_{i=0}^{D}$ is (β,γ)-rec

LEM 63 Given integer $D \geq 0$ and a sequence of scalars $\{\theta_i\}_{i=0}^{D}$ from $\mathbb{F}$, Given $\beta, \gamma \in \mathbb{F}$

(i) Assume $\{\theta_i\}_{i=0}^{D}$ is (β, γ)-rec. Then $\exists\, \delta \in \mathbb{F}$ s.t

$$P(\theta_{i-1}, \theta_i) = 0 \qquad 1 \leq i \leq D.$$

(ii) Assume $\exists\, \delta \in \mathbb{F}$ s.t

$$P(\theta_{i-1}, \theta_i) = 0 \qquad 1 \leq i \leq D$$

Further assume $\theta_{i-1} \neq \theta_{i+1}$ for $1 \leq i \leq D-1$, then $\{\theta_i\}_{i=0}^{D}$ is (β, γ)-rec.

Pf Define

$$p_i = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \qquad 1 \leq i \leq D$$

and observe

$$p_i - p_{i+1} = (\theta_{i-1} - \theta_{i+1})(\theta_{i-1} - \beta\theta_i + \theta_{i+1} - \gamma)$$

for $1 \leq i \leq D-1$. Result follows. $\square$

Proof of th 57 First assume $D \geq 3$

By LEM 60 (with $R = A^2$, $S = A$) $\exists$ $Z \in M$ s.t

$$A^2 A^* A - A A^* A^2 = Z A^* - A^* Z \qquad (*)$$

Recall $\{A^i\}_{i=0}^{D}$ is a basis for M so $\exists$ $p \in \mathbb{F}[\lambda]$

with degree $\leq D$ s.t $Z = p(A)$.

let d = degree of p. We show $d = 3$

First suppose $d > 3$, Multiply each term in $(*)$ on left by

E_d^* and on right by E_0^*. Evaluate using L58 to get

$$\underset{\overset{\#}{0}}{0} = \underset{\overset{\#}{0}}{c}\,(\theta_0^* - \theta_d^*)\ \underset{\substack{\#\\ 0\\ \text{by L58}}}{E_d^* A^d E_0^*} \qquad c = \text{leading coeff of } p$$

this is contradiction.

Next suppose $d < 3$. Multiply each term in $(*)$ on the

left by E_3^* and on the right by E_0^*. Evaluate using L58

to get

$$\underset{\overset{\#}{0}}{(\theta_1^* - \theta_2^*)}\ \underset{\substack{\#\\ 0\ \text{by L5.8}}}{E_3^* A^3 E_0^*} = 0$$

cont.

We have shown $d=3$, Abbrev $\beta = c^{-1}-1$

Now divide both sides of $(*)$ by c to find $\exists\ \gamma, \delta \in \mathbb{F}$ s.t

$$(\beta+1)(A^2A^*A - AA^*A^2) = A^3A^* - A^*A^3$$

$$-\gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

Rearranging terms we get TD1. To get TD2 pick i $(2 \le i \le D-1)$

Multiply each term in TD1 on left by E^*_{i-2} and on right

by E^*_{i+1} Simplify using L59 to get

$$0 = \underbrace{E^*_{i-2} A^3 E^*_{i+1}}_{\neq 0} \Big(\underbrace{\theta^*_{i-2} - (\beta+1)\theta^*_{i-1} + (\beta+1)\theta^*_i}_{\text{must be } 0} - \theta^*_{i+1} \Big)$$

So $\{\theta^*_i\}_{i=0}^{D}$ is β-rec.

So $\exists\ \gamma^* \in \mathbb{F}$ s.t. $\{\theta^*_i\}_{i=0}^{D}$ is (β, γ^*)-rec

So by L63 $\exists\ \delta^* \in \mathbb{F}$ s.t

$$p^*(\theta^*_{i-1}, \theta^*_i) = 0 \qquad 1 \le i \le D$$

Now $\beta, \gamma^*, \delta^*$ sat TD2 by L62*

We are done for $D \ge 3$

Now assume $D < 3$

Let $\beta \in \mathbb{F}$ (arbitrary)

If $D = 2$ define

$$\gamma = \theta_0 - \beta\theta_1 + \theta_2$$

and if $D \le 1$ let $\gamma \in \mathbb{F}$ (arb)

By construction $\{\theta_i\}_{i=0}^{D}$ is (β, γ)-rec

So by L63 (i) $\exists\ \delta \in \mathbb{F}$ s.t

$$P(\theta_{i-1}, \theta_i) = 0 \quad \text{for } 1 \le i \le D.$$

Now β, γ, δ satisfy TD1 by L62.

Interchanging A, A^* in above argument, $\exists\ \gamma^*, \delta^* \in \mathbb{F}$ s.t

$\beta, \gamma^*, \delta^*$ sat TD2 $\square$

Prop 64 Given $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ that satisfy

TD1, TD2

(i) the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are both equal to $\beta + 1$ for $2 \le i \le D-1$

(ii) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}$ $(1 \le i \le D-1)$

(iii) $\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ $(1 \le i \le D-1)$

(iv) $\delta = \theta^2_{i-1} - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i)$ $(1 \le i \le D)$

(v) $\delta^* = \theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1} + \theta^*_i)$ $(1 \le i \le D)$

Pf (iv) From TD1 and L62

(v) From TD2 and L62*

(ii) By L63 and (iv) above

(iii) Sim to (ii)

(i) $\{\theta_i\}_{i=0}^D$ is (β,γ)-rec by (ii) so $\{\theta_i\}_{i=0}^D$ is β-rec
Sim $\{\theta^*_i\}_{i=0}^D$ is β-rec. Result follows. □

COR 65 the scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ in Th 57 are unique provided $p \geq 3$.

Pf By Prop 64 $\square$

<u>Pf of Th 56</u>: Immed from Prop 64 (i) $\square$

Ex 66 take $\Gamma = H(D,N)$

Recall Γ has a Q-poly structure such that

$$\theta_i = \theta_i^* = (N-1)(D-i) - i \qquad (0 \le i \le D)$$

One checks that for this structure the parameters

$\beta, \gamma, \gamma^*, \delta, \delta^*$ from thm 57 are

$$\beta = 2, \qquad \gamma = 0, \qquad \gamma^* = 0$$

$$\delta = N^2, \qquad \delta^* = N^2.$$

TD1, TD2 become

$$[A,[A,[A,A^*]]] = N^2[A,A^*],$$

$$[A^*,[A^*,[A^*,A]]] = N^2[A^*,A]$$

these equations are called the Dolan-Grady relations.

Note 67 The Onsager algebra $\mathcal{O}$ is the Lie algebra over $\mathbb{F}$ defined by generators Y, Z and relations

$$[Y, [Y, [Y, Z]]] = 4[Y, Z],$$

$$[Z, [Z, [Z, Y]]] = 4[Z, Y]$$

where $[\ ,\ ]$ is Lie bracket. It turns out

$\mathcal{O}$ is ∞ dim'l,

$\mathcal{O}$ is used in the statistical mechanics of the Ising model.

By Ex 66 we see that for $H(D,N)$ the standard module is an $\mathcal{O}$-module on which Y, Z act as

$\frac{2A}{N}$, $\frac{2A^*}{N}$ resp.

Note 68 For $q \in \mathbb{F}$ $q \neq 0, 1, -1$

define the "q integer"

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2..$$

the (cubic) q-Serre relations in the variables Y, Z are

$$X^3 Y - [3]_q X^2 Y X + [3]_q X Y X^2 - Y X^3 = 0,$$

$$Y^3 X - [3]_q Y^2 X Y + [3]_q Y X Y^2 - X Y^3 = 0.$$

These are among the defining relations for the algebra $U_q(\widehat{sl_2})$. This is the quantum group for the cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The q-Serre relations are the same thing as the TD1, TD2 relations with $\beta = q^2 + q^{-2}$, $\gamma = 0$, $\gamma^* = 0$, $\delta = 0$, $\delta^* = 0$.

$\square$

Lec 20 Friday March 6

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam D

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$ and write $T = T(x)$ etc.

We now solve the equations in Thm 56 to get the eigenvalues and dual eigenvalues of Γ in closed form.

LEM 69 Given a finite sequence $\{\theta_i\}_{i=0}^{D}$ of scalars in $\mathbb{C}$, and given $\beta \in \mathbb{C}$. Then $\{\theta_i\}_{i=0}^{D}$ is β-rec iff $\exists$ $a, b, c \in \mathbb{C}$ such that

Case $\beta \neq \mp 2$ $\qquad \theta_i = a + bq^i + cq^{-i} \qquad (0 \leq i \leq D)$

where $q + q^{-1} = \beta$

Case $\beta = 2$ $\qquad \theta_i = a + bi + ci^2 \qquad (0 \leq i \leq D)$

Case $\beta = -2$ $\qquad \theta_i = a + b(-1)^i + ci(-1)^i \qquad (0 \leq i \leq D)$

Pf (For case $\beta \neq \mp 2$) Assume $D \geq 3$ else trivial.

Let L denote the set of all vectors $(\sigma_0, \sigma_1, \ldots, \sigma_D)$ in $\mathbb{C}^{D+1}$

that are β-rec, ie

$$\sigma_{i-2} - (\beta+1)\sigma_{i-1} + (\beta+1)\sigma_i - \sigma_{i+1} = 0 \qquad (2 \leq i \leq D-1) \qquad (*)$$

Obs L is a subspace of $\mathbb{C}^{D+1}$.

In $(*)$ $\sigma_0, \sigma_1, \sigma_2$ are free and $\sigma_3, \ldots, \sigma_D$ are det by $\sigma_0, \sigma_1, \sigma_2$

so $\dim L = 3$

Pick $0 \neq q \in \mathbb{C}$ s.t

$$\beta = q + q^{-1}$$

Obs $q \neq 1$, $q \neq -1$

One checks the three vectors

$$(1, 1, 1, \ldots, 1), \qquad (1, q, q^2, \ldots, q^D), \qquad (1, q^{-1}, q^{-2}, \ldots, q^{-D})$$

are in L and lin indep. So they form a

basis for L. Result follows. $\square$

No. Lec 20-3
Date 3/6/09

Note 70 Ref to L69, for $\beta \neq \pm 2$ sometimes we replace q by q^2 and adjust b, c to write

$$\theta_i = a + b q^{2i-D} + c q^{D-2i} \qquad (0 \le i \le D)$$

COR 71 Referring to Γ, assume $D \ge 3$ to avoid trivialities. Let $\beta \in \mathbb{F}$ be from Th 57. Then the eigenvalues $\{\theta_i\}_{i=0}^{D}$ of Γ and dual eigenvalues $\{\theta_i^*\}_{i=0}^{D}$ of Γ satisfy one of the following forms

Case I: $\beta \neq \mp 2$

$$\theta_i = a + b q^{2i-D} + c q^{D-2i}$$
$$\theta_i^* = a^* + b^* q^{2i-D} + c^* q^{D-2i}$$
$$0 \le i \le D$$

$$\beta = q^2 + q^{-2}$$

Case II: $\beta = 2$

$$\theta_i = a + bi + ci^2$$
$$\theta_i^* = a^* + b^* i + c^* i^2$$
$$0 \le i \le D$$

Case III $\beta = -2$

$$\theta_i = a + b(-1)^i + c i (-1)^i$$
$$\theta_i^* = a^* + b^*(-1)^i + c^* i (-1)^i$$
$$0 \le i \le D$$

No. Lec 20-4
Date 3/6/09

Caution Ref to Cor 71, possibly some of

$q, a, b, c, a^*, b^*, c^*$ are in $\mathbb{C} \setminus \mathbb{R}$

even though the θ_i, θ_i^* are all in $\mathbb{R}$

Note 72 Ref to Cor 71, for Case I

The parameters $\beta, \gamma, \gamma^*, \delta, \delta^*$ from Th 57 are:

$$\beta = q^2 + q^{-2}$$

$$\gamma = -a(q-q^{-1})^2$$

$$\gamma^* = -a^*(q-q^{-1})^2$$

$$\delta = -bc(q^2-q^{-2})^2 + a^2(q-q^{-1})^2$$

$$\delta^* = -b^*c^*(q^2-q^{-2})^2 + a^{*2}(q-q^{-1})^2$$

This is checked using Prop 64. Similar equations hold for cases II, III.

Note 73 Earlier we found a φ-poly structure for

$\Gamma = H(D,N)$. It was Case II with

$$a = a^* = (N-1)D$$

$$b = b^* = -N$$

$$c = c^* = 0$$

Obs our DRG Γ is bipartite iff $a_i = 0$ for $0 \le i \le D$

Call our Q-poly structure dual bipartite whenever

$a_i^* = 0$ for $0 \le i \le D$.

Th 74 Let the scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ be as in Th 57. Assume $D \ge 2$.

(i) Assume Γ is bipartite. Then

$$0 = A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^* (AA^* + A^* A) - \delta^* A \quad (*)$$

and $\gamma = 0$

(ii) Assume the Q-poly str is dual bipartite. Then

$$0 = A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (AA^* + A^* A) - \delta A^*$$

and $\gamma^* = 0$

pf (i) Let F = RHS of $(*)$ Show $F = 0$ Obs

$$F = \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* F E_j^*$$

For $0 \le i, j \le D$ show $E_i^* F E_j^* = 0$. Obs

$$E_i^* F E_j^* = P^*(\theta_i^*, \theta_j^*) \; E_i^* A E_j^*$$

where P^* is from Not 61

Case $|i-j| > 1$: $\quad E_i^* A E_j^* = 0$

Case $|i-j|=1$: $\rho^*(\theta_i^*, \theta_j^*)=0$ since the dual eigenvalues are β-rec

Case $i=j$: $E_i^* A E_i^* = 0$ since $a_i = 0$

In all cases

$$E_i^* F E_j^* = 0$$

so $F=0$.

Show $\gamma = 0$

Recall back in LEM 8 of Ch 1. We found

$H \in Mat_X(\mathbb{F})$ s.t

$$HA = -AH.$$

Replacing H by $-H$ if nec

$$H = \sum_{i=0}^{D} (-1)^i E_i^*$$

Obs $HA^* = A^* H$ and $H^2 = I$. By th 57 we have TD1:

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*]$$

Conjugate this by H to get

$$0 = \left[-A,\ (-A)^2A^* - \beta(-A)A^*(-A) + A^*(-A)^2 - \gamma(-AA^* - A^*A) - \delta A^*\right]$$

So

$$0 = \left[A,\ A^2A^* - \beta AA^*A + A^*A^2 + \gamma(AA^* + A^*A) - \delta A^*\right]$$

subtracting this from TD1.

$$0 = \gamma\left[A,\ AA^* + A^*A\right]$$

$$= \gamma\left[A^2, A^*\right]$$

But $\quad [A^2, A^*] \neq 0 \quad$ since

$$E_0^*[A^2, A^*]E_2^* = \underbrace{E_0^*A^2E_2^*}_{\neq 0}\ \underbrace{(\theta_2^* - \theta_0^*)}_{\neq 0}$$

So $\gamma = 0$

(ii) Sim. $\square$

Ex 75 Take $\Gamma = H(D,2)$ hypercube take $\mathbb{F}=\mathbb{C}$

Γ is bipartite. The Q-poly structure we found earlier

satisfies $q^h_{ij} = p^h_{ij}$ for $0 \le h,i,j \le D$

So this structure is dual bipartite.

Recall

$$\theta_i = \theta_i^* = D-2i \qquad (0 \le i \le D)$$

Here $\beta = 2$

$$\gamma = 0, \quad \gamma^* = 0 \qquad \delta = 4, \quad \delta^* = 4$$

So by Th 74

$$4A = A^{*2}A - 2A^*AA^* + AA^{*2}$$

$$= [A^*, [A^*, A]] \qquad (*)$$

$$4A^* = A^2A^* - 2AA^*A + A^*A^2$$

$$= [A, [A, A^*]] \qquad (**)$$

For notational convenience define A^ε by

$$[A, A^*] = 2iA^\varepsilon \qquad (i^2 = -1)$$

Then $(*)$, $(**)$ become

$$[A^*, A^\varepsilon] = 2iA$$

$$[A^\varepsilon, A] = 2iA^*$$

We now recognize the Lie algebra $sl_2(\mathbb{C})$

Recall $sl_2(\mathbb{C})$ is the Lie algebra consisting of all 2×2 matrices over $\mathbb{C}$ with trace 0, the Lie bracket is

$$[x,y] = xy - yx \qquad \forall x,y \in sl_2(\mathbb{C})$$

$sl_2(\mathbb{C})$ has a basis

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad a^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad a^\varepsilon = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and

$$[a, a^*] = 2ia^\varepsilon, \qquad [a^*, a^\varepsilon] = 2ia, \qquad [a^\varepsilon, a] = 2ia^*$$

Thus for $H(D,2)$ the standard module becomes an $sl_2(\mathbb{C})$-module s.t. a, a^*, a^ε act as A, A^*, A^ε resp.

Lecture 21 Monday March 9

Given DRG $\Gamma = (X, R)$ diam $D \geq 1$

Assume ordering $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$ write $T = T(x)$ etc.

We mention a handy formula.

LEM 76 With above notation $0 \leq i \leq D$,

(i) $c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta_1 \theta_i^*$

where θ_{-1}^*, θ_{D+1}^* are indets

(ii) $c_i^* \theta_{i-1} + a_i^* \theta_i + b_i^* \theta_{i+1} = \theta_1^* \theta_i$

where θ_{-1}, θ_{D+1} are indets.

Pf (i) By AW duality

$$u_i(\theta_1) = u_1^*(\theta_i^*)$$
$$= \frac{\theta_i^*}{\theta_0^*}$$

Result follows from this and the 3-term rec for the u_i

(ii) Sim $\square$

Setting $i=0$ in LEM 76(i) and using $k=\theta_0$

$$\frac{\theta_1}{\theta_0} = \frac{\theta_1^*}{\theta_0^*}$$

LEM 77 Assume Γ is Bipartite

(i) $$c_i = \frac{\theta_0}{\theta_0^*} \frac{\theta_1^* \theta_i^* - \theta_0^* \theta_{i-1}^*}{\theta_{i-1}^* - \theta_{i+1}^*} \qquad 1 \le i \le D-1$$

(ii) $c_D = \theta_0$

(iii) $b_i = \theta_0 - c_i \qquad (0 \le i \le D)$

(iv) $$\frac{\theta_{D-1}^*}{\theta_D^*} = \frac{\theta_1^*}{\theta_0^*}$$

Pf (i) In L76(i) Eval using $a_i = 0$,

$b_i = k - c_i$, and solve for c_i

(ii) $c_D = k - b_D = k = \theta_0$

(iii) $k = \theta_0$

(iv) Set $i=D$ in L76(i) $\square$

No. Lec 21-3
Date 3/9/09

LEM 78 Assume $\{E_i\}_{i=0}^{D}$ is dual bipartite.

(i) $c_i^* = \frac{\theta_0^*}{\theta_0} \frac{\theta_1 \theta_i - \theta_0 \theta_{i+1}}{\theta_{i-1} - \theta_{i+1}}$ $\quad 1 \le i \le D-1$

(ii) $c_D^* = \theta_0^*$

(iii) $b_i^* = \theta_0^* - c_i^*$ $\quad 0 \le i \le D$

(iv) $\frac{\theta_{D-1}}{\theta_0} = \frac{\theta_1}{\theta_0}$

Pf Sim to L77 $\square$

LEM 79 Assume Γ is bip and $\{E_i\}_{i=0}^{D}$ is dual bip.

Further assume $\beta = 2$. Then

(i) $\theta_i = \theta_i^* = D - 2i \qquad (0 \le i \le D)$

(ii) $c_i = c_i^* = i \qquad (0 \le i \le D)$

"It looks like $H(D, 2)$"

Pf Assume $D \ge 2$ else trivial.

We are in Case II: the θ_i, θ_i^* have form

$$\theta_i = a + bi + ci^2$$

$$\theta_i^* = a^* + b^* i + c^* i^2 \qquad 0 \le i \le D.$$

By Th 74 (i)

$$0 = \gamma$$

$$= \theta_0 - \underset{\substack{\shortparallel \\ 2}}{\beta}\, \theta_1 + \theta_2$$

$$= a - 2(a + b + c) + a + 2b + 4c$$

$$= 2c$$

So $c = 0$

Sim $c^* = 0$

the constraint

$$\frac{\theta_{D-1}}{\theta_0} = \frac{\theta_1}{\theta_0}$$

from L78 (iv) gives

$$b/a = -2/D$$

Similarly using L77 (iv)

$$b^*/a^* = -2/D$$

So far

$$\frac{\theta_i}{\theta_0} = 1 - \frac{2i}{D} \qquad 0 \le i \le D$$

$$\frac{\theta_i^*}{\theta_0^*} = 1 - \frac{2i}{D} \qquad 0 \le i \le D$$

For $1 \le i \le D-1$ solve for c_i using L77 (i) to get

$$c_i = \frac{i \ \theta_0}{D}$$

But $c_1 = 1$ so

$$\theta_0 = D$$

hence

$$\theta_i = D - 2i \qquad (0 \le i \le D)$$

Also

$$c_0 = \theta_0 = 0$$

$$c_i = i \qquad 1 \le i \le D-1$$

So

$$c_i = i \qquad (0 \le i \le D)$$

Similarly

$$\theta_i^* = D - 2i \qquad (0 \le i \le D)$$

$$c_i^* = i \qquad (0 \le i \le D) \qquad \square$$

Fin Ex TFAE

(i) Γ is bipartite and

$$c_i = i \qquad (0 \le i \le D)$$

(ii) Γ is $H(D,2)$

Pf hint (i)→(ii) Consider set of vectors

$$\left\{ E_1\hat{y} - E_1\hat{z} \;\middle|\; y,z \in X,\ yz \in R \right\}$$

Show For all u, v in this set either

$$u = \mp v \quad \text{or} \quad \langle u, v \rangle = 0$$

Use this to show $\Gamma = K_2 \times K_2 \times \cdots \times K_2$. $\square$

LEM 80 Assume Γ is bip and $\{E_i\}_{i=0}^{D}$ is dual bip. Further assume $\beta \neq \mp 2$

then $\exists\, 0 \neq q \in \mathbb{C}$ $(q^2 \neq 1,\ q^2 \neq -1)$ s.t

(i) $\theta_i = \theta_i^* =$

$$\left(q^{D-2} + q^{2-D}\right) \frac{q^{D-2i} - q^{2i-D}}{q^2 - q^{-2}} \qquad (0 \leq i \leq D)$$

(ii) $c_i = c_i^* =$

$$\frac{q^{D-2} + q^{2-D}}{q^{D-2i} + q^{2i-D}} \; \frac{q^{2i} - q^{-2i}}{q^2 - q^{-2}} \qquad (0 \leq i \leq D)$$

Pf Sim to pf of L79, except use Case I forms for θ_i, θ_i^* $\square$

LEM 81 Assume Γ is bip and $\{E_i\}_{i=0}^{D}$ is dual bip.

Further assume $\beta = -2$. Then D is even and

(i) $\theta_i = \theta_i^* = (-1)^i (D-2i)$ $\quad (0 \le i \le D)$

(ii) $c_i = c_i^* = i$ $\quad (0 \le i \le D)$

Pf Sim to pf of L79 except use Case III form for θ_i, θ_i^* $\square$

No. Lec 21-10
Date 3/9/09

Note Unique sol to LEM 8 I is $H(D,2)$ (D even)

with Q-poly str assoc with ordering

$D,\ 2-D,\ D-4,\ 6-D,\ \ldots$

of the eigenvalues.

(Fun ex to show this really is a Q-poly str.)

With ref to L80

$$\beta = q^2 + q^{-2}$$

$$\gamma = \gamma^* = 0$$

$$\delta = \delta^* = (q^{D-2} + q^{2-D})^2$$

By L74

$$A^2A^* - (q^2+q^{-2})AA^*A + A^*A^2 = (q^{D-2}+q^{2-D})^2 A^*$$

$$A^{*2}A - (q^2+q^{-2})A^*AA^* + AA^{*2} = (q^{D-2}+q^{2-D})^2 A$$

In cyclic form this looks as follows.

LEM 82 With ref to L80 $\exists\ A^\varepsilon \in T$ s.t.

$$qAA^* - q^{-1}A^*A = zA^\varepsilon$$

$$qA^*A^\varepsilon - q^{-1}A^\varepsilon A^* = zA$$

$$qA^\varepsilon A - q^{-1}AA^\varepsilon = zA^*$$

where

$$z = i(q^{D-2} + q^{2-D}),$$

$$i^2 = -1$$

pf Routine - just def A^ε using 1st equation. $\square$

Lecture 22 Wednesday March 11

$\mathbb{F} = \mathbb{C}$ Given bipartite DRG $\Gamma = (X, R)$

diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is dual bip Q-poly

Further assume $\beta \neq \mp 2$, let $q \in \mathbb{C}$ be as in L80

Fix $x \in X$ write $T = T(x)$ etc.

Last time we found $A^{\varepsilon} \in T$ s.t

$$q A A^{*} - q^{-1} A^{*} A = z A^{\varepsilon},$$

$$q A^{*} A^{\varepsilon} - q^{-1} A^{\varepsilon} A^{*} = z A,$$

$$q A^{\varepsilon} A - q^{-1} A A^{\varepsilon} = z A^{*}.$$

where $z = i(q^{D-2} + q^{2-D})$ $\quad i^2 = -1$

Next goal: show A^{ε} is an imaginary adj matrix

Search for $W \in M$ and $W^{*} \in M^{*}$

s.t

$$W A^{*} W^{-1} = W^{*-1} A W^{*} = A^{\varepsilon}$$

(just like we did for K_N)

Find W: For moment assume W exists. Write

$$W = \sum_{i=0}^{D} \alpha_i E_i \qquad \alpha_i \in \mathbb{C}$$

W^{-1} exists so $\alpha_i \neq 0$ $(0 \le i \le D)$ and

$$W^{-1} = \sum_{i=0}^{D} \alpha_i^{-1} E_i$$

Require

$$A^{\varepsilon} = W A^{*} W^{-1}$$

For $0 \le i, j \le D$

$$E_i A^{\varepsilon} E_j = E_i \left(W A^{*} W^{-1} \right) E_j$$

$$= E_i A^{*} E_j \quad \alpha_i / \alpha_j$$

Also

$$E_i A^{\varepsilon} E_j = E_i \left(\frac{q A A^{*} - q^{-1} A^{*} A}{z} \right) E_j$$

$$= E_i A^{*} E_j \quad \frac{q \theta_i - q^{-1} \theta_j}{z}$$

Since we assume our Q-poly str is dual bip

$$E_i A^{*} E_j = 0 \quad \text{if} \quad |i-j| \neq 1$$

Require

$$\alpha_i / \alpha_j = \frac{q\theta_i - q^{-1}\theta_j}{z} \quad \text{if} \quad |i-j| = 1 \qquad (0 \le i, j \le D)$$

Recall for $1 \le i \le D$

$$\underset{\substack{\| \\ -z^2}}{\delta} = \theta_{i-1}^2 - \underset{\substack{\| \\ q^2+q^{-2}}}{\beta}\,\theta_{i-1}\theta_i + \theta_i^2 - \underset{\substack{\| \\ 0}}{\gamma}(\theta_{i-1} + \theta_i)$$

$$-z^2 = (q\theta_{i-1} - q^{-1}\theta_i)(q^{-1}\theta_{i-1} - q\theta_i)$$

$$1 = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \; \frac{q\theta_{i-1} - q^{-1}\theta_i}{z}$$

So for $0 \le i, j \le D$

$$1 = \frac{q\theta_i - q^{-1}\theta_j}{z} \; \frac{q\theta_j - q^{-1}\theta_i}{z} \quad \text{if} \quad |i-j| = 1$$

Only requirement on $\{\alpha_i\}_{i=0}^{D}$ is

$$\frac{\alpha_i}{\alpha_{i-1}} = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \qquad 1 \le i \le D$$

By L80

$$\frac{q\theta_i - q^{-1}\theta_{i-1}}{z} = \mathbf{i}\, q^{2i-1-D} \qquad 1 \le i \le D$$

$$(\mathbf{i}^2 = -1)$$

Def 83 With above notation pick $0 \neq \alpha_0 \in \mathbb{C}$

and def $\{\alpha_i\}_{i=1}^{D}$ by

$$\frac{\alpha_i}{\alpha_{i-1}} = i^D q^{2i-1-D} \qquad (1 \le i \le D)$$

Put

$$W = \sum_{i=0}^{D} \alpha_i E_i$$

$$W^* = \sum_{i=0}^{D} \alpha_i E_i^*$$

Prop 84 With above not

(i) $W A^* W^{-1} = A^{\varepsilon}$

(ii) $W^{*-1} A W^* = A^{\varepsilon}$

pf (i) $A^{\varepsilon} = (E_0 + E_1 + \cdots + E_D) A^{\varepsilon} (E_0 + E_1 + \cdots + E_D)$

$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i A^{\varepsilon} E_j$$

Also

$$W A^* W^{-1} = \sum_{i=0}^{D} \sum_{j=0}^{D} E_i (W A^* W^{-1}) E_j$$

$$= \sum_{i=0}^{D} \sum_{j=0}^{D} \alpha_i / \alpha_j \, E_i A^* E_j$$

Fn $0 \le i, j \le D$

$$E_i A^{\varepsilon} E_j - \alpha_i/\alpha_j \, E_i A^* E_j$$

$$= \underbrace{E_i A^* E_j}_{0 \text{ if } |i-j| \neq 1} \Bigg(\underbrace{\frac{q\theta_i - q^{-1}\theta_j}{z} - \frac{\alpha_i}{\alpha_j}}_{0 \text{ if } |i-j| = 1} \Bigg)$$

$= 0$ ✓ pf (ii) very sim □

COR 85 With above not

A^{ε} is similar to A, A^{*}

In particular A^{ε} is diagonalizable with dist eigenvalues

$$\theta_i^{\varepsilon} = \theta_i = \theta_i^{*} \qquad (0 \le i \le D)$$

Prop 86 With above not

$$\begin{tikzcd} A \arrow[rr] & & A^* \arrow[dl] \\ & A^\varepsilon \arrow[ul] & \end{tikzcd}$$

ρ

where

$$\rho: \begin{array}{ccc} T & \longrightarrow & T \\ m & \longrightarrow & (ww^*)m(ww^*)^{-1} \end{array}$$

pf (just like LEM 51)

check $A \to A^*$:

$$ww^*A \overset{?}{=} A^*ww^*$$

By cnstr

$$wA^*w^{-1} = w^{*-1}Aw^*$$

so

$$w^*wA^* = Aw^*w$$

take trans

$$ww^*A = A^*ww^* \checkmark$$

check $A^* \to A^\varepsilon$:

$$WW^* A^* \stackrel{?}{=} A^\varepsilon WW^*$$

$$A^\varepsilon WW^* = WA^*W^{-1}WW^*$$

$$= WA^*W^*$$

$$= WW^*A^* \checkmark$$

check $A^\varepsilon \to A$

$$WW^*A^\varepsilon \stackrel{?}{=} AWW^*$$

$$WW^*A^\varepsilon = WW^*(W^{*-1}AW^*)$$

$$= WAW^*$$

$$= AWW^* \checkmark$$

Thm 87 With above not,

A^ε is an imaginary adj matrix for Γ

in sense of Def 55.

Pf. By Prop 86 any two of A, A^*, A^ε related the same way as A, A^*. $\square$

Problem. 88 In our study of K_N we obtained many results concerning $A, A^*, A^\varepsilon, W, W^*$.

Try to find similar results for the Bip/dual bip case.

[is W a q-exponential in A? See if W is an exponential in A, for $H(D,2)$. Repeat for Case III]

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ $\quad \Gamma = (X, R)$ any DRG diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$, write $T = T(x)$ etc.

Fix an irred T-module W.

Next goal is to carefully study W.

$r =$ endpt of W $\qquad$ $t =$ dual endpt of W

$d =$ diam of W $\qquad$ $d^* =$ dual diam of W.

LEM 89 With above not

(i) $E_i^* A E_j^* W \neq 0$ if $|i-j| = 1$ $\quad (r \leq i, j \leq r+d)$

(ii) Suppose W is thin. Then

$$E_r^* W + E_{r+1}^* W + \cdots + E_{r+i}^* W = E_r^* W + A E_r^* W + \cdots + A^i E_r^* W$$

$$(0 \leq i \leq d)$$

(iii) Suppose W is thin. Then $W = M E_r^* W$

(iv) Suppose W is thin. Then

$$E_j E_r^* W = E_j W \qquad (0 \leq j \leq D)$$

Moreover W is dual thin.

Pf (i) Suppose $\exists$ i, j $(r \le i, j \le r+d)$ s.t

$$E_i^* A E_j^* W = 0 \quad \text{and} \quad |i-j| = 1$$

If $i-j=1$ then

$$E_r^* W + \cdots + E_j^* W$$

is a non0 proper subspace of W that is inv under A, A^*, cont the irred of W.

If $j-i=1$ then

$$E_j^* W + \cdots + E_{r+d}^* W$$

is a non0 proper subspace of W that is inv under A, A^*, cont irred of W.

(ii) By (i) and since

$$E_i^* A E_j^* = 0 \quad \text{if } |i-j| > 1 \qquad (0 \le i, j \le D)$$

(iii) $\supseteq$: Since W is M-inv

$\subseteq$: Set $i=d$ in (ii) and use

$$W = \sum_{i=0}^{d} E_{r+i}^* W$$

(iv) For $0 \le j \le D$

$$E_j W = E_j M E_r^* W$$

$$= E_j E_r^* W \qquad E_j M = \mathrm{Span}(E_j)$$

Now $\dim E_j W \le \dim E_r^* W$

$$= 1$$

$\square$

L 90 With above not

(i) $E_i A^* E_j W \neq 0$ if $|i-j|=1$ $(t \le i, j \le t+d^*)$

(ii) Suppose W is dual thin. Then
$$E_t W + \cdots + E_{t+i} W = E_t W + A^* E_t W + \cdots + A^{*i} E_t W \quad (0 \le i \le d^*)$$

(iii) Suppose W is dual thin then $W = M^* E_t W$

(iv) Suppose W is dual thin then

$$E_j^* E_t W = E_j^* W \qquad 0 \le j \le D$$

Moreover W is thin.

$\square$

Lecture 23 Fri March 13

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$, write $T = T(x)$ etc.

Goal: study str of single irred T-module

LEM 91 the following hold for $0 \leq h \leq D$

(i) For $0 \neq w \in E_h^* V$

$$\left| \{ i \mid 0 \leq i \leq D,\ E_i w = 0 \} \right| \leq 2h$$

(ii) For $0 \neq w \in E_h V$

$$\left| \{ i \mid 0 \leq i \leq D,\ E_i^* w = 0 \} \right| \leq 2h$$

Pf (i) Suppose not. Then $\exists$ subset

$$\Omega \subseteq \{0, 1, \ldots, D\} \qquad |\Omega| = 2h+1$$

s.t.

$$E_i w = 0 \qquad \forall i \in \Omega$$

By constr $w = E_h^* w$

So $\forall i \in \Omega$

$$0 = E_h^* E_i E_h^* w$$

$$= E_h^* \left(|X|^{-1} m_i \sum_{\ell=0}^{D} u_\ell(\theta_i) A_\ell \right) E_h^* w$$

$$E_h^* A_\ell E_h^* = 0 \text{ for } \ell > 2h$$

so

$$0 = \sum_{\ell=0}^{2h} u_\ell(\theta_i) \quad E_h^* A_\ell E_h^* w$$

letting i range over Ω get system of $2h+1$ homogeneous linear equations in the unknowns

$$E_h^* A_\ell E_h^* w \qquad 0 \le \ell \le 2h$$

The coefficient matrix is essentially Vandermonde since poly u_ℓ has degree ℓ and the θ_i are distinct.

So the coeff matrix is nonsingular forcing

$$E_h^* A_\ell E_h^* w = 0 \qquad 0 \le \ell \le 2h$$

This is imposs since $E_h^* A_0 E_h^* w = w \neq 0$ $\square$

(ii) Sim

COR 92 Let W denote an irred T-module with endpt r, dual endpt t, diam d, dual diam d^*. Then

(i) $2r + d^* \geq D$

(ii) $2t + d \geq D$

pf (i) Fix $0 \neq w \in E_r^* W$. By L91(i)

$$2r \geq \left| \{ i \mid 0 \leq i \leq D, \ E_i w = 0 \} \right|$$

$$\geq \left| \{ i \mid 0 \leq i \leq D, \ E_i W = 0 \} \right|$$

$$= D - d^*$$

by def of d^*

(ii) Sim $\square$

Until further notice let W denote

a <u>thin</u> irred T-module with endpt r, dual endpt t,

diam d. : Obs dual diam $d^* = d$

LEM 93 With above notation

(i) For all $0 \neq \eta \in E_t W$ the vector

$E_i^* \eta$ is a basis for $E_i^* W$ for $r \leq i \leq r+d$.

Moreover

$$E_r^* \eta, \; E_{r+1}^* \eta, \; \ldots, \; E_{r+d}^* \eta$$

is a basis for W

(ii) For all $0 \neq \eta^* \in E_r^* W$ the vector

$E_i \eta^*$ is basis for $E_i W$ for $t \leq i \leq t+d$

Moreover

$$E_t \eta^*, \; E_{t+1} \eta^*, \; \ldots, \; E_{t+d} \eta^*$$

is a basis for W.

pf (i) By constr, for $0 \le i \le D$

$E_i^* W$ has dim 1 if $r \le i \le r+d$, and $E_i^* W = 0$

otherwise.

So for $r \le i \le r+d$ suf to show $E_i^* z \neq 0$.

By L90 (iv)

$$E_i^* W = E_i^* E_t W$$

$E_t W = \mathrm{Span}(z)$

$$= \mathrm{Span}\, E_i^* z$$

Result follows.

(ii) Sim. $\square$

DEF 94 With above notation

(i) By a <u>standard basis</u> for W we mean a sequence

$$E^*_r z,\ E^*_{r+1} z, \ldots, E^*_{r+d} z$$

where $0 \neq z \in E_t W$

(ii) By a <u>dual standard basis</u> for W we mean a sequence

$$E_t z^*,\ E_{t+1} z^*, \ldots, E_{t+d} z^*$$

where $0 \neq z^* \in E^*_r W$.

We mention how to recognize a standard basis

LEM 95 Let $\{w_i\}_{i=0}^d$ denote a sequence of vectors in W, not all 0

then $\{w_i\}_{i=0}^d$ is a standard basis for W iff both

(i) $w_i \in E^*_{r+i} W \qquad 0 \le i \le d$

(ii) $\sum_{i=0}^d w_i \in E_t W$

Pf $\rightarrow$ clear

$\leftarrow$ Define $\eta = \sum_{i=0}^d w_i$

So $\eta \in E_t W$

By constr

$$w_i = E^*_{r+i}\eta \qquad 0 \le i \le d$$

Obs $\eta \ne 0$ else $w_i = 0$ for $0 \le i \le d$. $\square$

A similar result holds for dual standard basis

Notation for an integer $d \geq 0$

$\mathrm{Mat}_{d+1}(\mathbb{F})$ is $\mathbb{F}$-algebra of all $d+1 \times d+1$ matrices over $\mathbb{F}$.

Rows / Cols indexed by $0, 1, \ldots, d$.

Here is another way to recognize the st. basis

LEM 96 Let $\{w_i\}_{i=0}^{d}$ denote any basis for W. Let B (resp B^*) denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ that represents A (resp A^*) rel $\{w_i\}_{i=0}^{d}$. Then $\{w_i\}_{i=0}^{d}$ is a st. basis for W iff both

(i) B has const row sum θ_t

(ii) $B^* = \mathrm{diag}(\theta_r^*, \theta_{r+1}^*, \ldots, \theta_{r+d}^*)$

Pf Put

$$z = \sum_{i=0}^{d} w_i$$

Obs

$$Az = \sum_{i=0}^{d} w_i \left(B_{i0} + B_{i1} + \cdots + B_{id} \right)$$

So

B has const row sum θ_t iff $z \in E_t W$

Result follows. $\square$

DEF 97

(i) We associate with W an $\mathbb{F}$-alg hom

$$\flat : T \to \mathrm{Mat}_{d+1}(\mathbb{F})$$

as follows. $\forall\, Y \in T$, $Y^\flat$ is the matrix rep Y rel a st basis for W.

(ii) We assoc with W a $\mathbb{F}$-alg hom

$$\#: T \to \mathrm{Mat}_{d+1}(\mathbb{F})$$

as follows. $\forall\, Y \in T$, $Y^\#$ is the matrix rep Y rel a dual st. basis for W

LEM 98

(i) $A^\flat$ has const row sum θ_t

(ii) $A^{*\flat} = \mathrm{diag}(\theta_r^*, \theta_{r+1}^*, \ldots, \theta_{r+d}^*)$

(iii) $A^\# = \mathrm{diag}(\theta_t, \ldots, \theta_{t+d})$

(iv) $A^{*\#}$ has const row sum θ_r^*

Pf By L96 and dual $\square$

DEF 99

(i) We def scalars

$$c_i(w),\quad a_i(w),\quad b_i(w) \qquad 0 \le i \le d$$

by

$$A^b = \begin{pmatrix} a_0(w) & b_0(w) & & & \\ c_1(w) & a_1(w) & b_1(w) & & 0 \\ & c_2(w) & \ddots & \ddots & \\ & 0 & \ddots & \ddots & b_{d-1}(w) \\ & & & c_d(w) & a_d(w) \end{pmatrix}$$

and $c_0(w) = 0,\ b_d(w) = 0$

(ii) We def scalars

$$c_i^*(w),\quad a_i^*(w),\quad b_i^*(w) \qquad 0 \le i \le d$$

by

$$A^{*\#} = \begin{pmatrix} a_0^*(w) & b_0^*(w) & & & \\ c_1^*(w) & a_1^*(w) & b_1^*(w) & & 0 \\ & c_2^*(w) & \ddots & \ddots & \\ & 0 & \ddots & \ddots & b_{d-1}^*(w) \\ & & & c_d^*(w) & a_d^*(w) \end{pmatrix}$$

and $c_0^*(w) = 0,\quad b_d^*(w) = 0$

LEM 100

(i) $b_i(w) \neq 0, \; b_i^*(w) \neq 0 \qquad 0 \le i \le d-1$

(ii) $c_i(w) \neq 0, \; c_i^*(w) \neq 0 \qquad 1 \le i \le d$

Pf follows from L89 (i) and L90 (i) $\square$

By L98 (i), (iii)

(i) $c_i(w) + a_i(w) + b_i(w) = \theta_t \qquad (0 \le i \le d)$

(ii) $c_i^*(w) + a_i^*(w) + b_i^*(w) = \theta_r^* \qquad (0 \le i \le d)$

LEM 101

(i) $a_i(W), b_i(W), c_i(W) \in \mathbb{R}$ $0 \le i \le d$

(ii) $a_i^*(W), b_i^*(W), c_i^*(W) \in \mathbb{R}$ $0 \le i \le d$.

pf (i) $a_i(W) \in \mathbb{R}$ since its an eigenvalue

for real sym matrix $E^*_{r+i} A E^*_{r+i}$

Now

$$b_i(W) + c_i(W) = \theta_t - a_i(W) \in \mathbb{R}$$

So $b_0(W) \in \mathbb{R}$ and $c_i(W) \in \mathbb{R} \to b_i(W) \in \mathbb{R}$ $1 \le i \le d-1$.

Also for $0 \le i \le d-1$

$$b_i(W) c_{i+1}(W) \in \mathbb{R}$$

since this is eigval of real sym matrix

$$E^*_{r+i} A E^*_{r+i+1} A E^*_{r+i}$$

so

$$b_i(W) \in \mathbb{R} \to c_{i+1}(W) \in \mathbb{R} \qquad 0 \le i \le d-1.$$

(ii) Sim. $\square$

LEM 102

Let $\{w_i\}_{i=0}^d$ denote a st. basis for W. Then

$\{w_i\}_{i=0}^d$ are mut orthog and

$$\|w_i\|^2 = \|w_0\|^2 \frac{b_0(W) b_1(W) \cdots b_{i-1}(W)}{c_1(W) c_2(W) \cdots c_i(W)} \qquad 0 \le i \le d$$

Pf. The $\{w_i\}_{i=0}^d$ are mut orthog since

$$V = E_0^* V + \cdots + E_D^* V \qquad \text{(orthog ds)}$$

Also for $0 \le i \le d-1$

$$\langle A w_i, w_{i+1} \rangle = \langle w_i, A w_{i+1} \rangle$$

$$\langle A w_i, w_{i+1} \rangle = \langle c_{i-1}(W) w_{i-1} + a_i(W) w_i + b_{i+1}(W) w_{i+1}, w_{i+1} \rangle = c_{i+1}(W) \|w_{i+1}\|^2$$

$$\langle w_i, A w_{i+1} \rangle = \langle w_i, c_{i+2}(W) w_{i+2} + a_{i+1}(W) w_{i+1} + b_i(W) w_i \rangle = b_i(W) \|w_i\|^2$$

result follows by ind on i $\square$

- Sim result holds for dual st. basis

(Spr break previous week)

No. Lec 24-1
Date 3/23/09

Lecture 24 Monday March 23

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$ write $T = T(x)$ etc.

Until further notice W is a thin irred T-module

endpt r, dual endpt t, diam d.

Fix $0 \neq z \in E_t W$, $0 \neq z^* \in E_r^* W$

Yields standard basis

$$\{E_{r+i}^* z\}_{i=0}^{d}$$

and dual standard basis

$$\{E_{t+i} z^*\}_{i=0}^{d}$$

for W.

Last time we saw

$$\| E_{r+i}^* z \|^2 = \| E_r^* z \|^2 \frac{b_0(W) b_1(W) \cdots b_{i-1}(W)}{c_1(W) c_2(W) \cdots c_i(W)} \qquad 0 \leq i \leq d$$

etc

DEF 103 For $0 \le i \le d$ def

$$k_i(W) = \frac{b_0(W)\, b_1(W) \cdots b_{i-1}(W)}{c_1(W)\, c_2(W) \cdots c_i(W)}$$

$$k_i^*(W) = \frac{b_0^*(W)\, b_1^*(W) \cdots b_{i-1}^*(W)}{c_1^*(W)\, c_2^*(W) \cdots c_i^*(W)}$$

— —

The following scalar will be useful

LEM 104 $\exists$ nonzero scalar $\nu = \nu(W)$ such that

on W,

$$\nu E_r^* E_t E_r^* = E_r^*$$

$$\nu E_t E_r^* E_t = E_t$$

Moreover $\nu \in \mathbb{R}$

Pf Let m denote the trace of $E_r^* E_t$ on W

claim

$$E_r^* E_t E_r^* = m E_r^*$$

on W. By L89 (iv)

$$E_t E_r^* W = E_t W$$

By L90 (iv)

$$E_r^* E_t W = E_r^* W$$

so

$$E_r^* E_t E_r^* W = E_r^* W$$

Since $E_r^* W$ has dim 1 $\exists$ $0 \neq \alpha \in \mathbb{F}$ s.t.

$$E_r^* E_t E_r^* = \alpha E_r^*$$

on W. Take the trace on W and use $\dim E_r^* W = 1$

to get $\alpha = m$. Claim proved.

We similarly have

$$E_t E_r^* E_t = m E_t$$

$\nu = m^{-1}$. Obs $\nu \in \mathbb{R}$ since E_r^*, E_t are real $\square$

No. Lec 24-4
Date 3/24/09

LEM 105 for $0 \le i \le d$

(i) $\| E^*_{r+i} \eta \|^2 = k_i(w)\, \nu^{-1} \| \eta \|^2$

(ii) $\| E_{t+i} \eta^* \|^2 = k_i^*(w)\, \nu^{-1} \| \eta^* \|^2$

pf (i) We saw

$$\| E^*_{r+i} \eta \|^2 = k_i(w) \| E^*_r \eta \|^2$$

Also

$$\| E^*_r \eta \|^2 = \langle \underset{E_t}{\overset{\uparrow}{E^*_r \eta}},\ \underset{E_t}{\overset{\uparrow}{E^*_r \eta}} \rangle \qquad E_t \eta = \eta$$

$$= \langle \eta,\ E_t E^*_r E_t \eta \rangle$$

$$= \langle \eta,\ \nu^{-1} \eta \rangle$$

$$= \nu^{-1} \| \eta \|^2$$

(ii) Sim. $\square$

LBM 106 We have

(i) $$E_r^* z = \frac{\langle z, z^* \rangle}{\| z^* \|^2} z^*$$

(ii) $$E_t z^* = \frac{\langle z^*, z \rangle}{\| z \|^2} z$$

(iii) $$\nu \, |\langle z, z^* \rangle|^2 = \| z \|^2 \, \| z^* \|^2$$

Pf (i) z^*, $E_r^* z$ each non 0 vectors in 1 dim'l space

$E_r^* W$ so $\exists\, 0 \neq \alpha \in \mathbb{F}$ s.t

$$E_r^* z = \alpha z^*$$

to get α take innerprod of each side with z^*

(ii) Sim.

(iii) Use (i), (ii) and

$$\nu E_r^* E_t E_r^* = E_r^* \quad \text{on } W \qquad \square$$

LEM 107 We have

$$(i) \quad \nu = \sum_{i=0}^{d} k_i(w)$$

$$(ii) \quad \nu = \sum_{i=0}^{d} k_i^*(w)$$

Pf (i)

$$z = \sum_{j=0}^{D} E_j^* z$$

$$= \sum_{i=0}^{d} E_{r+i}^* z$$

So

$$\| z \|^2 = \sum_{i=0}^{d} \| E_{r+i}^* z \|^2$$

$$= \| z \|^2 \sum_{i=0}^{d} k_i(w) \nu^{-1}$$

by L105 (i). Result follows.

(ii) Sim $\square$

Def 108 For $0 \le i \le d+1$ we def poly $v_i = v_i(w)$

in $\mathbb{F}[\lambda]$ by

$$v_0 = 1$$

$$\lambda v_i = b_{i-1}(w)\, v_{i-1} + a_i(w)\, v_i + c_{i+1}(w)\, v_{i+1}$$

$$0 \le i \le d$$

where $v_{-1} = 0$, $c_{d+1}(w) = 1$.

Obs each v_i has degree i

(the v_i^* are sim defined)

LEM 109

(i) $\quad v_i(A)\, E_r^* \eta = E_{r+i}^* \eta \qquad (0 \le i \le d)$

(ii) $\quad v_{d+1}(A)\, W = 0$

(iii) $\quad v_i^*(A^*)\, E_t \eta^* = E_{t+i}\, \eta^* \qquad (0 \le i \le d)$

(iv) $\quad v_{d+1}^*(A^*)\, W = 0$

Pf (i), (ii) Abr $\quad s_i = E_{r+i}^* \eta \quad$ fn $\; 0 \le i \le d+1$

By cnstr

$$A s_i = b_{i-1}(w) s_{i-1} + a_i(w) s_i + c_{i+1}(w) s_{i+1} \qquad 0 \le i \le d.$$

Comparing this with Def 108 gives

$$v_i(A)\, s_0 = s_i \qquad (0 \le i \le d+1)$$

But $s_{d+1} = 0$ so

$$v_{d+1}(A)\, s_0 = 0$$

Now $\quad v_{d+1}(A)\, W = v_{d+1}(A) M s_0$

$$= M \underbrace{v_{d+1}(A)\, s_0}_{0}$$

$$= 0$$

(iii), (iv) Sim $\qquad \square$

LEM 110

(i) V_{d+1} is a non 0 scalar mult of the min poly of A on W

(ii) the zeros of V_{d+1} are $\{\theta_{t+i}\}_{i=0}^{d}$

(iii) V^*_{d+1} is a non 0 scalar mult of the min poly of A^* on W

(iv) the zeros of V^*_{d+1} are $\{\theta^*_{r+i}\}_{i=0}^{d}$

Pf (i) Clear from L109 (i), (ii)

(ii) the eigenspaces of A on W are

$$E_{t+i}W \qquad 0 \le i \le d.$$

(iii), (iv) Sim. $\square$

the $v_i(W)$, $v_i^*(W)$ are normalized as follows.

LEM 111 For $0 \le i \le d$

(i) $v_i(\theta_t) = k_i(W)$

(ii) $v_i^*(\theta_r^*) = k_i^*(W)$

Pf (i)

$$
\begin{aligned}
k_i(W) \, \| E_r^* z \|^2 &= \| E_{r+i}^* z \|^2 \\
&= \langle z, E_{r+i}^* z \rangle \\
&= \langle z, v_i(A) E_r^* z \rangle \\
&= \langle v_i(A) z, E_r^* z \rangle \\
&= v_i(\theta_t) \langle z, E_r^* z \rangle \\
&= v_i(\theta_t) \, \| E_r^* z \|^2
\end{aligned}
$$

(ii) Sim $\square$

We now give the transition matrices between the standard and dual standard bases for W.

thm 112 For $0 \le j \le d$

(i) $$E^*_{r+j} \, z = \frac{\langle z, z^* \rangle}{\| z^* \|^2} \sum_{i=0}^{d} v_j(\theta_{t+i}) \, E_{t+i} \, z^*$$

(ii) $$E_{t+j} \, z^* = \frac{\langle z^*, z \rangle}{\| z \|^2} \sum_{i=0}^{d} v^*_j(\theta^*_{r+i}) \, E^*_{r+i} \, z$$

pf (i) $$E^*_{r+j} \, z = v_j(A) \, E^*_r z$$

$$= (E_0 + \cdots + E_0) \, v_j(A) \, E^*_r z$$

$$= \sum_{i=0}^{d} E_{t+i} \, v_j(A) \, E^*_r z$$

$$= \sum_{i=0}^{d} v_j(\theta_{t+i}) \, E_{t+i} \underbrace{E^*_r z}_{\frac{\langle z, z^* \rangle}{\| z^* \|^2} z^*} \quad \text{by 106}$$

(ii) Sim.

DEF 113 For $0 \le i \le d$ we def polys $u_i = u_i(W)$ in $\mathbb{F}[\lambda]$ by

$$u_0 = 1$$

$$\lambda u_i = c_i(W)\, u_{i-1} + a_i(W)\, u_i + b_i(W)\, u_{i+1} \qquad (0 \le i \le d-1)$$

where $u_{-1} = 0$

We obs u_i has deg i for $0 \le i \le d$

(polys u_i^* sim defined)

LEM 114 We have for $0 \le i \le d$

(i) $$u_i = \frac{v_i}{k_i(W)}$$

(ii) $$u_i^* = \frac{v_i^*}{k_i^*(W)}$$

Pf ex. $\square$

The u_i, u_i^* are normalized as follows.

LEM 118 For $0 \le i \le d$

(i) $u_i(\theta_0) = 1$

(ii) $u_i^*(\theta_0^*) = 1$

Pf By L113, L111. $\square$

thm 116 For $0 \le i, j \le d$

(i) $\langle E^*_{r+i} \eta, E_{t+j} \eta^* \rangle = \nu^{-1} k_i(w) k^*_j(w) u_i(\theta_{t+j}) \langle \eta, \eta^* \rangle$

(ii) $\langle E^*_{r+i} \eta, E_{t+j} \eta^* \rangle = \nu^{-1} k_i(w) k^*_j(w) u^*_j(\theta^*_{r+i}) \langle \eta, \eta^* \rangle$

(iii) $u_i(\theta_{t+j}) = u^*_j(\theta^*_{r+i})$ (Askey-Wilson duality)

Pf (i) Using th 112.

$$\langle E^*_{r+i} \eta, E_{t+j} \eta^* \rangle =$$

$$\frac{\langle \eta, \eta^* \rangle}{\| \eta^* \|^2} \sum_{h=0}^{d} \left\langle v_i(\theta_{t+h}) E_{t+h} \eta^*, E_{t+j} \eta^* \right\rangle$$

$$= \frac{\langle \eta, \eta^* \rangle}{\| \eta^* \|^2} \underbrace{v_i(\theta_{t+j})}_{k_i(w) u_i(\theta_{t+j})} \underbrace{\| E_{t+j} \eta^* \|^2}_{k^*_j(w) \nu^{-1} \| \eta^* \|^2}$$

(ii) Sim

(iii) By (i), (ii) $\square$

Here is the orthogonality for the v_i.

th 117 For $0 \le i, j \le d$

$$(i) \quad \sum_{h=0}^{d} v_i(\theta_{t+h})\, v_j(\theta_{t+h})\, k_h^*(w) = \delta_{ij}\, \nu\, k_i(w)$$

$$(ii) \quad \sum_{h=0}^{d} v_h(\theta_{t+i})\, v_h(\theta_{t+j})\, k_h^{-1}(w) = \delta_{ij}\, \nu\, (k_i^*)^{-1}(w)$$

Pf Use thm 112.

Here is the orthog for the u_i

th 118 For $0 \le i, j \le d$

(i) $$\sum_{h=0}^{d} u_i(\theta_{t+h})\, u_j(\theta_{t+h})\, k_h^*(W) = \delta_{ij}\, \nu\, k_i(W)^{-1}$$

(ii) $$\sum_{h=0}^{d} u_h(\theta_{t+i})\, u_h(\theta_{t+j})\, k_h(W) = \delta_{ij}\, \nu\, k_i^*(W)^{-1}$$

pf Use th 117 and def of u_i □

Here is the orthogonality for the u_i^*

th 119 For $0 \le i, j \le d$

(i) $$\sum_{h=0}^{d} u_i^*(\theta^*_{r+h})\, u_j^*(\theta^*_{r+h})\, k_h(W) = \delta_{ij}\, \nu\, k_i^*(W)^{-1}$$

(ii) $$\sum_{h=0}^{d} u_h^*(\theta^*_{r+i})\, u_h^*(\theta^*_{r+j})\, k_h^*(W) = \delta_{ij}\, \nu\, k_i(W)^{-1}$$

Pf Use th 118 and AW duality $\square$

Here is the orthogonality for the v_i^*

th 120 For $0 \le i, j \le d$

(i) $$\sum_{h=0}^{d} v_i^*(\theta^*_{r+h})\, v_j^*(\theta^*_{r+h})\, k_h(W) = \delta_{ij}\, k_i^*(W)\, \nu$$

(ii) $$\sum_{h=0}^{d} v_h^*(\theta^*_{r+i})\, v_h^*(\theta^*_{r+j})\, k_h^*(W)^{-1} = \delta_{ij}\, k_i(W)^{-1}\, \nu$$

Pf Use th 119 and $v_i^* = k_i^*(W)\, u_i^*$ $\square$

Lecture 25 Wednesday March 25

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given PRG $\Gamma = (X, R)$ diam $D \geq 2$

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly

Fix $x \in X$ write $T = T(x)$ etc

Fix a thin irred T-module W endpt r, dual endpt t

diam d. Fix $0 \neq \eta^* \in E_r^* W$, $0 \neq \eta \in E_t W$

Find $a_i(W)$, $b_i(W)$, $c_i(W)$ etc.

by L98:

$$c_i(W) + a_i(W) + b_i(W) = \theta_t \qquad (0 \leq i \leq d)$$

$$c_i^*(W) + a_i^*(W) + b_i^*(W) = \theta_r^* \qquad (0 \leq i \leq d)$$

For $d = 0$

$$a_0(W) = \theta_t$$
$$a_0^*(W) = \theta_r^*$$

nothing more to say.

Until further notice $d \geq 1$

We introduce a paramtr $\varphi_1 = \varphi_1(W)$

LEM 121 $\exists$ Nonzero $\varphi_1 = \varphi_1(W) \in \mathbb{F}$ s.t.

(i) $$b_0(W) = \frac{\varphi_1}{\theta^*_{r+1} - \theta^*_r}$$

(ii) $$b^*_0(W) = \frac{\varphi_1}{\theta_{t+1} - \theta_t}$$

Pf the polys $u_i = u_i(W)$ sat

$$\lambda u_i = c_i(W) u_{i-1} + a_i(W) u_i + b_i(W) u_{i+1} \qquad 0 \le i \le d$$

So for $i = 0$

$$\lambda u_0 = a_0(W) u_0 + b_0(W) u_1 \qquad u_0 = 1$$

$$u_1 = \frac{\lambda - a_0(W)}{b_0(W)}$$

Also

$$a_0(W) + b_0(W) = \theta_t$$

So

$$u_1 = 1 + \frac{\lambda - \theta_t}{b_0(W)}$$

Dually

$$u_1^* = 1 + \frac{\lambda - \theta_r^*}{b_0^*(w)}$$

By AW duality

$$u_1(\theta_{t+1}) = u_1^*(\theta_{r+1}^*)$$

gives

$$\frac{\theta_{t+1} - \theta_t}{b_0(w)} = \frac{\theta_{r+1}^* - \theta_r^*}{b_0^*(w)}$$

so

$$b_0(w)\,(\theta_{r+1}^* - \theta_r^*) = b_0^*(w)\,(\theta_{t+1} - \theta_t)$$

Call this common value φ_1 and obs $\varphi_1 \neq 0$

Result follows. $\square$

Lem 122 We have

(i) $$a_0(w) = \theta_t + \frac{\varphi_0}{\theta_r^* - \theta_m^*}$$

(ii) $$a_0^*(w) = \theta_r^* + \frac{\varphi_1}{\theta_t - \theta_{tM}}$$

(iii) $$u_1(w) = 1 + \frac{(\lambda - \theta_t)(\theta_m^* - \theta_r^*)}{\varphi_1}$$

(iv) $$u_1^*(w) = 1 + \frac{(\theta_{tM} - \theta_t)(\lambda - \theta_r^*)}{\varphi_1}$$

Pf (i) Use $a_0(w) = \theta_t - b_0(w)$

(ii) ✓

(iii) Use

$$u_1 = 1 + \frac{\lambda - \theta_t}{b_0(w)}$$

(iv) ✓

□

LEM 123 Fa osced

(i) $c_i(W)\left(\theta^*_{rri-1} - \theta^*_{rri}\right) - b_i(W)\left(\theta^*_{rri} - \theta^*_{rri+1}\right) =$

$$\left(\theta_{tH} - \theta_t\right)\left(\theta^*_{rri} - \theta^*_r\right) + \varphi_1$$

(ii) $c^*_i(W)\left(\theta_{tri-1} - \theta_{tri}\right) - b^*_i(W)\left(\theta_{tri} - \theta_{tri+1}\right) =$

$$\left(\theta^*_{rH} - \theta^*_r\right)\left(\theta_{tri} - \theta_t\right) + \varphi_1$$

pf (i) the polys $u_i = u_i(W)$ sat

$$\lambda u_i = c_i(W) u_{i-1} + a_i(W) u_i + b_i(W) u_{i+1}$$

take $\lambda = \theta_{tH}$ use AW duality

$$u_j(\theta_{tH}) = u^*_1(\theta^*_{rj})$$
$$= 1 + \frac{(\theta_{tH} - \theta_t)(\lambda - \theta^*_r)}{\varphi_1}$$

Also elim $a_i(W)$ using

$$c_i(W) + a_i(W) + b_i(W) = \theta_t$$

(ii) Sim $\square$

Example so far:

the Bip or dual Bip cases.

Assume Γ is Bip

So $\qquad a_i = 0 \qquad\qquad 0 \le i \le D$

$$E_i^* A E_i^* = 0$$

Def

$$R = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^* \qquad \text{"raising matrix"}$$

$$L = \sum_{i=1}^{D} E_{i-1}^* A E_i^* \qquad \text{"Lowering matrix"}$$

Obs

$$A = R + L$$

$$R E_i^* V \subseteq E_{i+1}^* V \quad (0 \le i \le D),$$

$$L E_i^* V \subseteq E_{i-1}^* V \quad (0 \le i \le D)$$

ne $E_{-1}^* = 0$, $E_{D+1}^* = 0$

Also

$$E_{i+1}^* R = R E_i^* \qquad 0 \le i \le D$$

$$E_{i-1}^* L = L E_i^* \qquad 0 \le i \le D$$

For Γ bip we show each irred T-mod is thin.

LEM 124 For Γ bip let W denote an irred T-module with dual endpt t. Fix any $0 \neq \eta \in E_t W$. Then

(i) $R E^*_{i-1} \eta + L E^*_{i+1} \eta = \theta_t E^*_i \eta \qquad 0 \le i \le D$

(ii) Suppose η is an eigenvector for $E_t A^* E_t$. Then

$$\theta^*_{i-1} R E^*_{i-1} \eta + \theta^*_{i+1} L E^*_{i+1} \eta =$$

$$(\theta_{t+1} \theta^*_i - \alpha\, \theta_{t+1} + \alpha\, \theta_t) E^*_i \eta \qquad 0 \le i \le D$$

where α is the eigval of $E_t A^* E_t$ and

$\theta^*_{-1}, \theta_{D+1}, \theta^*_{D+1}$ are indets.

Pf (i)

$$R E^*_{i-1} \eta + L E^*_{i+1} \eta = E^*_i R \eta + E^*_i L \eta$$

$$= E^*_i (R + L) \eta$$

$$= E^*_i A \eta$$

$$= \theta_t E^*_i \eta$$

(iii) Recall

$$A^* E_t W \subseteq E_t W + E_{t+1} W$$

z is eigenvector for $E_t A^* E_t$ with eigenvalue α so

$$(A^* - \alpha I) z \in E_{t+1} W$$

So

$$(A - \theta_{t+1} I)(A^* - \alpha I) z = 0$$

So

$$\theta^*_{i+1} R E^*_{i+1} z \quad + \quad \theta^*_{i+1} L E^*_{i+1} z$$

$$= (R E^*_{i+1} + L E^*_{i+1}) A^* z$$

$$= (E^*_i R + E^*_i L) A^* z$$

$$= E^*_i A \quad A^* z$$

$$= E^*_i A (A^* - \alpha I) z \quad + \quad \alpha E^*_i A z$$

$$= \theta_{t+1} E^*_i (A^* - \alpha I) z \quad + \quad \theta_t \alpha E^*_i z$$

$$= (\theta_{t+1} \theta^*_i - \theta_{t+1} \alpha + \theta_t \alpha) E^*_i z$$

LEM 125 For Γ bipartite, each irred T-module is thin.

Pf. Let W denote an ~~thin~~ irred T-module with endpt t. By constr $E_t W$ is nonzero and invariant under $E_t A^* E_t$

$E_t A^* E_t$ is real symmetric so it is diagonalizable on $E_t W$. So $\exists\ 0 \neq z \in E_t W$ that is an eigenvector for $E_t A^* E_t$. By L124

$$R E^*_{i-1} z + L E^*_{i+1} z \in \mathrm{Span}(E^*_i z)$$

$$\theta^*_{i-1} R E^*_{i-1} z + \theta^*_{i+1} L E^*_{i+1} z \in \mathrm{Span}(E^*_i z)$$

for $0 \le i \le D$ where $\theta^*_{-1}, \theta^*_{D+1}$ indets

Since $\{\theta^*_i\}_{i=0}^{D}$ are mut dist this gives

$$R E^*_i z \in \mathrm{Span}(E^*_{i+1} z) \qquad (0 \le i \le D)$$

$$L E^*_i z \in \mathrm{Span}(E^*_{i-1} z) \qquad (0 \le i \le D)$$

def $W' = \mathrm{Span}\{ E_i^* \eta \mid 0 \le i \le D \}$

So $W' \subseteq W$

claim $W' = W$.

Obs $W' \neq 0$ since

$$0 \neq \eta = \sum_{i=0}^{D} E_i^* \eta \in W'$$

By construction $A^* W' \subseteq W'$

By construction

$$R W' \subseteq W', \qquad L W' \subseteq W'$$

and $A = R + L$ so

$$A W' \subseteq W'$$

Now W' is T-module.

Now $W' = W$ by the irred of W.

Now for $0 \le i \le D$

$$E_i^* W = \mathrm{Span}(E_i^* \eta)$$

has dim ≤ 1 so W is thin. $\square$

LEM 126 Assume the Q-poly structure $\{E_i\}_{i=0}^{D}$ is dual bip. Then each irred T-module is thin.

Pf. Very similar to pf for bip just replace $A \leftrightarrow A^*$, $E_i \leftrightarrow E_i^*$ everywhere. $\square$