

$F = \mathbb{R} \text{ or } \mathbb{C}$       Given DRG  $\Gamma = (X, R)$     diam  $D$

Assume  $\{E_i\}_{i=0}^D$  is  $\mathbb{Q}$ -poly ordering of the prim idempotents of  $\Gamma$

Fix  $x \in X$  write  $T = T(x)$  etc.

Recall our goal is to prove thm 56, 57.

Notation 61

(i) Given  $\beta, \gamma, \delta \in F$  define a 2-variable polynomial

$$P(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \delta$$

(ii) Given  $\beta, \gamma^*, \delta^* \in F$  define a 2-variable poly

$$P^*(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma^*(\lambda + \mu) - \delta^*$$

LEM 62 Given  $\beta, \gamma, \delta \in \mathbb{F}$

$$0 = \left[ A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* A A^*) - \delta A^* \right] \quad (*)$$

iff

$$P(\theta_i, \theta_j) = 0 \quad \text{for } 1 \leq i \leq d.$$

Pf. Let  $C = \text{RHS of } (*)$

$$\begin{aligned} C &= (E_0 + E_1 + \dots + E_d) C (E_0 + E_1 + \dots + E_d) \\ &= \sum_{i=0}^d \sum_{j=0}^d E_i C E_j \end{aligned}$$

For  $0 \leq i, j \leq d$  use  $E_i A = \theta_i E_i$  and  $A E_j = \theta_j E_j$

to get

$$E_i C E_j = (\theta_i - \theta_j) P(\theta_i, \theta_j) E_i A^* E_j$$

$\Rightarrow$ : For  $0 \leq i \leq d$  show  $P(\theta_i, \theta_i) = 0$ .

$C = 0$  so

$$0 = E_i C E_i$$

$$= \underbrace{(\theta_i - \theta_i)}_{\neq 0} P(\theta_i, \theta_i) \underbrace{E_i A^* E_i}_{\neq 0}$$

$$\text{so } 0 = P(\theta_{i-1}, \theta_i)$$

$\Leftarrow$ :  $P$  is symmetric in its arguments so

$$P(\theta_i, \theta_{i-1}) = 0 \quad 1 \leq i \leq n$$

To show  $C = 0$  show

$$E_i C E_j = 0 \quad 0 \leq i, j \leq n$$

Given  $i, j$

$$\text{If } |i-j| > 1 \text{ then } E_i A^* E_j = 0 \text{ so } E_i C E_j = 0$$

$$\text{If } |i-j| = 1 \text{ then } P(\theta_i, \theta_j) = 0 \text{ so } E_i C E_j = 0$$

$$\text{If } i = j \text{ then } \theta_i - \theta_j = 0 \text{ so } E_i C E_j = 0$$

In each case  $E_i C E_j = 0$  so  $C = 0$ .  $\square$

For the moment let  $\{\theta_i\}_{i=0}^{\infty}$  be any sequence of scalars in  $\mathbb{F}$ . Given  $\beta \in \mathbb{F}$ , call this sequence

$\beta$ -recurrent whenever

$$\theta_{i-2} - (\beta+1)\theta_{i-1} + (\beta+1)\theta_i - \theta_{i+1} = 0$$

for  $2 \leq i < \infty$ . Given  $\beta, \gamma \in \mathbb{F}$  call  $\{\theta_i\}_{i=0}^{\infty}$

$(\beta, \gamma)$ -recurrent whenever

$$\theta_{i+1} - \beta\theta_i + \theta_{i+2} = \gamma$$

for  $1 \leq i < \infty$ . Obs  $\forall \beta \in \mathbb{F}$  TFAE

(i)  $\{\theta_i\}_{i=0}^{\infty}$  is  $\beta$ -rec

(ii)  $\exists \gamma \in \mathbb{F}$  s.t.  $\{\theta_i\}_{i=0}^{\infty}$  is  $(\beta, \gamma)$ -rec

LEM 63 Given integer  $d \geq 0$  and a sequence of scalars

$\{\theta_i\}_{i=0}^d$  from  $\mathbb{F}$ , Given  $\beta, \gamma \in \mathbb{F}$

(i) Assume  $\{\theta_i\}_{i=0}^d$  is  $(\beta, \gamma)$ -rec then  $\exists \delta \in \mathbb{F}$  s.t.

$$P(\theta_{i-1}, \theta_i) = 0 \quad 1 \leq i \leq d.$$

(ii) Assume  $\exists \delta \in \mathbb{F}$  s.t.

$$P(\theta_{i-1}, \theta_i) = 0 \quad 1 \leq i \leq d$$

Further assume  $\theta_{i-1} \neq \theta_{i+1}$  for  $1 \leq i \leq d-1$ , then  $\{\theta_i\}_{i=0}^d$

is  $(\beta, \gamma)$ -rec

Pf define

$$p_i = \theta_{i-1}^2 - \beta \theta_{i-1} \theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \quad 1 \leq i \leq d$$

and observe

$$p_i - p_{i+1} = (\theta_{i-1} - \theta_{i+1})(\theta_{i-1} - \beta \theta_i + \theta_{i+1} - \gamma)$$

for  $1 \leq i \leq d-1$ . Result follows.  $\square$

Proof of th 57 First assume  $D \geq 3$

By LEM 60 (with  $R = A^2$ ,  $S = A$ )  $\exists Z \in M$  s.t.

$$A^2 A^* A - A A^* A^2 = Z A^* - A^* Z \quad (*)$$

Recall  $\{A^i\}_{i=0}^D$  is a basis for  $M$  so  $\exists p \in \mathbb{F}[\lambda]$

with degree  $\leq D$  s.t.  $Z = p(A)$ .

Let  $d = \text{degree of } p$ . We show  $d = 3$ .

First suppose  $d > 3$ . Multiply each term in (\*) on left by

$E_d^*$  and on right by  $E_0^*$ . Evaluate using L58 to get

$$0 = c (\underbrace{\theta_0^*}_{\neq 0} - \underbrace{\theta_d^*}_{\neq 0}) \underbrace{E_d^* A^d E_0^*}_{\neq 0} \quad c = \text{leading coeff of } p$$

by L58

this is contradiction.

Next suppose  $d < 3$ . Multiply each term in (\*) on the

left by  $E_3^*$  and on the right by  $E_0^*$ . Evaluate using L58

to get

$$(\underbrace{\theta_1^* - \theta_2^*}_{\neq 0}) \underbrace{E_3^* A^3 E_0^*}_{\neq 0} = 0$$

by L5.8

cont.

We have shown  $d=3$ , Abbrev  $\beta = c^2 - 1$

Now divide both sides of (\*) by  $c$  to find  $\exists \gamma, \delta \in \mathbb{F}$  s.t.

$$(\beta + 1)(A^2 A^* A - A A^* A^2) = A^3 A^* - A^* A^3 \\ - \gamma(A^2 A^* - A^* A^2) - \delta(A A^* - A^* A)$$

Rearranging terms we get TD1. To get TD2 put  $i'$  ( $2 \leq i' \leq n-1$ )

Multiply each term in TD1 on left by  $E_{i',2}^*$  and on right

by  $E_{i'n}^*$  Simplify using L59 to get

$$0 = \underbrace{E_{i',2}^* A^3 E_{i'n}^*}_{\neq 0} \left( \underbrace{\theta_{i',2}^* - (\beta + 1)\theta_{i'n}^* + (\beta + 1)\theta_{i'}^* - \theta_{i'n}^*}_{\text{must be 0}} \right)$$

So  $\{\theta_{i'}^*\}_{i'=0}^n$  is  $\beta$ -rec.

So  $\exists \gamma^* \in \mathbb{F}$  s.t.  $\{\theta_{i'}^*\}_{i'=0}^n$  is  $(\beta, \gamma^*)$ -rec

So by L63  $\exists \delta^* \in \mathbb{F}$  s.t.

$$p^*(\theta_{i'}^*, \theta_{i'}^*) = 0 \quad 1 \leq i' \leq n$$

Now  $\beta, \gamma^*, \delta^*$  sat TD2 by L62\*

We are done for  $D \geq 3$

Now assume  $D < 3$

Let  $\beta \in \mathbb{F}$  (arbitrary)

If  $D=2$  define

$$\gamma = \theta_0 - \beta\theta_1 + \theta_2$$

and if  $D \leq 1$  let  $\gamma \in \mathbb{F}$  (arb)

By construction  $\{\theta_i\}_{i=0}^D$  is (p.r.)-rec

So by LG3 (i)  $\exists \delta \in \mathbb{F}$  s.t.

$$P(\theta_i, \theta_i) = 0 \quad \text{for } i \geq 0.$$

Now  $\beta, \gamma, \delta$  satisfy TD1 by LG2.

Interchanging  $A, A^*$  in above argument,  $\exists \gamma^*, \delta^* \in \mathbb{F}$  s.t.

$\beta, \gamma^*, \delta^*$  sat TD2

□



Prop 64. Given  $\beta, \gamma, \gamma^*, \delta, \delta^* \in F$  that satisfy

TD1, TD2

(i) the expressions

$$\frac{\theta_{i+2} - \theta_{i+1}}{\theta_{i+1} - \theta_i}, \quad \frac{\theta_{i+2}^* - \theta_{i+1}^*}{\theta_{i+1}^* - \theta_i^*}$$

are both equal to  $\beta + \gamma$  for  $2 \leq i \leq n-1$

(ii)  $\gamma = \theta_{i+1} - \beta \theta_i + \theta_{i-1} \quad (1 \leq i \leq n-1)$

(iii)  $\gamma^* = \theta_{i+1}^* - \beta \theta_i^* + \theta_{i-1}^* \quad (1 \leq i \leq n-1)$

(iv)  $\delta = \theta_{i+1}^2 - \beta \theta_{i+1} \theta_i + \theta_i^2 - \gamma (\theta_{i+1} + \theta_i) \quad (1 \leq i \leq n)$

(v)  $\delta^* = \theta_{i+1}^{*2} - \beta \theta_{i+1}^* \theta_i^* + \theta_i^{*2} - \gamma^* (\theta_{i+1}^* + \theta_i^*) \quad (1 \leq i \leq n)$

PF (iv) From TD1 and LG2

(vi) From TD2 and LG2<sup>\*</sup>

(vii) By LG3 and (vi) above

(viii) Sim to (iii)

(i)  $\{\theta_i\}_{i=0}^n$  is  $(\beta, \gamma)$ -rec by (ii) so  $\{\theta_i^*\}_{i=0}^n$  is  $\beta$ -rec  
 Sim  $\{\theta_i^*\}_{i=0}^n$  is  $\beta$ -rec. Result follows.  $\square$

COR 65 the scalars  $\beta, \gamma, \delta, \delta^*$  in A57

are unique provided P23.

Pf By Prop 64

□

Pf of A56 : Immed from Prop 64 (i)

□

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Ex 66 take  $\Gamma = H(0, N)$

Recall  $\Gamma$  has a  $Q$ -poly structure such that

$$\theta_i = \theta_i^* = (N-1)(0-i) - i \quad (0 \leq i \leq 0)$$

One checks that for this structure the parameters

$\beta, \gamma, \gamma^*, \delta, \delta^*$  from thm 57 are

$$\beta = 2, \quad \gamma = 0, \quad \gamma^* = 0$$

$$\delta = N^2, \quad \delta^* = N^2$$

$T01, T02$  become

$$[A, [A, [A, A^*]]] = N^2 [A, A^*],$$

$$[A^*, [A^*, [A^*, A]]] = N^2 [A^*, A]$$

these equations are called the Dolan-Grady relations.

Note 67 The Onsager algebra  $\mathcal{O}$  is the Lie algebra over  $\mathbb{F}$  defined by generators  $Y, Z$  and relations

$$[Y, [Y, [Y, Z]]] = 4[Y, Z],$$

$$[Z, [Z, [Z, Y]]] = 4[Z, Y]$$

where  $[, ]$  is Lie bracket. It turns out

$\mathcal{O}$  is  $\infty$  dim'l.

$\mathcal{O}$  is used in the statistical mechanics of the Ising model.

By Ex 66 we see that for  $H(D, N)$  the standard module is an  $\mathcal{O}$ -module on which  $Y, Z$  act as

$$\frac{2A}{N}, \quad \frac{2A^*}{N} \quad \text{resp.}$$

Note 68 For  $q \in \mathbb{F}$   $q \neq 0, 1, -1$

define the "q integer"

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n = 0, 1, 2, \dots$$

the (cubic) q-Serre relations in the variables  $X, Z$  are

$$X^3 Y - [3]_q X^2 Y X + [3]_q X Y X^2 - Y X^3 = 0,$$

$$Y^3 X - [3]_q Y^2 X Y + [3]_q Y X Y^2 - X Y^3 = 0.$$

These are among the defining relations for the algebra

$u_q(\widehat{sl}_2)$ . This is the quantum group for the cartesian matrix

$$\begin{pmatrix} z & -z \\ -z & z \end{pmatrix}.$$

The q-Serre relations are the same thing as the TD1, TD2 relations

$$\text{with } \beta = q^2 + q^{-2}, \quad r = 0, \quad r^{\vee} = 0, \quad s = 0, \quad s^{\vee} = 0.$$



$F = \mathbb{R} \text{ or } \mathbb{C}$  Given DRG  $\Gamma = (X, R)$  dim  $D$

Assume  $\{E_i\}_{i=0}^D$  is  $Q$ -poly

Fix  $x \in X$  and write  $T = T(x)$  etc.

We now solve the equations in Thm 56 to get the eigenvalues and dual eigenvalues of  $\Gamma$  in closed form.

LEM 69 Given a finite sequence  $\{\theta_i\}_{i=0}^D$  of scalars in  $\mathbb{C}$ , and given  $\beta \in \mathbb{C}$ . Then  $\{\theta_i\}_{i=0}^D$  is  $\beta$ -rec

$\iff \exists a, b, c \in \mathbb{C}$  such that

Case  $\beta \neq \pm 2$   $\theta_i = a + b\zeta^i + c\eta^i \quad (0 \leq i \leq D)$

where  $\zeta + \eta = \beta$

Case  $\beta = 2$   $\theta_i = a + bi + ci^2 \quad (0 \leq i \leq D)$

Case  $\beta = -2$   $\theta_i = a + b(-1)^i + c(-1)^i \quad (0 \leq i \leq D)$

Pf (For case  $\beta \neq \pm 2$ ) Assume  $d \geq 3$  else trivial.

Let  $L$  denote the set of all vectors  $(\sigma_0, \sigma_1, \dots, \sigma_d)$  in  $\mathbb{C}^{d+1}$

that are  $\beta$ -rec, i.e.

$$\sigma_{i+2} - (\beta+1)\sigma_{i+1} + (\beta-1)\sigma_i = 0 \quad (2 \leq i \leq d-1) \quad (*)$$

Obs  $L$  is a subspace of  $\mathbb{C}^{d+1}$ .

In  $(*)$   $\sigma_0, \sigma_1, \sigma_2$  are free and  $\sigma_3, \dots, \sigma_d$  are det by  $\sigma_0, \sigma_1, \sigma_2$

so  $\dim L = 3$

Pick  $q \in \mathbb{C}$  s.t.

$$\beta = q + q^{-1}$$

Obs  $q \neq 1, q \neq -1$

One checks the three vectors

$$(1, 1, 1, \dots, 1), \quad (1, q, q^2, \dots, q^d), \quad (1, q^{-1}, q^{-2}, \dots, q^{-d})$$

are in  $L$  and lin indep. So they form a

basis for  $L$ . Result follows.  $\square$

Note 70 Ref to L69, for  $\beta \neq \pm 2$  sometimes

we replace  $q$  by  $q^2$  and adjust  $b, c$  to write

$$\theta_i = a + bq^{2i-2} + cq^{2-2i} \quad (0 \leq i \leq D)$$

COR 71 Referring to  $\Gamma$ , assume  $D \geq 3$  to avoid trivialities.

Let  $\beta \in \mathbb{F}$  be from Th 57. Then the eigenvalues  $\{\theta_i\}_{i=0}^D$

of  $\Gamma$  and dual eigenvalues  $\{\theta_i^*\}_{i=0}^D$  of  $\Gamma$  satisfy one of the

following forms

Case I:  $\beta \neq \pm 2$

$$\begin{aligned} \theta_i &= a + bq^{2i-2} + cq^{2-2i} \\ \theta_i^* &= a^* + b^*q^{2i-2} + c^*q^{2-2i} \end{aligned} \quad 0 \leq i \leq D$$

$$\beta = q^2 + q^{-2}$$

Case II:  $\beta = 2$

$$\begin{aligned} \theta_i &= a + bi + ci^2 \\ \theta_i^* &= a^* + b^*i + c^*i^2 \end{aligned} \quad 0 \leq i \leq D$$

Case III:  $\beta = -2$

$$\begin{aligned} \theta_i &= a + b(-1)^i + c i (-1)^i \\ \theta_i^* &= a^* + b^*(-1)^i + c^* i (-1)^i \end{aligned} \quad 0 \leq i \leq D$$



Caution Ref to Cor 71, possibly some of

$q, a, b, c, a^*, b^*, c^*$  are in  $\mathbb{C} \setminus \mathbb{R}$

even though the  $a_i, a_i^*$  are all in  $\mathbb{R}$

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Note 72 Ref to Cor 71, for Case I

The parameters  $\beta, \gamma, \gamma^*, \delta, \delta^*$  from th 57 are:

$$\beta = q^2 + q^{-2}$$

$$\gamma = -a(q - q^{-1})^2$$

$$\gamma^* = -a^*(q - q^{-1})^2$$

$$\delta = -bc(q^2 - q^{-2})^2 + a^2(q - q^{-1})^2$$

$$\delta^* = -b^*c^*(q^2 - q^{-2})^2 + a^{*2}(q - q^{-1})^2$$

this is checked using Prop 64. Similar equations hold for cases

II, III.

Note 73 Earlier we found a  $\mathcal{Q}$ -poly structure for

$\Gamma^* = H(0, N)$ . It was Case II with

$$a = a^* = (N-1)0$$

$$b = b^* = -N$$

$$c = c^* = 0$$

Obs our DRG  $\Gamma$  is bipartite iff  $a_i = 0$  for  $0 \leq i \leq D$

Call our  $\mathbb{Q}$ -poly structure dual bipartite whenever

$$a_i^* = 0 \text{ for } 0 \leq i \leq D.$$

th 74 Let the scalars  $\beta, \gamma, \gamma^*, \delta, \delta^*$  be as in th 57. Assume  $D \geq 2$ .

(i) Assume  $\Gamma$  is bipartite. Then

$$0 = A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^* (A A^* A^* A) - \delta^* A \quad (*)$$

and  $\delta = 0$

(ii) Assume The  $\mathbb{Q}$ -poly str is dual bipartite. Then

$$0 = A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* A^* A) - \delta A^*$$

and  $\delta^* = 0$

pf (i) Let  $F = \text{RHS of } (*)$  Show  $F = 0$  Obs

$$F = \sum_{i=0}^D \sum_{j=0}^D E_i^* F E_j^*$$

For  $0 \leq i, j \leq D$  show  $E_i^* F E_j^* = 0$ . Obs

$$E_i^* F E_j^* = P^*(\theta_i^*, \theta_j^*) E_i^* A E_j^*$$

where  $P^*$  is from Not 61

Case  $|i-j| > 1$ :  $E_i^* A E_j^* = 0$

Case  $(i-1) = 1$ :  $P^*(\theta_i^*, \theta^*) = 0$  since

the dual eigenvalues are  $\beta$ -rec

Case  $i=1$ :  $E_i^* A E_i^* = 0$  since  $a_i = 0$

In all cases

$$E_i^* F E_i^* = 0$$

so  $F = 0$ .

Show  $X = 0$

Recall back in LEM 8 of Ch 1. We found

$$H \in \text{Mat}_X(\mathbb{F}) \text{ s.t.}$$

$$HA = -AH$$

Replacing  $H$  by  $-H$  if nec

$$H = \sum_{i=0}^p (-1)^i E_i^*$$

Obs  $HA^* = A^*H$  and  $H^2 = I$ . By Th 57 we have TD1:

$$0 = \left[ A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \delta A^* \right]$$

Conjugate the by H to get

$$0 = \left[ -A, (-A)^2 A^\dagger - \beta(-A) A^\dagger (-A) + A^\dagger (-A)^2 - \gamma(-A A^\dagger - A^\dagger A) - \delta A^\dagger \right]$$

So

$$0 = \left[ A, A^2 A^\dagger - \beta A A^\dagger A + A^\dagger A^2 + \gamma(A A^\dagger + A^\dagger A) - \delta A^\dagger \right]$$

Subtracting the from TDI.

$$0 = \gamma \left[ A, A A^\dagger + A^\dagger A \right]$$

$$= \gamma \left[ A^2, A^\dagger \right]$$

But  $[A^2, A^\dagger] \neq 0$  since

$$E_0^\dagger [A^2, A^\dagger] E_2^\dagger = \underbrace{E_0^\dagger A^2 E_2^\dagger}_{\neq 0} (\underbrace{\delta_2^\dagger - \delta_0^\dagger}_{\neq 0})$$

$$\text{So } \gamma = 0$$

(iii) Sim.

□

Ex 75 Take  $\Gamma = H(0, 2)$  hypercube take  $\mathbb{F} = \mathbb{C}$

$\Gamma$  is bipartite. The  $\mathbb{Q}$ -poly structure we found earlier

$$\text{satisfies } q_{i,j}^h = p_{i,j}^h \quad \text{for } 0 \leq h, i, j \leq 0$$

So this structure is dual bipartite.

Recall

$$\theta_i^* = \theta_i^* = 0 - 2i \quad (0 \leq i \leq 0)$$

Here

$$\beta = 2$$

$$\gamma = 0, \quad \gamma^* = 0 \quad \delta = 4, \quad \delta^* = 4$$

So by Th 74

$$\begin{aligned} 4A &= A^{*2}A - 2A^*AA^* + AA^{*2} \\ &= [A^*, [A^*, A]] \end{aligned} \quad (*)$$

$$\begin{aligned} 4A^* &= A^2A^* - 2AA^*A + A^*A^2 \\ &= [A, [A, A^*]] \end{aligned} \quad (**)$$

For notational convenience define  $A^\varepsilon$  by

$$[A, A^*] = 2iA^\varepsilon \quad (i^2 = -1)$$

then  $(*)$ ,  $(**)$  become

$$[A^*, A^E] = 2\sigma A$$

$$[A^E, A] = 2iA^*$$

We now recognize the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$

Recall  $\mathfrak{sl}_2(\mathbb{C})$  is the Lie algebra consisting of all  $2 \times 2$  matrices over  $\mathbb{C}$  with trace 0, the Lie bracket is

$$[x, y] = xy - yx \quad \forall x, y \in \mathfrak{sl}_2(\mathbb{C})$$

$\mathfrak{sl}_2(\mathbb{C})$  has a basis

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a^E = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and

$$[a, a^*] = 2ia^E, \quad [a^*, a^E] = 2ia, \quad [a^E, a] = 2ia^*$$

Thus for  $H(p, 2)$  the standard module becomes an

$\mathfrak{sl}_2(\mathbb{C})$ -module it  $a, a^*, a^E$  act as  $A, A^*, A^E$  resp.





Given DRG  $\Gamma = (X, R)$  diam  $D \geq 1$

Assume ordering  $\{E_i\}_{i=0}^D$  is  $\mathcal{Q}$ -poly

Fix  $x \in X$  write  $T = T(x)$  etc.

We mention a handy formula.

LEM 76 With above notation

posi<sup>ve</sup>.

$$(i) \quad c_i \theta_{i+}^* + a_i \theta_i^* + b_i \theta_{i-}^* = \theta_i^*$$

where  $\theta_{i+}^*, \theta_{i-}^*$  are unlets

$$(ii) \quad c_i^* \theta_{i+} + a_i^* \theta_i + b_i^* \theta_{i-} = \theta_i^*$$

where  $\theta_{i+}, \theta_{i-}$  are unlets.

Pf (i) By AW duality

$$\begin{aligned} u_i(\theta_i) &= u_i^*(\theta_i^*) \\ &= \frac{\theta_i^*}{\theta_0^*} \end{aligned}$$

Result follows from this and the 3-term rec for the  $u_i$

(iii) Sim

□

Setting  $i=0$  in LEM 76 (i) and using  $k=\theta_0$

$$\frac{\theta_1}{\theta_0} = \frac{\theta_1^*}{\theta_0^*}$$

LEM 77. Assume  $\Gamma$  is Bipartite

$$(i) \quad c_i = \frac{\theta_0}{\theta_0^*} \frac{\theta_1^* \theta_i^* - \theta_0^* \theta_i^*}{\theta_i^* - \theta_0^*} \quad (1 \leq i \leq n)$$

$$(ii) \quad c_0 = \theta_0$$

$$(iii) \quad b_i = \theta_0 - c_i \quad (0 \leq i \leq n)$$

$$(iv) \quad \frac{\theta_0^*}{\theta_0^*} = \frac{\theta_1^*}{\theta_0^*}$$

Pf (i) In L76 (i) Eval using  $a_i = 0$ ,

$$b_i = k - c_i, \quad \text{and solve for } c_i$$

$$(ii) \quad c_0 = k - b_0 = k = \theta_0$$

$$(iii) \quad k = \theta_0$$

(iv) Set  $i=0$  in L76 (i)

□

LEM 78 Assume  $\{E_i\}_{i=0}^D$  is dual bipartite.

$$(i) \quad c_i^* = \frac{\theta_0^*}{\theta_0} \frac{\theta_1 \theta_i - \theta_0 \theta_{i+1}}{\theta_{i+1} - \theta_{i-1}} \quad 1 \leq i \leq D-1$$

$$(ii) \quad c_0^* = \theta_0^*$$

$$(iii) \quad b_i^* = \theta_0^* - c_i^* \quad 0 \leq i \leq D$$

$$(iv) \quad \frac{\theta_{D-1}}{\theta_0} = \frac{\theta_1}{\theta_0}$$

Pf Sim to L77

□

LEM 79 Assume  $\Gamma$  is bip and  $\{E_i\}_{i=0}^D$  is dual bip.

Further assume  $\beta = 2$ . Then

$$(i) \quad \theta_i = \theta_i^* = D - 2i \quad (0 \leq i \leq D)$$

$$(ii) \quad c_i = c_i^* = i \quad (0 \leq i \leq D)$$

"It looks like  $H(0, 2)$ "

Pf Assume  $D \geq 2$  else trivial.

We are in Case II: the  $a_i, b_i$  have form

$$\theta_i = a + bi + ci^2$$

$$0 \leq i \leq D.$$

$$\theta_i^* = a^* + b^*i + c^*i^2$$

By Th 74 (i)

$$0 = \gamma$$

$$= \theta_0 - \beta \theta_1 + \theta_2$$

$$= a - 2(a + b + c) + a + 2b + 4c$$

$$= 2c$$

So  $c = 0$

Sim  $c^* = 0$

the constraint

$$\frac{\theta_{01}}{\theta_0} = \frac{\theta_1}{\theta_0}$$

from L78 (iv) gives

$$b/a = -z/d$$

Similarly using L77 (iv)

$$b^*/a^* = -z^*/d$$

So far

$$\frac{\theta_i}{\theta_0} = 1 - \frac{z_i}{d} \quad 0 \leq i \leq n$$

$$\frac{\theta_i^*}{\theta_0^*} = 1 - \frac{z_i^*}{d} \quad 0 \leq i \leq n$$

For  $1 \leq i \leq n-1$  solve for  $c_i$  using L77 (i) to get

$$c_i = \frac{c^* \theta_0}{d}$$

But  $c_i = 1$  so

$$\theta_0 = d$$

hence

$$\theta_i = b - z_i \quad (0 \leq i \leq n)$$

Also

$$c_0 = \theta_0 = 0$$

$$c_i = i \quad (1 \leq i \leq n-1)$$

So

$$c_i = i \quad (0 \leq i \leq n)$$

Similarly

$$\theta_i^* = 0 - 2i \quad (0 \leq i \leq n)$$

$$c_i^* = i \quad (0 \leq i \leq n)$$

□

From EX TFAE

(i)  $\Gamma$  is bipartite and

$$c_i = i \quad (0 \leq i \leq n)$$

(ii)  $\Gamma$  is  $H(0, 2)$

PF hint (i)  $\rightarrow$  (ii) Consider set of vectors

$$\left\{ E_i \hat{y} - E_i \hat{z} \mid y_i, z_i \in X, y_i, z_i \in \mathbb{R} \right\}$$

Show For all  $u, v$  in this set either

$$u = \pm v \text{ or } \langle u, v \rangle = 0$$

Use this to show  $\Gamma = K_2 \times K_2 \times \dots \times K_2$ .

□

LEM 80 Assume  $\Gamma$  is bip and  $\{E_i\}_{i=0}^D$  is

dual bip. Further assume  $\beta \neq \pm 2$

then  $\exists \alpha, q \in \mathbb{C}$  ( $q^2 \neq 1, q^2 \neq -1$ ) s.t

$$(i) \quad \theta_i = \theta_i^* =$$

$$\left( q^{0-2i} + q^{2-0} \right) \frac{q^{0-2i} - q^{2i-0}}{q^2 - q^{-2}} \quad (0 \leq i \leq D)$$

$$(ii) \quad c_i = c_i^* =$$

$$\frac{q^{0-2} + q^{2-0}}{q^{0-2i} + q^{2i-0}} \frac{q^{2i} - q^{-2i}}{q^2 - q^{-2}} \quad (0 \leq i \leq D)$$

Pf Sim to pf of L79, except use Case I forms for  $\theta_i, c_i$

□



LEM 8) Assume  $\Gamma$  is bip and  $\{E_i\}_{i=0}^p$  is dual bip

Further assume  $\beta = -2$ . then  $D$  is even and

$$(i) \quad \theta_i = \theta_i^* = (-1)^i (0 - 2i) \quad (0 \leq i \leq p)$$

$$(ii) \quad c_i = c_i^* = i \quad (0 \leq i \leq p)$$

Pf Sim to pf of L79 except use Case III from for  $e_i, e_i^*$   $\square$

---

Note Unique sol to LEM 81 is  $H(p, z)$  (0 even)

with  $\mathbb{Q}$ -poly str assoc with ordering

$D, 2-0, 0-4, 6-0, \dots$

of the eigenvalues.

(Fun ex to show this really is a  $\mathbb{Q}$ -poly str.)

---

With ref to L80

$$\beta = q^2 + q^{-2}$$

$$\gamma = \gamma^* = 0$$

$$\delta = \delta^* = (q^{p-2} + q^{2-p})^2$$

By L74

$$A^2 A^* - (q^2 + q^{-2}) A A^* A + A^* A^2 = (q^{p-2} + q^{2-p})^2 A^*$$

$$A^* A^2 - (q^2 + q^{-2}) A^* A A^* + A A^* A^2 = (q^{p-2} + q^{2-p})^2 A$$

In cyclic form this looks as follows.

LEM 82 With ref to L80  $\exists A^E \in T$  s.t.

$$q A A^* - q^{-1} A^* A = z A^E$$

$$q A^* A^E - q^{-1} A^E A^* = z A$$

$$q A^E A - q^{-1} A A^E = z A^*$$

where

$$z = c(q^{p-2} + q^{2-p}),$$

$$c^2 z = 1$$

Pf Routine - just def  $A^E$  using 1st equation.



$\mathbb{F} = \mathbb{C}$  Given bipartite DRG  $\Gamma = (X, R)$

dim  $D \geq 2$

Assume  $\{E_i\}_{i=0}^D$  is dual bip  $\mathbb{Q}$ -alg

Further assume  $\beta \neq \pm 2$ , let  $q \in \mathbb{C}$  be as in L80

Fix  $x \in X$  into  $T = T(x)$  etc.

Last time we found  $A^\varepsilon \in T$  s.t.

$$qAA^\vee - q^{-1}A^\vee A = zA^\varepsilon,$$

$$qA^\vee A^\varepsilon - q^{-1}A^\varepsilon A^\vee = zA,$$

$$qA^\varepsilon A - q^{-1}AA^\varepsilon = zA^\vee.$$

where  $z = i(q^{D-2} + q^{2-D})$   $i^2 = -1$

Next goal: show  $A^\varepsilon$  is an imaginary adj matrix

Search for  $W \in M$  and  $W^\vee \in M^*$

s.t.

$$WA^\vee W^{-1} = W^{*\vee} A W^\vee = A^\varepsilon$$

(just like we did for  $W$ )

Find  $W$ : For moment assume  $W$  exists. Write

$$W = \sum_{i=0}^D \alpha_i E_i \quad \alpha_i \in \mathbb{C}$$

$W^{-1}$  exists so  $\alpha_i \neq 0$  ( $0 \leq i \leq D$ ) and

$$W^{-1} = \sum_{i=0}^D \alpha_i^{-1} E_i$$

Require

$$A^E = WA^*W^{-1}$$

For  $0 \leq i, j \leq D$

$$\begin{aligned} E_i A^E E_j &= E_i (WA^*W^{-1}) E_j \\ &= E_i A^* E_j \quad \alpha_i/\alpha_j \end{aligned}$$

Also

$$\begin{aligned} E_i A^E E_j &= E_i \left( \frac{qAA^* - q^*A^*A}{z} \right) E_j \\ &= E_i A^* E_j \frac{q\theta_i - q^*\theta_j}{z} \end{aligned}$$

Since we assume our  $Q$ -poly str is dual bip

$$E_i A^* E_j = 0 \quad \text{if } |i-j| \neq 1$$

Require

$$L_i/d_i = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \quad \& \quad |i-1|=1 \quad (0 \leq i, 1 \leq 0)$$

Recall for  $1 \leq i \leq 0$

$$\delta = \underbrace{\theta_{i-1}^2}_{-z^2} - \underbrace{\beta \theta_{i-1} \theta_i}_{q^2 + q^{-2}} + \underbrace{\theta_i^2}_{0} - \underbrace{\gamma(\theta_{i-1} + \theta_i)}_0$$

$$-z^2 = (q\theta_{i-1} - q^{-1}\theta_i)(q^{-1}\theta_{i-1} - q\theta_i)$$

$$1 = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \frac{q\theta_{i-1} - q^{-1}\theta_i}{z}$$

So for  $0 \leq i, 1 \leq 0$

$$1 = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \frac{q\theta_{i-1} - q^{-1}\theta_i}{z} \quad \& \quad |i-1|=1$$

Only requirement on  $\{\alpha_i\}_{i=0}^p$  is

$$\frac{\alpha_i}{\alpha_{i-1}} = \frac{q\theta_i - q^{-1}\theta_{i-1}}{z} \quad 1 \leq i \leq 0$$

By L 80

$$\frac{q\theta_i - q^{-1}\theta_{i-1}}{z} = \prod_{r=0}^{i-1} q^{2i-1-0} \quad 1 \leq i \leq 0$$

$$\left( \prod_{r=0}^{i-1} q^{2i-1-0} = 1 \right)$$

Def 83

With above notation

put  $0 \neq \alpha_0 \in \mathbb{C}$ and def  $\{\alpha_i\}_{i=1}^p$  by

$$\frac{\alpha_i}{\alpha_{i-1}} = \alpha^{\frac{1}{q}} \eta^{2i-1-p} \quad (1 \leq i \leq p)$$

Put

$$W = \sum_{i=0}^p \alpha_i E_i$$

$$W^* = \sum_{i=0}^p \alpha_i \bar{E}_i^*$$

Prop 84 With above not

$$(i) \quad WA^*W^{-1} = A^\varepsilon$$

$$(ii) \quad W^{-1}AW^* = A^\varepsilon$$

$$\begin{aligned} \text{pf (i)} \quad A^\varepsilon &= (E_0 + E_1 + \dots + E_D) A^\varepsilon (E_0 + E_1 + \dots + E_D) \\ &= \sum_{i=0}^D \sum_{j=0}^D E_i A^\varepsilon E_j \end{aligned}$$

Also

$$\begin{aligned} WA^*W^{-1} &= \sum_{i=0}^D \sum_{j=0}^D E_i (WA^*W^{-1}) E_j \\ &= \sum_{i=0}^D \sum_{j=0}^D \alpha_i / \alpha_j E_i A^* E_j \end{aligned}$$

For  $0 \leq i, j \leq D$

$$E_i A^\varepsilon E_j - \alpha_i / \alpha_j E_i A^* E_j$$

$$= \underbrace{E_i A^\varepsilon E_j}_{\substack{0 \neq \\ 1 \neq 1}} \left( \underbrace{\frac{\alpha_i - \alpha_j}{z}}_{0 \neq \quad 1 \neq 1} - \frac{\alpha_i}{\alpha_j} \right)$$

= 0 ✓

pf (ii) Very sim

□



COR 85 With above not

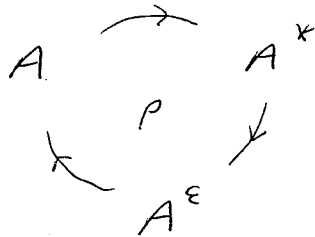
$A^\varepsilon$  is similar to  $A, A^*$

In particular  $A^\varepsilon$  is diagonalizable with dist eigenvalues

$$\theta_i^\varepsilon = \theta_i = \theta_i^*$$

$$(0 \leq i \leq n)$$

Prop 86 With above not



where

$$T \longrightarrow T$$

P:

$$m \longrightarrow (ww^*)_m (ww^*)^{-1}$$

Pf (just like LEM51)

check  $A \rightarrow A^*$ :

$$ww^*A \stackrel{?}{=} A^*ww^*$$

By const

$$wA^*w^{-1} = w^{-1}Aw^*$$

so

$$w^*wA^* = A^*w^*w$$

take trans

$$ww^*A = A^*ww^* \quad \checkmark$$

check  $A^\vee \rightarrow A^\varepsilon$ :

$$W W^\vee A^\vee \stackrel{?}{=} A^\varepsilon W W^\vee$$

$$\begin{aligned} A^\varepsilon W W^\vee &= W A^\vee W^\vee W W^\vee \\ &= W A^\vee W^\vee \\ &= W W^\vee A^\vee \quad \checkmark \end{aligned}$$

check  $A^\varepsilon \rightarrow A$

$$W W^\vee A^\varepsilon \stackrel{?}{=} A W W^\vee$$

$$\begin{aligned} W W^\vee A^\varepsilon &= W W^\vee (W^\vee \rightarrow A W^\vee) \\ &= W A W^\vee \\ &= A W W^\vee \quad \checkmark \end{aligned}$$

Thm 87 With above not,

$A^\varepsilon$  is an imaginary adj matrix for  $\Gamma$

in sense of Def 55.

Pf. By Prop 86 any two of  $A, A^\vee, A^\varepsilon$  related the same way as  $A, A^\vee$ .  $\square$

Problem. 88 In our study of  $K_N$  we obtained

many results concerning  $A, A^*, A^E, W, W^*$ .

Try to find similar results for the Bip/dual bip case.

[ is  $W$  a  $q$ -exponential in  $A$ ? See if  $W$  is an  
exponential in  $A$ , for  $H(2,2)$ . Repeat for Case III ]

$$F = \mathbb{R} \text{ or } \mathbb{C} \quad \Gamma = (X, R) \quad \text{any PRG diam } D \geq 2$$

Assume  $\{E_i\}_{i=0}^D$  is  $\mathcal{Q}$ -alg

Fix  $x \in X$ , write  $T \cong T(x)$  etc.

Fix an irred  $T$ -module  $W$ .

Next goal is to carefully study  $W$ .

$r = \text{endpt of } W$

$t = \text{dual endpt of } W$

$d = \text{diam of } W$

$d^* = \text{dual diam of } W$ .

LEM 89

With above not

$$(i) \quad E_i^* A E_j^* W \neq 0 \quad \text{if } |i-j| = 1 \quad (r \leq i, j \leq r+d)$$

(ii) Suppose  $W$  is then. then

$$E_r^* W + E_{r+1}^* W + \dots + E_{r+d}^* W = E_r^* W + A E_r^* W + \dots + A^d E_r^* W$$

$$(0 \leq i \leq d)$$

(iii) Suppose  $W$  is then. then  $W = M E_r^* W$

(iv) Suppose  $W$  is then. then

$$E_j E_r^* W = E_j W \quad (0 \leq j \leq D)$$

Moreover  $W$  is dual then.

Pf (i) Suppose  $\exists i, j$  ( $r \leq i < r+d$ ) s.t

$$E_i^* A E_j^* W = 0 \quad \text{and} \quad |i-j| = 1$$

If  $i-j = 1$  then

$$E_r^* W + \dots + E_j^* W$$

is a non 0 proper subspace of  $W$  that is inv under  $A, A^*$ , cont the inv of  $W$ .

If  $j-i = 1$  then

$$E_j^* W + \dots + E_{r+d}^* W$$

is a non 0 proper subspace of  $W$  that is inv under  $A, A^*$ , cont inv of  $W$ .

(iii) By (i) and since

$$E_i^* A E_j^* = 0 \quad \text{if} \quad |i-j| > 1 \quad (0 \leq i, j \leq d)$$

(iii)  $\geq$ : Since  $W$  is  $M$ -inv

$\leq$ : Set  $i=d$  in (iii) and use

$$W = \sum_{i=0}^d E_{r+i}^* W$$

(iv)  $F_n \quad 0 \leq n \leq D$

$$E_2 W = E_2 M E_r^* W$$

$$= E_2 E_r^* W$$

$$E_2 M = \text{Span}(E_2)$$

$$\text{Now } \dim E_2 W \leq \dim E_r^* W$$

$$= 1$$

□

L 90 With above not

(i)  $E: A^* E_7 W \neq 0$  if  $|c_1| = 1$  ( $0 \leq i, j \leq t+d$ )

(ii) Suppose  $W$  is dual then then

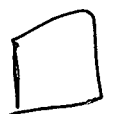
$$E_7 W + \dots + E_{t+i} W = E_7 W + A^* E_7 W + A^{*i} E_7 W \quad (\text{as is } d^*)$$

(iii) Suppose  $W$  is dual then then  $W = M^* E_7 W$

(iv) Suppose  $W$  is dual then then

$$E_7^* E_7 W = E_7^* W \quad 0 \leq j \leq d$$

Then  $W$  is Num.





$F = \mathbb{R} \text{ or } \mathbb{C}$  Given DRG  $\Gamma = (X, R)$  diam  $D \geq 2$

Assume  $\{E_i\}_{i=0}^D$  is  $\mathbb{Q}$ -poly

Fix  $x \in X$ , write  $T = T(x)$  etc.

Goal: study str of simple unred  $T$ -module

LEM 91 The following hold for  $0 \leq h \leq D$

(i)  $\forall a \neq w \in E_h^* V$

$$\left| \left\{ i \mid 0 \leq i \leq D, E_i w = 0 \right\} \right| \leq 2h$$

(ii)  $\forall a \neq w \in E_h V$

$$\left| \left\{ i \mid 0 \leq i \leq D, E_i^* w = 0 \right\} \right| \leq 2h$$

Pf (i) Suppose not, then  $\exists$  subset

$$\Omega \subseteq \{0, 1, \dots, D\} \quad |\Omega| = 2h + 1$$

s.t

$$E_i w = 0 \quad \forall i \in \Omega$$

By constr  $w = E_h^* w$

So  $\forall i \in \Omega$

$$0 = E_h^* E_i E_h^* w$$

$$= E_h^* \left( |X|^{-1} m_i \sum_{l=0}^D u_l(\theta_i) A_l \right) E_h^* w$$

$$E_h^* A_l E_h^* = 0 \text{ for } l > 2h$$

so

$$0 = \sum_{l=0}^{2h} u_l(\theta_i) E_h^* A_l E_h^* w$$

Letting  $i$  range over  $\Omega$  get system of  $2ht$  homogeneous

linear equations in the unknowns

$$E_h^* A_l E_h^* w \quad 0 \leq l \leq 2h$$

the coefficient matrix is essentially Vandermonde since

poly  $u_l$  has degree  $l$  and the  $\theta_i$  are distinct.

So the coeff matrix is nonsingular forcing

$$E_h^* A_l E_h^* w = 0 \quad 0 \leq l \leq 2h.$$

This is imposs since

$$E_h^* A_0 E_h^* w = w \neq 0$$

(ii) Sim

□