Lecture 13 Wednesday Feb 18

2. The subconstituent algebra T

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Given DRG $\Gamma = (X, R)$ with diam D st. module $V = \mathbb{F}^X$

Fix $x \in X$ and write $M^* = M^*(x)$ etc.

Def 1 Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ generated by M and M^*.

We call T the <u>subconstituent algebra</u> of Γ with respect to x

With reference to Def 1, by a <u>T-module</u> we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$.

The T-module W is called <u>irreducible</u> whenever $W \neq 0$ and W contains no T-module other than $0, W$.

Given T-modules W, W' by an <u>isomorphism of T modules</u> from W to W' we mean a vector space iso $\sigma : W \to W'$ s.t.

$$(\sigma B - B\sigma) W = 0 \qquad \forall B \in T.$$

The T-modules W, W' are <u>isomorphic</u> whenever there exists a T-module iso from W to W'.

We have a few comments on T

LEM 2

(i) T is generated by A, $\{E_i^*\}_{i=0}^{D}$

(ii) Assume $\{E_i\}_{i=0}^{D}$ is Q-poly. Then T is generated by A, A^*.

Pf (i) A generates M and $\{E_i^*\}_{i=0}^{D}$ is a basis for M^*

(ii) A^* generates M^* $\square$

LEM 3 $\quad \forall\, B \in T$

(i) $\; B^t \in T$

(ii) $\; \overline{B} \in T$

Pf $\quad T$ is generated by real symmetric matrices by LEM 2 (i) $\square$

Given subspaces $\quad U \subseteq W \subseteq V$

obs $\quad \exists$ unique subspace $U' \subseteq W$ s.t.

$$W = U + U' \qquad \text{(orthog dir sum)}$$

We have

$$U' = \{ w \in W \,/\, \langle u, w \rangle = 0 \quad \forall u \in U \}$$

Call U' the <u>orthogonal complement of U in W</u>

LEM 4 Given a T-module W and a T-submodule $U \subseteq W$. the orthogonal complement of U in W is a T-module.

Pf Call it U'. Given $B \in T$ and $u' \in U'$ check $Bu' \in U'$. $\forall u \in U$

$$\langle Bu', u\rangle \overset{?}{=} 0$$

$$\overset{\shortparallel}{\langle u', \underbrace{\bar{B}^t u}_{\in U} \rangle} = 0 \checkmark \qquad \begin{array}{l} \bar{B}^t \in T \\ \bar{B}^t u \in U \end{array}$$

$\square$

COR 5 Each T-module is an orthog direct sum of irred T-modules. In particular V is an orthog direct sum of irred T-modules.

$\square$

LEM 6 For a T-module W

(i) W is the orthog direct sum of the non0 spaces among $\{E_i^* W \mid 0 \le i \le D\}$

(ii) W is the orthogonal direct sum of the non0 spaces among $\{E_i W \mid 0 \le i \le D\}$

Pf (i) The $E_i^* W$ are mutually orthog since $E_i^* W \subseteq E_i^* V$ and the $E_i^* V$ are mutually orthog.

Check $W = \sum_{i=0}^{D} E_i^* W$

$\supseteq$: Each $E_i^* \in T$ and $TW \subseteq W$

$\subseteq$ $\forall w \in W$

$$w = Iw = \sum_{i=0}^{D} E_i^* w$$

(ii) Sim to (i) $\square$

We will return to T-modules in a moment.

LEM 7

(i) For $0 \le i,j \le D$

$$A_i E_j^* V \subseteq \sum_{\substack{0 \le h \le D \\ p_{ij}^h \ne 0}} E_h^* V$$

(ii) For $0 \le i \le D$

$$A E_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V$$

(iii) For $0 \le i,j \le D$

$$A_i^* E_j V \subseteq \sum_{\substack{0 \le h \le D \\ q_{ij}^h \ne 0}} E_h V.$$

(iv) Assume $\{E_i\}_{i=0}^D$ is Q-poly. Then for $0 \le i \le D$

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V$$

Pf (i) $\forall\, y \in \Gamma_j(x)$

$$A_i \hat{y} = \sum_{z \in \Gamma_i(y)} \hat{z}$$

$\forall\, z \in \Gamma_i(y)$ let $h = \partial(x,z)$ and obs $p_{ij}^h \ne 0$

(ii) Set $i=1$ in (i)

(iii) $\forall v \in E_j V$

$$A_i^* v = |X| E_i \hat{x} \circ v \qquad \text{by L 91}$$

$$\in E_i V \circ E_j V$$

$$= \sum_{\substack{0 \le h \le D \\ q_{ij}^h \ne 0}} E_h V \qquad \text{by Th 92.}$$

(iv) Set $i=1$ in (iii) $\square$

COR 8 For an irred T-module W

(i) $A E_i^* W \subseteq E_{i-1}^* W + E_i^* W + E_{i+1}^* W \qquad (0 \le i \le D)$

(ii) Assume $\{E_i\}_{i=0}^{D}$ is Q-poly. Then

$$A^* E_i W \subseteq E_{i-1} W + E_i W + E_{i+1} W \qquad (0 \le i \le D)$$

Pf By L7 (ii), (iv). $\square$

Def 9 Given an irred T-module W

(i) the diameter $d = d(W)$ is

$$d = \left| \left\{ i \mid 0 \le i \le D,\ E_i^* W \ne 0 \right\} \right| - 1$$

(ii) the endpoint $r = r(W)$ is

$$r = \min \left\{ i \mid 0 \le i \le D,\ E_i^* W \ne 0 \right\}$$

(iii) the dual diameter is

$$d^* = \left| \left\{ i \mid 0 \le i \le D,\ E_i W \ne 0 \right\} \right| - 1$$

(iv) the dual endpt $t = t(W)$ is

$$t = \min \left\{ i \mid 0 \le i \le D,\ E_i W \ne 0 \right\}$$

[t is defined up to given ordering of the E_i]

LEM 10 Given an irred T-module W with endpt r and diam d

Then

$$E_i^* W \neq 0 \quad \text{iff} \quad r \leq i \leq r+d \qquad (0 \leq i \leq D)$$

pf By contruction $E_i^* W = 0$ for $0 \leq i < r$ and $E_r^* W \neq 0$

Suppose $\exists i$ $(r < i \leq r+d)$ s.t. $E_i^* W = 0$

Set $W' = E_r^* W + E_{r+1}^* W + \cdots + E_{i-1}^* W$.

By constr $W' \neq 0$.

$M^* W' \subseteq W'$

Also $AW' \subseteq W'$ by Cor 8 (i)

so W' is T-module

Now $W' = W$ by the irred of W.

But this contradicts the diameter of W.

So $E_i^* W \neq 0$ $r \leq i \leq r+d$.

Now $E_i^* W = 0$ $r+d < i \leq D$ by the def of d. $\square$

LEM 11 Assume the ordering $\{E_i\}_{i=0}^{D}$ is Q-poly.

Let W denote an irreducible T-module with endpt t and dual diameter d^*. Then

$$E_i W \neq 0 \quad \text{iff} \quad t \leq i \leq d^* + t \qquad (0 \leq i \leq D)$$

pf Similar to the pf of L10 $\square$

Def 12 Given an irred T-module W.

(i) Call W <u>thin</u> whenever

$$\dim E_i^* W \le 1 \qquad (0 \le i \le D)$$

(ii) Call W <u>dual thin</u> whenever

$$\dim E_i W \le 1 \qquad (0 \le i \le D)$$

We now consider a certain T-module called the <u>primary</u> T-module. Recall $\mathbb{1} = \sum_{y \in X} \hat{y}$

For $0 \le i \le D$ define

$$\mathbb{1}_i = \sum_{y \in \Gamma_i(x)} \hat{y}$$

Obs

$$\mathbb{1}_i = E_i^* \mathbb{1}$$

$$= A_i \hat{x}$$

LEM 13 For $0 \le j \le D$

$$|X| E_j \hat{x} = A_j^* \mathbb{1}$$

(call this vector $\mathbb{1}_j^*$)

Pf Recall

$$A_j^* \mathbb{1} = |X| E_j \hat{x} \circ \mathbb{1}$$
$$= |X| E_j \hat{x} \qquad \square$$

Prop 14 $M\hat{x} = M^*\mathbb{1}$ is a thin, dual-thin irreducible T-module with basis $\{\mathbb{1}_i\}_{i=0}^{D}$ and basis $\{\mathbb{1}_i^*\}_{i=0}^{D}$. The module has endpt 0, dual endpt 0, diameter D, dual diameter D.

Pf. $M\hat{x} = M^*\mathbb{1}$ by L13; call this space W. $MW \subseteq W$ since $W = M\hat{x}$. $M^*W \subseteq W$ since $W = M^*\mathbb{1}$. So W is T-module. Suppose W is reducible. By Cor 5 W is ods of irred T-modules. Since $\hat{x} \in W$ these modules cannot all be orthog $\hat{x}$. So one of them has ~~endpt~~ 0 and hence contains $\hat{x}$.

But then it contains $M\hat{x} = W$. So it equals W.

Concerning the remaining assertions, for $0 \le i \le D$

$$E_i^* W = E_i^* M^*$$
$$= \text{Span } E_i^* \mathbb{1} \qquad \text{has dim } 1$$

$$E_i W = E_i M \hat{x}$$
$$= \text{Span } E_i \hat{x} \qquad \text{has dim } 1. \qquad \square$$

$\square$

No. Lec 14-1
Date 2/20/09

Lecture 14 Friday February 20

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Fix a DRG $\Gamma = (X, R)$ diam D

Fix $x \in X$ and write $T = T(x)$, etc.

Last time we considered the irred T-module

$$M\hat{x} = M^* \mathbb{1}$$

Call this T-module <u>primary</u> (or <u>trivial</u>)

We begin a careful study of the primary T-module.

To this end we first obtain some "reduction rules"

involving E_0 and E_0^*.

No. Lec 14-2
Date 2/20/09

Lem 15 The following hold for $0 \le i, j \le D$

(i) $$E_0 A_i^* E_j = \delta_{ij} E_0 A_i^*$$

(ii) $$E_0 E_i^* E_j = |X|^{-1} k_i u_i(\theta_j) E_0 A_j^*$$

(iii) $$E_0 A_i^* A_j = k_j u_j(\theta_i) E_0 A_i^*$$

(iv) $$E_0 E_i^* A_j = \sum_{h=0}^{D} p_{ij}^h E_0 E_h^*$$

Pf (i) Recall

$$E_0 A_i^* E_j = 0 \text{ iff } q_{ij}^0 = 0$$

If $i \neq j$ then $q_{ij}^0 = 0$ so $E_0 A_i^* E_j = 0$

Also

$$E_0 A_i^* = E_0 A_i^* (E_0 + \cdots + E_D)$$

$$= E_0 A_i^* E_i$$

(ii) Use

$$E_i^* = |X|^{-1} \sum_{h=0}^{D} v_i(\theta_h) A_h^*$$

and (i) above.

(iii) Use

$$A_j = \sum_{h=0}^{D} v_j(\theta_h) E_h$$

and (i) above.

(iv) Recall by L 76

$$E_0 E_0^* B = E_0 B \rho \qquad \forall B \in M$$

so

$$E_0 E_0^* A_j = E_0 E_j^*$$

Now

$$E_0 E_i^* A_j = E_0 E_0^* A_i A_j$$

$$= \sum_{h=0}^{D} p_{ij}^h E_0 E_0^* A_h$$

$$= \sum_{h=0}^{D} p_{ij}^h E_0 E_h^* \qquad \square$$

We also have the dual equations

LEM 16 The following hold for $0 \le i, j \le D$.

(i) $E_0^* A_i E_j^* = \delta_{ij} E_0^* A_i$

(ii) $E_0^* E_i E_j^* = |X|^{-1} m_i u_j(\theta_i) E_0^* A_j$

(iii) $E_0^* A_i A_j^* = m_j u_i(\theta_j^*) E_0^* A_i$

(iv) $E_0^* E_i A_j^* = \sum_{h=0}^{D} q_{ij}^h E_0^* E_h$

pf Similar to the proof of L15 $\square$

We mention some special cases of LEM 15, 16

LEM 17 For $0 \le j \le D$

(i) $E_0 E_0^* E_j = |X|^{-1} E_0 A_j^*$

(ii) $E_0 E_0^* A_j = E_0 E_j^*$

(iii) $E_0^* E_0 E_j^* = |X|^{-1} E_0^* A_j$

(iv) $E_0^* E_0 A_j^* = E_0^* E_j$

pf Use L15, L16 $\square$

LEM 18

(i) $E_0 E_i^* E_0 = |X|^{-1} k_i E_0$ $(0 \le i \le D)$

(ii) $E_0 E_0^* E_0 = |X|^{-1} E_0$

(iii) $E_0^* E_i E_0^* = |X|^{-1} m_i E_0^*$ $(0 \le i \le D)$

(iv) $E_0^* E_0 E_0^* = |X|^{-1} E_0^*$

pf use L15, 16 $\square$

Note more reduction rules come by taking the transpose in L15–17

The following reduction rules are of a slightly different nature

LEM 19 For $0 \le i, j \le D$

(i) $A_i E_0^* A_j = |X| E_i^* E_0 E_j^*$

(ii) $E_i E_0^* A_j = A_i^* E_0 E_j^*$

(iii) $A_i E_0^* E_j = E_i^* E_0 A_j^*$

(iv) $E_i E_0^* E_j = |X|^{-1} A_i^* E_0 A_j^*$

Pf (i) $A_i E_0^* A_j = |X| A_i E_0^* E_0 E_j^*$ by L17 (iii)

$= |X| E_i^* E_0 E_j^*$ by L17 (ii) transpose

(ii) - (iv) Similar. $\square$

LEM 20 We have

$$\sum_{i=0}^{D} k_i^{-1} E_i^* E_0 E_i^* = \sum_{j=0}^{D} m_j^{-1} E_j E_0^* E_j \qquad (*)$$

Pf LHS $= |X|^{-1} \sum_{i=0}^{D} k_i^{-1} A_i E_0^* A_i$

$$= |X|^{-1} \sum_{i=0}^{D} k_i^{-1} \left(\sum_{r=0}^{D} v_i(\theta_r) E_r \right) E_0^* \left(\sum_{s=0}^{D} v_i(\theta_s) E_s \right)$$

$$= |X|^{-1} \sum_{r=0}^{D} \sum_{s=0}^{D} E_r E_0^* E_s \left(\underbrace{\sum_{i=0}^{D} k_i^{-1} v_i(\theta_r) v_i(\theta_s)}_{\delta_{rs} m_r^{-1} |X|} \right)$$

$$= \sum_{j=0}^{D} m_j^{-1} E_j E_0^* E_j$$

□

DEF 21 Define $e_0 = e_0(x)$ to be $|X|$ times the common value in $(*)$. We obs $e_0 \in T$ and

$$\overline{e_0} = e_0, \qquad e_0^t = e_0$$

For the moment let W denote the primary T-module

Obs by LEM 4

$$V = W + W^{\perp} \qquad \text{(orthog dir sum of } T\text{-modules)}$$

where $W^{\perp}$ = orthog complement of W in V.

LEM 22 With above notation

(i) $(e_0 - I)W = 0$,

(ii) $e_0 W^{\perp} = 0$.

In other words e_0 acts on V as the orthogonal projection $V \to W$.

pf (i) W has basis $\{\mathbb{1}_i\}_{i=0}^{D}$

For $0 \le h \le D$

$$e_0 \mathbb{1}_h = |X| \sum_{i=0}^{D} k_i^{-1} E_i^* E_0 E_i^* \mathbb{1}_h$$

$$= |X| k_h^{-1} E_h^* E_0 \underset{A_h \hat{x}}{\underset{\shortparallel}{\mathbb{1}_h}}$$

$$= |X| k_h^{-1} E_h^* \underbrace{v_h(\theta_0)}_{k_h} E_0 \hat{x}$$

$$= |X| E_h^* E_0 \hat{x}$$

$$= E_h^* J \hat{x}$$

$$= E_h^* \mathbb{1}$$

$$= \mathbb{1}_h$$

(ii) Given $v \in W^\perp$ show $e_0 v = 0$

$e_0 \in T$ and $W^\perp$ is T-module so $e_0 v \in W^\perp$

By construction $e_0 \in M E_0^* M$ so

$$e_0 v \in M E_0^* M V$$
$$\subseteq M E_0^* V$$
$$\subseteq M \hat{x} \qquad E_0^* V = \mathrm{Span}(\hat{x})$$
$$= W$$

Now $e_0 v \in W \cap W^\perp = 0$ $\square$

Recall the center

$$Z(T) = \{ B \in T \mid Bt = tB \ \forall t \in T \}$$

is a subalgebra of T.

COR 23 We have

(i) $e_0 \in Z(T)$

(ii) $e_0^2 = e_0$

(iii) $e_0 V$ = primary module

(iv) rank $e_0 = D+1$

Pf (i) $\forall B \in T$ the expression $Be_0 - e_0 B$

vanishes on W and $W^\perp$, where W = prim module.

But $V = W + W^\perp$ so $Be_0 - e_0 B$ vanishes on V.

(ii), (iii) By L22

(iv) Prim module has dim $D+1$. $\square$

From now on we denote the primary module by $e_0 V$

Recall e_0V has a basis $\{\mathbb{1}_i\}_{i=0}^{D}$ and a basis $\{\mathbb{1}_i^*\}_{i=0}^{D}$

LEM 24 For $0 \le i,j \le D$,

(i) $\langle \mathbb{1}_i, \mathbb{1}_j \rangle = \delta_{ij} k_i$

(ii) $\langle \mathbb{1}_i^*, \mathbb{1}_j^* \rangle = \delta_{ij} |X| m_i$

(iii) $\langle \mathbb{1}_i, \mathbb{1}_j^* \rangle = k_i m_j u_i(\theta_j)$

pf (i) clear

(ii) $$\text{LHS} = |X|^2 \langle E_i \hat{x}, E_j \hat{x} \rangle$$
$$= |X|^2 \langle \hat{x}, E_i E_j \hat{x} \rangle$$
$$= \delta_{ij} |X|^2 \langle x, E_i \hat{x} \rangle$$
$$= \delta_{ij} |X|^2 \Big(\underbrace{(x,x)\text{-entry of } E_i}_{m_i |X|^{-1}} \Big)$$

(iii) $$\text{LHS} = |X| \langle A_i \hat{x}, E_j \hat{x} \rangle$$
$$= |X| \langle \hat{x}, A_i E_j \hat{x} \rangle$$
$$= |X| v_i(\theta_j) \langle \hat{x}, E_j \hat{x} \rangle$$
$$= |X| v_i(\theta_j) \; |X|^{-1} m_j$$
$$= m_j v_i(\theta_j) = k_i m_j u_i(\theta_j)$$ $\square$

We now give the action of the T-generators on the basis $\{\mathbb{1}_i\}_{i=0}^{D}$

LEM 25 For $0 \le i, j \le D$

(i) $E_i^* \mathbb{1}_j = \delta_{ij} \mathbb{1}_j$

(ii) $A_i^* \mathbb{1}_j = m_i u_j(\theta_i) \mathbb{1}_j$

(iii) $E_i \mathbb{1}_j = |X|^{-1} m_i k_j u_j(\theta_i) \sum_{h=0}^{D} u_h(\theta_i) \mathbb{1}_h$

(iv) $A_i \mathbb{1}_j = \sum_{h=0}^{D} p_{ij}^h \mathbb{1}_h$

Pf (i) clear

(ii) Use (i) and $A_i^* = m_i \sum_{h=0}^{D} u_h(\theta_i) E_h^*$

(iii) Use
$$E_i \mathbb{1}_j = E_i A_j \hat{x} = v_j(\theta_i) E_i \hat{x}$$

and

$$v_j(\theta_i) = k_j u_j(\theta_i), \qquad E_i \hat{x} = |X|^{-1} m_i \sum_{h=0}^{D} u_h(\theta_i) A_h \hat{x} = |X|^{-1} m_i \sum_{h=0}^{D} u_h(\theta_i) \mathbb{1}_h$$

(iv)
$$A_i \mathbb{1}_j = A_i A_j \hat{x} = \sum_{h=0}^{D} p_{ij}^h A_h \hat{x} = \sum_{h=0}^{D} p_{ij}^h \mathbb{1}_h \qquad \square$$

We now give the action of the T-generators on the basis $\{ \mathbb{1}_i^* \}_{i=0}^{D}$

LEM 26 For $0 \le i, j \le D$

(i) $E_i \mathbb{1}_j^* = \delta_{ij} \mathbb{1}_j^*$

(ii) $A_i \mathbb{1}_j^* = k_i u_i(\theta_j) \mathbb{1}_j^*$

(iii) $E_i^* \mathbb{1}_j^* = |X|^{-1} k_i m_j u_i(\theta_j) \sum_{h=0}^{D} u_i(\theta_h) \mathbb{1}_h^*$

(iv) $A_i^* \mathbb{1}_j^* = \sum_{h=0}^{D} q_{ij}^h \mathbb{1}_h^*$

Pf Similar to L25 $\square$

We now show how to transition between the bases

$\{\mathbb{1}_i\}_{i=0}^{D}$ and $\{\mathbb{1}_i^*\}_{i=0}^{D}$

LEM 27 For $0 \le j \le D$

(i) $$\mathbb{1}_j = |X|^{-1} k_j \sum_{i=0}^{D} u_j(\theta_i)\, \mathbb{1}_i^*$$

(ii) $$\mathbb{1}_j^* = m_j \sum_{i=0}^{D} u_i(\theta_j)\, \mathbb{1}_i$$

pf (i) $\mathbb{1}_j = A_j \hat{x}$

Recall $A_j = k_j \sum_{i=0}^{D} u_j(\theta_i)\, E_i$

and $E_i \hat{x} = |X|^{-1} \mathbb{1}_i^* \quad \forall i$

(ii) Similar $\square$

Referring to LEM 27

as special cases we mention

$$\hat{x} = \mathbb{1}_0 = |X|^{-1} \sum_{i=0}^{D} \mathbb{1}_i^*$$

and

$$\mathbb{1} = \mathbb{1}_0^* = \sum_{i=0}^{D} \mathbb{1}_i$$

We emphasize the role of the polynomials

LEM 28

(i) We have

$$\mathbb{1}_i = v_i(A)\,\mathbb{1}_0 \qquad (0 \le i \le D)$$

(ii) Assume $\{E_i\}_{i=0}^{D}$ is Q-poly. Then

$$\mathbb{1}_i^* = v_i^*(A^*)\,\mathbb{1}_0^* \qquad (0 \le i \le D)$$

Pf (i) $\quad \mathbb{1}_i = A_i \hat{x}$

Recall $\quad A_i = v_i(A)$

and $\quad \hat{x} = \mathbb{1}_0$

(ii) $\quad \mathbb{1}_i^* = A_i^* \mathbb{1}$

Recall $\quad A_i^* = v_i^*(A^*)$

and

$$\mathbb{1} = \mathbb{1}_0^*$$

$\square$

We mention a special case of LEM 25, 26

LEM 29

(i) With respect to the bases $\{\mathbb{1}_i\}_{i=0}^D$ and $\{\mathbb{1}_i^*\}_{i=0}^D$ the matrices representing A are

$$\begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{D-1} \\ \mathbf{0} & & & c_D & a_D \end{pmatrix} \qquad \begin{pmatrix} \theta_0 & & & \mathbf{0} \\ & \theta_1 & & \\ & & \ddots & \\ \mathbf{0} & & & \theta_D \end{pmatrix}$$

respectively.

(ii) Assume $\{E_i\}_{i=0}^D$ is Q-polynomial. Then with respect to the bases $\{\mathbb{1}_i\}_{i=0}^D$ and $\{\mathbb{1}_i^*\}_{i=0}^D$ the matrices rep A^* are

$$\begin{pmatrix} \theta_0^* & & & \mathbf{0} \\ & \theta_1^* & & \\ & & \ddots & \\ \mathbf{0} & & & \theta_D^* \end{pmatrix} \qquad \begin{pmatrix} a_0^* & b_0^* & & & \mathbf{0} \\ c_1^* & a_1^* & b_1^* & & \\ & c_2^* & \ddots & \ddots & \\ & & \ddots & \ddots & b_{D-1}^* \\ \mathbf{0} & & & c_D^* & a_D^* \end{pmatrix}$$

Pf clear $\square$

Aside. In the Q-polynomial case the actions of A and A^* on E_0V give an example of a linear-algebraic object called a <u>Leonard pair</u>. A Leonard pair is defined as follows.

For the moment let $\mathbb{F}$ be any field.

Let m denote a square matrix over $\mathbb{F}$.

Call m <u>tridiagonal</u> whenever each non 0 entry of m is on the diagonal, the subdiagonal, or the superdiagonal.

Assume m is tridiagonal. Call m <u>irreducible</u> whenever each entry on the subdiagonal is non 0 and each entry on the superdiagonal is non 0.

DEF 30 Given a finite-dimensional non 0 vector space V over $\mathbb{F}$

A <u>Leonard pair</u> on V is an ordered pair of linear trans $A: V \to V$ and $A^*: V \to V$ s.t.

(i) $\exists$ basis for V wrt which the matrix representing A is irred tridiag and the matrix representing A^* is diag.

(ii) $\exists$ basis for V wrt which the matrix representing A^* is irred tridiag and the matrix representing A is diag.

Back to DRG's $\quad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Given DRG $\Gamma = (X, R)$ diam D

Fix $x \in X$, write $T = T(x)$ etc.

COR 31 $\quad$ Referring to LEM 29 assume $\{E_i\}_{i=0}^{D}$ is Q-polynomial. Then the pair A, A^* acts on the primary module $e_0 V$ as a Leonard pair.

Pf $\quad$ Imed from L29, Def 30 $\quad \square$

We classified the LP's in 2000. We will return to LP's shortly. $\quad \square$

Lecture 15 Monday February 23

No. Lec 15-1
Date 2/23/09

We have some comments on general irred T-modules.

Prop 32 Let W, U denote nonorthogonal irreducible T-modules. Then W, U are isomorphic as T-modules.

Pf. $\forall w \in W$ let $\sigma(w)$ denote the orthogonal projection of w into U. So $\sigma(w)$ is the unique vector in V s.t.

$$\sigma(w) \in U \qquad \text{and} \qquad w - \sigma(w) \in U^{\perp}$$

Show

$$\sigma : \begin{array}{ccc} W & \to & U \\ w & \to & \sigma(w) \end{array}$$

is a T-module iso.

<u>claim 1</u>:

$$(B\sigma - \sigma B) W = 0 \qquad \forall B \in T$$

<u>pf cl</u>: $\forall w \in W$

$$B\sigma(w) \in U \qquad \text{since } U \text{ is T-module}$$

and

$$Bw - B\sigma(w) \in U^{\perp} \qquad \text{since } U^{\perp} \text{ is T-module}$$

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so

$$B\sigma(w) = \sigma(Bw)$$

claim 1 is proved.

claim 2 $\sigma: W \to U$ is injective.

pf claim 2 Let K denote the kernel of σ on W.

Using claim 1 K is a T-submodule of W

so by the irred of W

$$K = 0 \text{ or } K = W$$

But $K \neq W$ since $\langle W, U \rangle \neq 0$ so $K = 0$

claim 3 $\sigma: W \to U$ is surjective

pf claim 3 Let Im denote the image of σ on W.

Using claim 1 Im is a T-submodule of U.

By the irred of U,

$$Im = 0 \text{ or } Im = U.$$

But $Im \neq 0$ since $\langle W, U \rangle \neq 0$ so $Im = U$. $\square$

Notation Let $\Psi = \Psi(x)$ denote the set of isomorphism classes of irreducible T-modules.

The elements of Ψ are called types

For $\phi \in \Psi$ define

V_ϕ = subspace of V spanned by the irreducible T-modules of type ϕ.

Call V_ϕ the ϕ-homogeneous component of V

Observe V_ϕ is a T-module.

By Prop 32

$$V = \sum_{\phi \in \Psi} V_\phi \qquad \text{(orthog dir sum of } T\text{-modules)}$$

Given $\phi \in \Psi$ and an irreducible T-module $W \subseteq V_\phi$ the dimension, diameter, endpt etc of W depends only on ϕ. So we often denote these by

$\dim \phi$, $d(\phi)$, $r(\phi)$ etc.

Write V_ϕ as an orthog direct sum of irreducible T-modules

$$V_\phi = W_1 + W_2 + \cdots + W_m \qquad (*)$$

[this decomp is not unique]

For $1 \le i \le m$ W_i has type ϕ so

$$\dim(V_\phi) = m \dim(\phi)$$

ie

$$m = \frac{\dim(V_\phi)}{\dim(\phi)}$$

In particular m is independent of which decomposition is used in $(*)$

We call m the <u>multiplicity of ϕ in V</u>

For example the primary module has multiplicity 1.

Another view of T

For any subgroup $G \subseteq \mathrm{Aut}(\Gamma)$

Recall the centralizer algebra

$$C_G = \{ B \in \mathrm{Mat}_X(\mathbb{F}) \mid B\sigma = \sigma B \ \forall \sigma \in G \}$$

Put

$$G = \text{Stabilizer of } x \text{ in } \mathrm{Aut}(\Gamma)$$

Then

$$T \subseteq C_G$$

Since

$$A \in C_{\mathrm{Aut}(\Gamma)} \subseteq C_G$$

$$E_i^* \in C_G \qquad 0 \le i \le D$$

and $\quad T$ gen by $\quad A, \{E_i^*\}_{i=0}^{D}$

As we will see, in some cases

$$T = C_G.$$

In any case we view T as a <u>combinatorial</u> <u>analog</u>

<u>of C_G.</u>

Extended example

Until further notice $\Gamma = (X, R)$ will be the complete graph K_n

Fix $x \in X$ write $T = T(x)$ etc. $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

We will describe T.

For $n = 2$, Γ consists of single edge and $T \cong \mathrm{Mat}_{2\times 2}(\mathbb{F})$

Now assume $n \geq 3$.

We have

$$E_0 = n^{-1} J \qquad\qquad E_1 = I - E_0 = I - n^{-1} J$$

$$A_0 = I \qquad\qquad A = J - I$$

$$A_0 = E_0 + E_1 \qquad\qquad E_0 = \frac{1}{n} A_0 + \frac{1}{n} A$$

$$A = (n-1) E_0 - E_1 \qquad\qquad E_1 = \frac{n-1}{n} A_0 - \frac{1}{n} A$$

$$E_0^* = \begin{array}{cc} & \begin{array}{cc} x & \Gamma(x) \end{array} \\ \begin{array}{c} x \\ \Gamma(x) \end{array} & \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \end{array} \qquad\qquad E_1^* = I - E_0^* = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{matrix} \end{array}\right)$$

$$A_0^* = I \qquad\qquad A^* = nE_0^* - A_0^* \qquad (A^* = A_1^*)$$

$$= \left(\begin{array}{c|c} n-1 & 0 \\ \hline 0 & \begin{matrix} -1 & & 0 \\ & -1 & \\ 0 & & \ddots \\ & & & -1 \end{matrix} \end{array}\right)$$

$$A_0^* = E_0^* + E_1^* \qquad\qquad E_0^* = \frac{1}{n} A_0^* + \frac{1}{n} A^*$$

$$A^* = (n-1) E_0^* - E_1^* \qquad\qquad E_1^* = \frac{n-1}{n} A_0^* - \frac{1}{n} A^*$$

By L18

$$n E_0^* E_0 E_0^* = E_0^*, \qquad n E_0 E_0^* E_0 = E_0$$

the idempotent e_0 from Def 21 is

$$e_0 = n\left(E_0^* E_0 E_0^* + k^{-1} E_1^* E_0 E_1^* \right) \qquad k = n-1$$

$$= \left(\begin{array}{c|ccc} 1 & & 0 & \\ \hline & \ddots & & \\ 0 & & \frac{1}{n-1} & \\ & & & \ddots \end{array} \right)$$

$$= \frac{n}{n-1}\left(E_0 + E_0^* - E_0 E_0^* - E_0^* E_0 \right)$$

LEM 33 We have

(i) $(e_0 V)^\perp = E_1 V \cap E_1^* V$

(ii) Each 1-diml subspace of $E_1 V \cap E_1^* V$ is

an irred T-module on which A, A^* act as $-I$

Pf (i) $\subseteq$:

$$E_0 V \subseteq e_0 V$$

so

$$(e_0 V)^\perp \subseteq (E_0 V)^\perp = E_1 V$$

Sim

$$(e_0 V)^\perp \subseteq E_1^* V$$

$\supseteq$: Obs $E_1 V \cap E_1^* V$ is a T-module

so it is a dsum of irred T-modules.

These modules have endpt 1 so are not primary.

These modules are orthog to $e_0 V$ by Prop 32

(ii) $E_1 V$ is eigspace for A with eigval -1, A, A^* gen T.
$E_1^* V$ is eigspace for A^* with eigval -1, □

COR 34

(i) Up to iso $\exists$ unique non primary irred T-module W.

(ii) W has dim 1,

endpt 1, dual endpt 1,

diam 0, dual diam 0

(iii) Each of A, A^* act on W as $-I$,

(iv) W appears in V with mult $n-2$

(v) the corresp homogenous component of V is $E_1 V \cap E_1^* V$

Pf Routine using L33. $\square$

Notation For K_N denote set of iso classes of T-modules by

$$\Psi = \{\underset{\text{primary}}{0}, \underset{\text{non primary}}{1}\}$$

obs $e_1 := I - e_0$ is the projection onto the non-prim homog component.

LEM 35 The matrices

$$E_0^*, \quad E_1^*, \quad E_0^* J E_1^*, \quad E_1^* J E_0^*, \quad E_1^* J E_1^* - E_1^* \qquad (*)$$

form a basis for T that is orthogonal wrt to inner product $\langle , \rangle$ from L50. Moreover $\dim(T) = 5$.

Pf Obs

$$E_0^* = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \qquad E_1^* = \left(\begin{array}{c|ccc} 0 & & 0 & \\ \hline & 1 & & 0 \\ 0 & & \ddots & \\ & 0 & & 1 \end{array}\right)$$

$$E_0^* J E_1^* = \left(\begin{array}{c|cccc} 0 & 1 & 1 & \cdots & 1 \\ \hline 0 & & & 0 & \end{array}\right) \qquad E_1^* J E_0^* = \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & \\ 1 & \\ \vdots & 0 \\ 1 & \end{array}\right)$$

$$E_1^* J E_1^* - E_1^* \quad \left(\begin{array}{c|cccc} 0 & & & 0 & \\ \hline & 0 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 1 & \\ & 1 & 1 & \ddots & \\ & \vdots & & & 0 \end{array}\right)$$

Above five matrices have non 0 entries in mutually disjoint positions, so they are mutually orthogonal. One checks the span of $*$ is closed under mult and contains A, A^*, so it is T. $\square$

LEM 36 Each of the following is a basis for T

(i) $I, E_0, E_0^*, E_0E_0^*, E_0^*E_0$

(ii) I, A, A^*, AA^*, A^*A

Pf Ex

LEM 37 the $\mathbb{F}$-algebra T is iso to

$$\mathrm{Mat}_{2\times 2}(\mathbb{F}) \oplus \mathbb{F}$$

Pf We display an algebra iso

$$\sigma : \mathrm{Mat}_{2\times 2}(\mathbb{F}) \oplus \mathbb{F} \to T$$

y	$\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&0\\0&1\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}, 1\right\}$
$\sigma(y)$	E_0^*	$\dfrac{E_0^* J E_1^*}{n-1}$	$\dfrac{E_1^* J E_0^*}{n-1}$	$\dfrac{E_1^* J E_1^*}{n-1}$	e_1

$\square$

Recall $\mathrm{Aut}(K_n)$ is symmetric group S_n

Let G denote the stabilizer of x in $\mathrm{Aut}(K_n)$

and obs G iso S_{n-1}

LEM 38 With above notation

$$T = C_G \qquad (\text{Centralizer algebra of } G)$$

pf One checks C_G has basis consisting of the matrices

(*) in L35. $\square$

We now show that for K_n, T has an inner automorphism that swaps A, A^*

We define $\Delta = \Delta(x) \in \mathrm{Mat}_X(\mathbb{F})$ by

$$\Delta = \left(\begin{array}{c|cccc} 1 & 1 & \cdots & & 1 \\ \hline 1 & 1-n & 1 & 1 & \cdots \\ \vdots & 1 & 1-n & 1 & \\ \vdots & 1 & 1 & \ddots & \\ \vdots & \vdots & & & \\ 1 & & & & 1-n \end{array}\right)$$

Obs

$$\Delta^t = \Delta, \qquad \overline{\Delta} = \Delta$$

One checks Δ is invertible with

$$\Delta^{-1} = n^{-1} \left(\begin{array}{c|cccc} 1 & 1 & 1 & \cdots & 1 \\ \hline 1 & -1 & & & 0 \\ 1 & & -1 & & \\ \vdots & & 0 & \ddots & \\ 1 & & & & -1 \end{array}\right)$$

One checks

$$\Delta = n(E_0 - E_1^*) = n(E_0^* - E_1)$$
$$= n(E_0 + E_0^* - I) = A + A^* + (2-n)I$$

LEM 39 We have

(i) $A\Delta = \Delta A^*$

(ii) $A^*\Delta = \Delta A$

Pf (i) Simplify each side using

$$\Delta = n(E_0 + E_0^* - I),$$

$$A = nE_0 - I, \qquad A^* = nE_0^* - I.$$

(ii) By (i) and since each of Δ, A, A^* is symmetric. $\square$

COR 40 For $i = 0, 1$

(i) $\Delta E_i^* V = E_i V$

(ii) $\Delta E_i V = E_i^* V$

Pf By L39 and since Δ^{-1} exists $\square$

Lecture 16 Wednesday February 25

By L39 we see $\Delta^2 \in Z(T)$

One finds

$$\Delta^2 = n \left(\begin{array}{c|cccc} 1 & & 0 & & \\ \hline & n-1 & -1 & -1 & \cdots \\ 0 & -1 & n-1 & -1 & \\ & -1 & -1 & n-1 & \\ & \vdots & & & \ddots \\ & & & & n-1 \end{array} \right)$$

$$= n^2 \left(I - E_0 - E_0^* + E_0 E_0^* + E_0^* E_0 \right)$$

$$= n e_0 + n^2 e_1$$

$\forall y \in X$ write

$$\check{y} = \Delta \hat{y}$$

Since Δ^{-1} exists

$$\{ \check{y} \mid y \in X \} \quad \text{is a basis for } V$$

By Cor 40

$$\check{x} \quad \text{is basis for } E_0 V$$

$$\{ \check{y} \mid y \in \Gamma(x) \} \quad \text{is basis for } E_1 V$$

By the def of Δ

$$\check{x} = \mathbb{1}$$

$$= n E_0 \hat{x}$$

Also $\forall y \in \Gamma(x)$

$$\check{y} = \mathbb{1} - n \hat{y}$$

$$= -n E_1 \hat{y}$$

Next goal Recall the function algebra $V, \circ$ for K_n.

What does $\circ$ look like with respect to the basis $\{\check{y} \mid y \in X\}$?

LEM 41 Referring to K_n

(i) $\forall y \in X$

$$E_1\hat{y} \circ E_1\hat{y} = \frac{n-2}{n} E_1\hat{y} + \frac{n-1}{n^2} \mathbb{1}$$

(ii) For distinct $y, z \in X$

$$E_1\hat{y} \circ E_1\hat{z} = -\frac{1}{n} E_1\hat{y} - \frac{1}{n} E_1\hat{z} - \frac{1}{n^2} \mathbb{1}$$

Pf (i) Recall $E_1 = I - E_0$, $E_0 = n^{-1} J$ so

$$E_1\hat{y} = \hat{y} - n^{-1}\mathbb{1}$$

$$\underbrace{\left(\hat{y} - \tfrac{1}{n}\mathbb{1}\right) \circ \left(\hat{y} - \tfrac{1}{n}\mathbb{1}\right)}_{\hat{y} - \frac{1}{n}\hat{y} - \frac{1}{n}\hat{y} + \frac{1}{n^2}\mathbb{1}} \overset{?}{=} \frac{n-2}{n}\left(\hat{y} - \tfrac{1}{n}\mathbb{1}\right) + \frac{n-1}{n^2}\mathbb{1}$$

(ii) Sim $\square$

LEM 42 Referring to K_n.

(i) $\check{x} \circ u = u \quad \forall u \in V$

(ii) For $y \in \Gamma(x)$

$$\check{y} \circ \check{y} = (2-n)\check{y} + (n-1)\check{x}$$

(iii) For dist $y, z \in \Gamma(x)$

$$\check{y} \circ \check{z} = \check{y} + \check{z} - \check{x}$$

pf (i) $\check{x} = \mathbb{1}$

(ii) By L42 (i) and $\check{y} = -n E_1 \hat{y}$

(iii) Sim $\square$

For K_n find the Norton algebra structure on $E_1 V$

LEM 43 Ref to K_n

(i) $\forall y \in \Gamma(x)$

$$\check{y} * \check{y} = (2-n)\check{y}$$

(ii) $\forall$ dist. $y, z \in \Gamma(x)$

$$\check{y} * \check{z} = \check{y} + \check{z}$$

pf Recall $\forall u, v \in E_1 V$

$$u * v = E_1(u \circ v)$$

Now use L43. $\square$

$\square$

Lecture 17 Friday February 27

Assume $\Gamma = (X, R)$ is K_n, $n \geq 3$

$\mathbb{F} = \mathbb{C}$ Fix $x \in X$ write $T = T(x)$ etc.

To motivate our next results, consider this problem:

Find invertible

$$W \in M, \quad W^* \in M^*$$

such that

$$W A^* W^{-1} = (W^*)^{-1} A W^*$$

Obs each of W, W^* is defined up to multiplication by a nonzero scalar,

Write

$$W = (r E_0 + I) f$$

$$W^* = (r^* E_0^* + I) f^*$$

Nonzero $r, r^*, f, f^* \in \mathbb{F}$ $r \neq -1$, $r^* \neq -1$

f, f^* free. Find r, r^*.

LEM 44 With above notation TFAE

(i) $$W A^* W^{-1} = (W^*)^{-1} A W^*$$

(ii) $r = r^*$ and $n = -\dfrac{r^2}{r+1}$

pf (i) $\to$ (ii) Obs

$$W^* W A^* = A W^* W$$

Expanding this we get

$$\left(r^* E_0^* + I\right)\left(r E_0 + I\right)\left(n E_0^* - I\right) = \left(n E_0 - I\right)\left(r^* E_0^* + I\right)\left(r E_0 + I\right)$$

Write each side as a linear comb of basis

$$I,\ E_0,\ E_0^*,\ E_0 E_0^*,\ E_0^* E_0$$

Using

$$n E_0 E_0^* E_0 = E_0, \qquad n E_0^* E_0 E_0^* = E_0^*$$

Compare coeffs.

(ii) $\to$ (i) Routine verification. $\square$

In what follows we assume W, W^* satisfy conditions (i), (ii) in LEM 44. For convience we take

$$f=1, \qquad f^*=1$$

and make a variable substitution

$$r = q^2-1 \qquad q \in \mathbb{F}$$

Recall $r \neq 0, -1$ so

$$q \neq 0, \qquad q \neq 1, \qquad q \neq -1$$

Obs

$$n = \frac{-r^2}{r+1}$$

$$= -(q-q^{-1})^2$$

We have

$$W = (q^2-I)E_0 + I$$

$$= q^2 E_0 + E_1$$

$$W^{-1} = q^{-2}E_0 + E_1$$

Sim

$$W^* = q^2 E_0^* + E_1^*, \qquad (W^*)^{-1} = q^{-2}E_0^* + E_1^*$$

Note $\quad q$ is defined up to sign and reciprocal.

A given solution q can be replaced by

$$-q, \quad q^{-1}, \quad -q^{-1}$$

We have

$$W(-q) = W(q) \qquad W^*(-q) = W^*(q)$$

$$W(q^{-1}) = (W(q))^{-1} \qquad W^*(q^{-1}) = (W^*(q))^{-1}$$

LEM 45 $\quad$ We have

$$W = \frac{q^{-1}A_0 + qA_1}{q^{-1} - q}$$

$$W^{-1} = \frac{qA_0 + q^{-1}A_1}{q - q^{-1}}$$

pf $\quad$ To verify each equation use

$$A_0 = I, \quad A_1 = nE_0 - I \qquad \square$$

COR 46 We have

$$W \circ W^{-1} = n^{-1} J$$

entrywise

"nW^{-1} is inverse of W in the algebra $M, \circ$"

Pf Use L45 and recall $A_i \circ A_j = \delta_{ij} A_i \quad \forall i,j$ $\square$

Here is another curious fact.

LEM 47 We have

$$W W^* W = W^* W W^*$$

pf Write each side as a linear combination of the basis $I, E_0, E_0^*, E_0 E_0^*, E_0^* E_0$ and compare coeffs.

Detail: Each side is

term	I	E_0	E_0^*	$E_0 E_0^*$	$E_0^* E_0$
coeff	1	q^2-1	q^2-1	$q^2(q-q^{-1})^2$	$q^2(q-q^{-1})^2$

$\square$

Note the matrix

$$ww^*w = w^*ww^*$$

is

$$\frac{-q}{q-q^{-1}}\left(\begin{array}{c|ccccc}
q^2 & q^2 & q^2 & \cdots & & q^2 \\
\hline
q^2 & q^{-2} & 1 & 1 & \cdots & \\
q^2 & 1 & q^{-2} & 1 & & \\
\vdots & 1 & 1 & q^{-2} & & \\
\vdots & \vdots & & & \ddots & \\
q^2 & & & & & q^{-2}
\end{array}\right)$$

Here is another view

LEM 48 the matrix

$$WW^*W = W^*WW^*$$

is equal to

$$\Delta\left(\frac{-q^3}{q-q^{-1}}e_0 + \frac{1}{(q-q^{-1})^2}e_1\right) \qquad (*) \quad \checkmark$$

Pf Write $(*)$ in the basis $I, E_0, E_0^*, E_0E_0^*, E_0^*E_0$ using

$$\Delta = n(E_0 + E_0^* - I),$$

$$e_0 = \frac{n}{n-1}\left(E_0 + E_0^* - E_0E_0^* - E_0^*E_0\right)$$

$$e_1 = I - e_0$$

$\square$

LEM 49 We have

$$(ww^*w)^2 = (ww^*)^3$$

$$= -q^6 e_0 + e_1$$

$$\in Z(T)$$

pf Obs

$$(ww^*w)^2 = (ww^*w)(w^*ww^*)$$

$$= (ww^*)^3$$

By L48

$$(ww^*w)^2 = \Delta^2 \left(\frac{-q^3}{q-q^{-1}} e_0 + \frac{1}{(q-q^{-1})^2} e_1 \right)^2$$

Reduce this using

$$\Delta^2 = n e_0 + n^2 e_1$$

$$e_0^2 = e_0, \qquad e_0 e_1 = e_1 e_0 = 0, \qquad e_1^2 = e_1$$

$$n = -(q-q^{-1})^2$$

and get $-q^6 e_0 + e_1$ $\square$

Motivated by LEM 49 we consider the following automorphism of T:

$$\rho : \begin{array}{ccc} T & \longrightarrow & T \\ m & \longrightarrow & (ww^*)\, m\, (ww^*)^{-1} \end{array}$$

By L49 ρ has order 3.

We will return to ρ shortly.

DEF 50 Recall

$$W A^* W^{-1} = (W^*)^{-1} A W^*$$

Denote this common value by A^{ε}

~

One checks

$$A^{\varepsilon} = \left(\begin{array}{c|ccccc}
0 & q^{-2} & q^{-2} & \cdots & & q^{-2} \\
\hline
q^{2} & 0 & 1 & 1 & & \\
q^{2} & 1 & 0 & 1 & \cdots & \\
\vdots & 1 & 1 & 0 & & \\
 & & & & \ddots & \\
 & & \vdots & & & \\
q^{2} & & & & & 0
\end{array}\right)$$

Note In terms of the bases for T in LEM36

we have

$A^\varepsilon =$

term	I	E_0	E_0^*	$E_0 E_0^*$	$E_0^* E_0$
coeff	-1	$-(q-q^{-1})^2$	$-(q-q^{-1})^2$	$-q(q-q^{-1})^3$	$q^{-1}(q-q^{-1})^3$

$$A^\varepsilon = \frac{qAA^* - q^{-1}A^*A}{q^{-1}-q}$$

Obs

$$I, A, A^*, A^\varepsilon$$

are lin indep.

LEM 51 the automorphism $\rho : T \to T$ sends

$$\begin{tikzcd} A \arrow[rr] & & A^* \arrow[dl] \\ & A^{\varepsilon} \arrow[ul] \arrow[u, phantom, "\rho"] & \end{tikzcd}$$

pf Check $A \to A^*$

Write $F = WW^*$

$$FAF^{-1} \overset{?}{=} A^*$$

$$WW^* A \overset{?}{=} A^* WW^*$$

Recall

$$WA^*W^{-1} = (W^*)^{-1} A W^*$$

so

$$W^*WA^* = AW^*W$$

taking the transpose gives

$$WW^*A = A^*WW^* \quad \checkmark$$

The checks that $A^* \to A^{\varepsilon} \to A$ are sim. $\square$

LEM 52 We have

(i) $$\frac{qAA^* - q^{-1}A^*A}{q - q^{-1}} = -A^\varepsilon$$

(ii) $$\frac{qA^*A^\varepsilon - q^{-1}A^\varepsilon A^*}{q - q^{-1}} = -A$$

(iii) $$\frac{qA^\varepsilon A - q^{-1}AA^\varepsilon}{q - q^{-1}} = -A^*$$

pf (i) We saw this above LEM 51

(ii), (iii) Apply ρ, ρ^2 to (i) and use LEM 51 $\square$

In terms of A, A^* LEM 52 asserts the following.

COR 53 We have

✓ (i) $$A^2A^* - (q^2+q^{-2})AA^*A + A^*A^2 + (q-q^{-1})^2A^* = 0,$$

✓ (ii) $$A^{*2}A - (q^2+q^{-2})A^*AA^* + AA^{*2} + (q-q^{-1})^2A = 0.$$

pf In LEM 52 (ii), (iii) elim A^ε using LEM52 (i) $\square$

Notation For any $\mathbb{F}$-algebra A

an <u>antiautomorphism</u> of A is an iso of $\mathbb{F}$-vector spaces

$$\sigma : A \to A$$

such that

$$\sigma(ab) = \sigma(b)\sigma(a) \qquad \forall a, b \in A.$$

auts / antiauts are related as follows

Composition	aut	antiaut
aut	aut	antiaut
antiaut	antiaut	aut

The set $AAut(A)$ of all auts/antiauts of A

forms a group under composition.

Prop 54

(i) the map

$$m \longrightarrow W m^t W^{-1}$$

is an antiaut of T that sends

$$A \to A, \qquad A^* \to A^\varepsilon, \qquad A^\varepsilon \to A^*$$

(ii) the map

$$m \longrightarrow (W^*)^{-1} m^t W^*$$

is an antiaut of T that sends

$$A^* \to A^*, \qquad A^\varepsilon \to A, \qquad A \to A^\varepsilon$$

(iii) the map

$$m \longrightarrow W W^* W m^t (W W^* W)^{-1}$$

is an antiaut of T that sends

$$A^\varepsilon \to A^\varepsilon, \qquad A \to A^*, \qquad A^* \to A$$

pf just check it! $\square$

Conclusion

- Let G denote the subgroup of $AAut(T)$ generated by the three antiauts in Prop 54
- One checks G consists of these 3 antiauts, together with id, ρ, ρ^2. So $|G| = 6$
- By Prop 54 each element of G permutes A, A^*, A^ε
- the action of G on A, A^*, A^ε induces a group iso

$$G \longrightarrow S_3$$

Note W is an example of a spin model

See F. Jaeger, M. Matsumoto, K. Nomura

"Bose-Mesner algebras related to type II matrices and spin models"

J. Algebraic Combinatorics 8 (1998) 39-72

Note For $n \neq 4$ each of the following is a basis for T

(i) $e_0, e_1, A, A^*, A^\varepsilon$

(ii) $e_0, e_1, E_0, E_0^*, E_0^\varepsilon$

On the primary T-module each of A, A^* has trace $n-2$

" " nonprimary thin T-module - - - -1

Define

$$\Phi = \frac{n-2}{2} e_0 - e_1$$

$$= \frac{n}{2} e_0 - I$$

Obs each of $A - \Phi$, $A^* - \Phi$ has trace 0 on each

thin T-module.

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LEM Ref to K_n

$$\Phi = \frac{n^2 A^* + n(n-2) A - [A, [A, A^*]]}{2n(n-1)}$$

$$\Phi = \frac{n^2 A + n(n-2) A^* - [A^*, [A^*, A]]}{2n(n-1)}$$

Recall $\Lambda = -(q-q^{-1})^2$

LEM

Col

(i) $$\frac{q+q^{-1}}{q-q^{-1}}\left[A-\Phi,\ A^{*}-\Phi\right] =$$

$$(q^2+q^{-2})(A-\Phi) + (q^2+q^{-2})(A^{*}-\Phi) - 2\,(A^{\varepsilon}-\Phi)$$

(ii) $$\frac{q+q^{-1}}{q-q^{-1}}\left[A^{*}-\Phi,\ A^{\varepsilon}-\Phi\right] =$$

$$(q^2+q^{-2})(A^{*}-\Phi) + (q^2+q^{-2})(A^{\varepsilon}-\Phi) - 2\,(A-\Phi)$$

(iii) $$\frac{q+q^{-1}}{q-q^{-1}}\left[A^{\varepsilon}-\Phi,\ A-\Phi\right] =$$

$$(q^2+q^{-2})(A^{\varepsilon}-\Phi) + (q^2+q^{-2})(A-\Phi) - 2\,(A^{*}-\Phi)$$

pf (i) write each side in the basis $I,\ E_0,\ E_0^{*},\ E_0E_0^{*},\ E_0^{*}E_0$

(ii), (iii) Apply ρ, ρ^2 to (i) $\square$

Lecture 18 Monday March 2

Given a DRG $\Gamma = (X, R)$ diam D $\qquad \mathbb{F} = \mathbb{C}$

Fix $x \in X$, write $T = T(x)$ etc

Assume $\{E_i\}_{i=0}^{D}$ is a Q-poly ordering of the primitive idempotents of Γ, and let A^* denote the corresp dual adj matrix.

Recall for $0 \le i, j \le D$ s.t $|i-j| > 1$

$$E_i A^* E_j = 0, \qquad E_i^* A E_j^* = 0$$

DEF 55 With the above notation, by an <u>imaginary adjacency matrix</u> for Γ we mean a matrix $A^\varepsilon \in T$ such that

(i) A^ε is diagonalizable with exactly $D+1$ distinct eigenvalues $\{\theta_i^\varepsilon\}_{i=0}^{D}$

(ii) For $0 \le i, j \le D$ s.t $|i-j| > 1$,

$$E_i A^\varepsilon E_j = 0, \qquad E_i^* A^\varepsilon E_j^* = 0$$

$$E_i^\varepsilon A E_j^\varepsilon = 0, \qquad E_i^\varepsilon A^* E_j^\varepsilon = 0$$

where E_r^ε is the prim idempotent of A^ε for θ_r^ε

(iii) Any two of A, A^*, A^ε generate T.

Problem Which Q-poly DRG's possess an imag adj matrix A^ε? To what extent is A^ε unique?

For some simple examples of Γ find all the imag adj matrices. Start with $D=2, D=3, \ldots$

Next goal: show Hamming graph $H(D,N)$ is Q-polynomial

Let $\Gamma = (X, R)$ denote $H(D,N)$. View $\Gamma = K_N \times K_N \times \cdots \times K_N$

Let $\tilde{\Gamma} = (\tilde{X}, \tilde{R})$ denote K_N (all objects for K_N get $\sim$)

So

$$X = \underbrace{\tilde{X} \times \tilde{X} \times \cdots \times \tilde{X}}_{D}$$

$$\mathrm{Mat}_X(\mathbb{F}) = \mathrm{Mat}_{\tilde{X}}(\mathbb{F}) \otimes \mathrm{Mat}_{\tilde{X}}(\mathbb{F}) \otimes \cdots \otimes \mathrm{Mat}_{\tilde{X}}(\mathbb{F})$$

Pick $\tilde{x} \in \tilde{X}$ for base vertex. Base vertex for X will be

$$x := (\tilde{x}, \tilde{x}, \ldots, \tilde{x})$$

Recall for $0 \le j \le D$ the jth largest eigenvalue of Γ is

$$\theta_j = (N-1)(D-j) - j$$

Let E_j be corresp prim idempotent for Γ. Show $\{E_j\}_{j=0}^{D}$

is Q-poly. The matrices $\{A_j\}_{j=0}^{D}$, $\{E_j\}_{j=0}^{D}$, $\{A_j^*\}_{j=0}^{D}$, $\{E_j^*\}_{j=0}^{D}$

are described as follows.

Matrix for $H(D,N)$	Description
A_j	$\sum t_1 \otimes t_2 \otimes \cdots \otimes t_D$ $t_i \in \{\tilde{A}_0, \tilde{A}_1\}$ $\quad 1 \le i \le D$ $j = \lvert \{ i \mid t_i = \hat{A}_1 \} \rvert$
E_j	$\sum t_1 \otimes t_2 \otimes \cdots \otimes t_D$ $t_i \in \{\tilde{E}_0, \tilde{E}_1\}$ $\quad 1 \le i \le D$ $j = \lvert \{ i \mid t_i = \tilde{E}_1 \} \rvert$
A_j^*	$\sum t_1 \otimes t_2 \otimes \cdots \otimes t_D$ $t_i \in \{\widetilde{A_0^*}, \widetilde{A_1^*}\}$ $\quad 1 \le i \le D$ $j = \lvert \{ i \mid t_i = \widetilde{A_1^*} \} \rvert$
E_j^*	$\sum t_1 \otimes t_2 \otimes \cdots \otimes t_D$ $t_i \in \{\widetilde{E_0^*}, \widetilde{E_1^*}\}$ $\quad 1 \le i \le D$ $j = \lvert \{ i \mid t_i = \widetilde{E_1^*} \} \rvert$

Recall $\tilde{\Delta} \in Mat_{\tilde{X}}(\mathbb{F})$ from LEM 39

For $i = 0, 1$ we saw

$$\tilde{\Delta} \tilde{A}_i \tilde{\Delta}^{-1} = \tilde{A}_i^*$$

$$\tilde{\Delta} \tilde{E}_i \tilde{\Delta}^{-1} = \tilde{E}_i^*$$

$$\tilde{\Delta} \tilde{A}_i^* \tilde{\Delta}^{-1} = \tilde{A}_i$$

$$\tilde{\Delta} \tilde{E}_i^* \tilde{\Delta}^{-1} = \tilde{E}_i$$

Define

$$\Delta = \underbrace{\tilde{\Delta} \otimes \tilde{\Delta} \otimes \cdots \otimes \tilde{\Delta}}_{D}$$

$$\in Mat_X(\mathbb{F})$$

Obs

$$\Delta^{-1} = \tilde{\Delta}^{-1} \otimes \tilde{\Delta}^{-1} \otimes \cdots \otimes \tilde{\Delta}^{-1}$$

and for $0 \le j \le D$

$$\Delta A_j \Delta^{-1} = A_j^*$$

$$\Delta E_j \Delta^{-1} = E_j^*$$

$$\Delta A_j^* \Delta^{-1} = A_j$$

$$\Delta E_j^* \Delta^{-1} = E_j$$

therefore $\{E_j\}_{j=0}^{D}$ is Q-poly with

$$q_{ij}^h = p_{ij}^h \qquad (0 \leq h, i, j \leq D)$$

$$\theta_j^* = \theta_j \qquad (0 \leq j \leq D). \qquad \square$$

Project on $H(D,N)$

$\forall y \in X$ define $\check{y} = \Delta \hat{y}$

so that

$$\{\check{y} \mid y \in X\} \qquad (1)$$

is a basis for st module V, and

for $0 \le j \le D$

$$\{\check{y} \mid y \in \Gamma_j(x)\} \qquad (2)$$

is a basis for $E_j V$

(i) Describe the function algebra product $\circ$ on V

Specifically $\forall y, z \in X$ find

$\check{y} \circ \check{z}$ as a linear comb of (1)

(ii) For $0 \le j \le D$ describe the Norton algebra product $*$ on $E_j V$. Specifically, $\forall y, z \in \Gamma_j(x)$ find

$\check{y} * \check{z}$ as a linear comb of (2)

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Projects

Problem For $H(D,2)$ find the central prim idempotents for T as a linear comb of the basis $E_h^* A_i E_j^*$

Problem. For $H(D,N)$ extend the above result.

Problem For $H(D,N)$ find a central element of T which is quadratic in A, A^*.

To do this extend Def 14.3 in Junie Go's paper on $H(D,2)$.

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, R)$ diam D

Assume $\{E_i\}_{i=0}^{D}$ is Q-poly ordering of primitive idempotents of Γ

Fix $x \in X$ write $T = T(x)$ etc.

Next goal is to prove following two theorems.

Thm 56 With above assumptions, the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \le i \le D-1$.

Thm 57 With above assumptions, $\exists\, \beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ s.t.

$$0 = \left[A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*\right] \qquad \text{TD1}$$

$$0 = \left[A^*,\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A\right] \qquad \text{TD2}$$

$[r,s] = rs - sr$

TD1, TD2 called the <u>tridiagonal relations</u>

—

We will prove Thm 56, Thm 57 after a few lemmas.

LEM 58 For $0 \le i, j, r \le D$

(i) $$E_i^* A^r E_j^* = \begin{cases} 0 & \text{if } r < |i-j| \\ \neq 0 & \text{if } r = |i-j| \end{cases}$$

(ii) $$E_i (A^*)^r E_j = \begin{cases} 0 & \text{if } r < |i-j| \\ \neq 0 & \text{if } r = |i-j| \end{cases}$$

pf (i) Recall

$$E_i^* A_r E_j^* = 0 \quad \text{iff} \quad p^r_{ij}$$

A_r = poly in A of degree r

$$p^r_{ij} = \begin{cases} 0 & \text{if } r < |i-j| \\ \neq 0 & \text{if } r = |i-j| \end{cases}$$

Result follows. $\square$

(ii) Sim

It turns out th 56, th 57 follow from LEM 58 alone.

In what follows we will give a sequence of results.

Each result has a dual obtained by interchanging $A \leftrightarrow A^*$

$E_i \leftrightarrow E_i^*$ $(0 \le i \le D)$. We wont explicitly state each dual

but we understand it is true.

LEM 59 For $0 \le i, j, r, s \le D$

$$E_i^* A^r A^* A^s E_j^* = \begin{cases} \theta_{j+s}^* \; E_i^* A^{r+s} E_j^* & \text{if } i-j = r+s \\ \theta_{j-s}^* \; E_i^* A^{r+s} E_j^* & \text{if } j-i = r+s \\ 0 & \text{if } |i-j| > r+s \end{cases}$$

pf Obs

$$E_i^* A^r A^* A^s E_j^* = E_i^* A^r \left(\sum_{h=0}^{D} \theta_h^* E_h^* \right) A^s E_j^*$$

and use L58 (i)

$\square$

LEM 60

$$\text{Span}\{RA^*S - SA^*R \mid R, S \in M\}$$
$$= \{YA^* - A^*Y \mid Y \in M\}$$

pf. Abbrev

$$L_i = E_0 + E_1 + \cdots + E_i \qquad (0 \le i \le D)$$
$$E_{D+1} = 0, \quad E_{-1} = 0$$

<u>Claim:</u> for $0 \le i \le D$
$$E_i A^* E_{i+1} - E_{i+1} A^* E_i = L_i A^* - A^* L_i$$

<u>pf cl</u> For $0 \le j \le D$

$$E_j A^* = E_j A^* (E_0 + E_1 + \cdots + E_D)$$
$$= E_j A^* (E_{j-1} + E_j + E_{j+1}) \qquad (*)$$

Sim
$$A^* E_j = (E_{j-1} + E_j + E_{j+1}) A^* E_j \qquad (**)$$

Now sum each of $*$, $**$ over $j = 0, 1, \ldots, i$ and take the difference to get the claim.

Recall $\{E_i\}_{i=0}^{p}$ a basis for M so $\{L_i\}_{i=0}^{D}$ is a basis for M.

Now

$$\mathrm{Span}\{RA^*S - SA^*R \mid R, S \in M\}$$

$$= \mathrm{Span}\{E_iA^*E_j - E_jA^*E_i \mid 0 \le i, j \le D\}$$

$$= \mathrm{Span}\{E_iA^*E_{i+1} - E_{i+1}A^*E_i \mid 0 \le i \le D\}$$

$$= \mathrm{Span}\{L_iA^* - A^*L_i \mid 0 \le i \le D\}$$

$$= \{YA^* - A^*Y \mid Y \in M\}$$

$\square$

$\square$