

2. The subconstituent algebra  $T$ 

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

Given DRG  $\Gamma = (X, R)$  with diam  $D$  st. module  $V = \mathbb{F}^X$

Fix  $x \in X$  and write  $M^x = M^x(x)$  etc.

Def 1 Let  $T = T(x)$  denote the subalgebra of  $\text{Mat}_x(\mathbb{F})$  generated by  $M$  and  $M^*$ .

We call  $T$  the subconstituent algebra of  $\Gamma$  with respect to  $x$

With reference to Def 1, by a  $T$ -module we mean a subspace  $W \subseteq V$  such that  $BW \subseteq W$  for all  $B \in T$ .

The  $T$ -module  $W$  is called irreducible whenever  $W \neq 0$

and  $W$  contains no  $T$ -module other than  $0$  or  $W$ .

Given  $T$ -modules  $W, W'$  by an isomorphism of  $T$ -modules

from  $W$  to  $W'$  we mean a vector space iso  $\sigma: W \rightarrow W'$  s.t.

$$(\sigma B - B\sigma)W = 0 \quad \forall B \in T.$$

The  $T$ -modules  $W, W'$  are isomorphic whenever there exists a  $T$ -module iso from  $W$  to  $W'$ .

We have a few comments on  $T$ .

LEM 2

(i)  $T$  is generated by  $A, \{E_i\}_{i=0}^p$

(ii) Assume  $\{E_i\}_{i=0}^p$  is  $\mathbb{Q}$ -poly. Then  $T$  is generated by  $A, A^*$ .

Pf (i)  $A$  generates  $M$  and  $\{E_i\}_{i=0}^p$  is a basis for  $M^*$

(ii)  $A^*$  generates  $M^*$

□

LEM 3  $\forall B \in T$

(i)  $B^t \in T$

(ii)  $\bar{B} \in T$

Pf  $T$  is generated by real symmetric matrices by LEM 2 (i)  $\square$

Given subspaces  $U \subseteq W \subseteq V$

obs  $\exists$  unique subspace  $U' \subseteq W$  s.t.

$$W = U + U' \quad (\text{orthog dis sum})$$

We have

$$U' = \{w \in W \mid \langle u, w \rangle = 0 \quad \forall u \in U\}$$

Call  $U'$  the orthogonal complement of  $U$  in  $W$

LEM 4 Given a  $T$ -module  $W$  and a  $T$ -submodule  $U \subseteq W$ . The orthogonal complement of  $U$  in  $W$  is a  $T$ -module.

Pf Call it  $U'$ . Given  $B \in T$  and  $u' \in U'$

check  $Bu' \in U'$ .  $\forall u \in U$

$$\langle Bu', u \rangle = 0$$

$$\langle u', \overline{B}^t u \rangle = 0$$

$$\overline{B}^t \in T$$

$$\overline{B}^t u \in U$$

□

COR 5 Each  $T$ -module is an orthog direct sum of

irred  $T$ -modules. In particular  $V$  is an orthog

direct sum of irred  $T$ -modules.

□

LEM 6 For a  $T$ -module  $W$

(i)  $W$  is the orthog direct sum of the non 0 spaces  
among  $\{E_i^*W \mid 0 \leq i \leq D\}$

(ii)  $W$  is the orthogonal direct sum of the non 0 spaces  
among  $\{E_i W \mid 0 \leq i \leq D\}$

Pf (i). The  $E_i^*W$  are mutually orthog since  $E_i^*W \subseteq E_i^*V$

and the  $E_i^*V$  are mutually orthog.

Check  $W = \sum_{i=0}^D E_i^*W$

$\geq$ : Each  $E_i^* \in T$  and  $TW \subseteq W$

$\leq \forall w \in W$

$$\begin{aligned} W &= Iw \\ &= \sum_{i=0}^D E_i^*W \end{aligned}$$

(ii) Sim to (i)

□

We will return to  $T$ -modules in a moment.

LEM 7

(i)  $\forall n \ 0 \leq i, j \leq n$

$$A_i E_j^* V \subseteq \sum_{\substack{0 \leq h \leq n \\ \Gamma_{ij}^h \neq \emptyset}} E_h^* V$$

(ii)  $\forall n \ 0 \leq i \leq n$

$$A E_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V$$

(iii)  $\forall n \ 0 \leq i, j \leq n$

$$A_i^* E_j V \subseteq \sum_{\substack{0 \leq h \leq n \\ \Gamma_{ij}^h \neq \emptyset}} E_h V$$

(iv) Assume  $\{E_i\}_{i=0}^n$  is  $Q$ -poly. then for  $0 \leq i \leq n$

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V$$

Pf (i)  $\forall \gamma \in \Gamma_{ij}(x)$

$$A_i \hat{\gamma} = \sum_{z \in \Gamma_{ij}^1(\gamma)} \hat{z}$$

$\forall z \in \Gamma_{ij}^h(\gamma)$  let  $h = \partial(x, z)$  and obs  $\Gamma_{ij}^h \neq \emptyset$

(iii) Set  $i=1$  in (ii)

(iii)  $\forall v \in E_1 V$

$$A_i^* v = |X| E_i \hat{x} \circ v$$

by L 91

$$\in E_i V \circ E_1 V$$

$$= \sum_{\substack{0 \leq h \leq p \\ p_{12}^h \neq 0}} E_h V$$

by Th 92.

(iv) Set  $i=1$  in (iii)

□

COR 8 For an unred  $T$ -module  $W$

$$(i) \quad A E_i^{\vee} W \subseteq E_{i+1}^{\vee} W + E_i^{\vee} W + E_{i-1}^{\vee} W \quad (0 \leq i \leq n)$$

(ii) Assume  $\{E_i\}_{i=0}^n$  is  $Q$ -poly. Then

$$A^{\vee} E_i W \subseteq E_{i+1} W + E_i W + E_{i-1} W \quad (0 \leq i \leq n)$$

Pf By L7 (ii), (iv),

□



Def 9 Given an unred  $T$ -module  $W$

(i) The diameter  $d = d(W)$  is

$$d = \left| \left\{ i \mid 0 \leq i \leq 0, E_i^* W \neq 0 \right\} \right| - 1$$

(ii) The endpoint  $r = r(W)$  is

$$r = \min \left\{ i \mid 0 \leq i \leq 0, E_i^* W \neq 0 \right\}$$

(iii) The dual diameter is

$$d^* = \left| \left\{ i \mid 0 \leq i \leq 0, E_i W \neq 0 \right\} \right| - 1$$

(iv) The dual endpt  $t = t(W)$  is

$$t = \min \left\{ i \mid 0 \leq i \leq 0, E_i W \neq 0 \right\}$$

[  $t$  is defined up to given ordering of the  $E_i$  ]

LEM 10 Given an irred  $T$ -module  $W$  with index  $r$  and diam  $d$

Then

$$E_i^* W \neq 0 \quad \forall \quad r \leq i \leq r+d \quad (0 \leq i \leq d)$$

Pf By constructn  $E_i^* W = 0$  for  $0 \leq i < r$  and  $E_r^* W \neq 0$

Suppose  $\exists i$  ( $r < i \leq r+d$ ) s.t.  $E_i^* W = 0$

$$\text{Set } W' = E_r^* W + E_{r+1}^* W + \dots + E_{i-1}^* W.$$

By constr  $W' \neq 0$

$$M^* W' \subseteq W'$$

Also  $AW' \subseteq W'$  by Cor 8 (i)

so  $W'$  is  $T$ -module.

Now  $W' = W$  by the irred of  $W$ .

But this contradicts the diameter of  $W$ .

So  $E_i^* W \neq 0$  for  $r \leq i \leq r+d$ .

Now  $E_i^* W = 0$  for  $r+d < i \leq d$  by the def of  $d$ .  $\square$

LEM 11 Assume the ordering  $\{E_i\}_{i=0}^p$  is  $\mathcal{Q}$ -poly.

Let  $W$  denote an irreducible  $T$ -module with center  $t$  and dual diameter  $d^*$ . Then

$$E_i W \neq 0 \quad \text{iff} \quad t \leq i \leq d^* + t \quad (0 \leq i \leq p)$$

pf Similar to the pf of L10

□

Def 12 Given an mod  $T$ -module  $W$ .

(i) Call  $W$  thin whenever

$$\dim E_i^* W \leq 1 \quad (0 \leq i \leq \rho)$$

(ii) Call  $W$  dual thin whenever

$$\dim E_i W \leq 1 \quad (0 \leq i \leq \rho)$$

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We now consider a certain  $T$ -module called the

primary  $T$ -module. Recall  $\mathbb{1} = \sum_{\gamma \in X} \hat{\gamma}$

For  $0 \leq i \leq \rho$  define

$$\mathbb{1}_i = \sum_{\gamma \in \Gamma_i^*(x)} \hat{\gamma}$$

obs

$$\begin{aligned} \mathbb{1}_i &= E_i^* \mathbb{1} \\ &= A_i \hat{x} \end{aligned}$$

LEM 13 For  $0 \leq j \leq 0$

$$|X| E_j \hat{x} = A_j^x \mathbb{1}$$

(call this vector  $\mathbb{1}_j^*$ )

Pf. Recall

$$\begin{aligned} A_j^x \mathbb{1} &= |X| E_j \hat{x} \circ \mathbb{1} \\ &= |X| E_j \hat{x} \quad \square \end{aligned}$$

Prop 14  $M \hat{x} = M^x \mathbb{1}$  is a thin, dual-thin irreducible

T-module with basis  $\{\mathbb{1}_i\}_{i=0}^0$  and basis  $\{\mathbb{1}_i^*\}_{i=0}^0$ .

The module has endpt 0, dual endpt 0, diameter 0,

dual diameter 0.

Pf.  $M \hat{x} = M^x \mathbb{1}$  by L13; call this space  $W$ .  $MW \subseteq W$

since  $W = M \hat{x}$ .  $M^x W \subseteq W$  since  $W = M^x \mathbb{1}$ . So  $W$  is

T-module. Suppose  $W$  is reducible. By Cor 5  $W$  is ods of

irred T-modules. Since  $\hat{x} \in W$  these modules cannot all

be orthog  $\hat{x}_i$ . So one of them has endpt 0 and hence contains  $\hat{x}_i$ .

But then it contains  $M\hat{x} = W$ . So it equals  $W$ .

Concerning the remaining assertions, for  $0 \in \mathcal{E} \in \mathcal{E} \in \mathcal{E}$

$$\begin{aligned} E_i^* W &= E_i^* M^* \\ &= \text{Span } E_i^* \mathbb{1} \quad \text{has dim } 1 \end{aligned}$$

$$\begin{aligned} E_i W &= E_i M \hat{x} \\ &= \text{Span } E_i \hat{x} \quad \text{has dim } 1. \quad \square \end{aligned}$$



$\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  Fix a DRG  $\Gamma = (X, R)$  diam  $D$

Fix  $x \in X$  and write  $T = T(x)$ , etc.

Last time we considered the irred  $T$ -module

$$M \hat{x} = M^* \mathbb{1}$$

Call this  $T$ -module primary (or trivial)

We begin a careful study of the primary  $T$ -module.

To this end we first obtain some "reduction rules"

involving  $E_0$  and  $E_0^\vee$ .

Lem 15 The following hold for  $0 \leq i, j \leq 0$ .

$$(i) \quad E_0 A_i^* E_j = \delta_{ij} E_0 A_i^*$$

$$(ii) \quad E_0 E_i^* E_j = |X|^{-1} k_i u_i(\theta_j) E_0 A_j^*$$

$$(iii) \quad E_0 A_i^* A_j = k_j u_j(\theta_i) E_0 A_i^*$$

$$(iv) \quad E_0 E_i^* A_j = \sum_{h=0}^0 p_{ij}^h E_0 E_h^*$$

Pf (i) Recall

$$E_0 A_i^* E_j = 0 \text{ iff } q_{ij}^0 = 0$$

$$\text{If } i \neq j \text{ then } q_{ij}^0 = 0 \implies E_0 A_i^* E_j = 0$$

Also

$$\begin{aligned} E_0 A_i^* &= E_0 A_i^* (E_0 + \dots + E_0) \\ &= E_0 A_i^* E_i \end{aligned}$$

(ii) Use

$$E_i^* = |X|^{-1} \sum_{h=0}^0 v_i(\theta_h) A_h^*$$

and (i) above.



(iii) Use

$$A_j = \sum_{h=0}^{\infty} v_j(\theta_h) E_h$$

and (i) above,

(iv) Recall by L76

$$E_0 E_0^* B = E_0 B P \quad \forall B \in M$$

so

$$E_0 E_0^* A_j = E_0 E_j^*$$

Now

$$\begin{aligned} E_0 E_i^* A_j &= E_0 E_0^* A_i A_j \\ &= \sum_{h=0}^{\infty} p_{ij}^h E_0 E_0^* A_h \\ &= \sum_{h=0}^{\infty} p_{ij}^h E_0 E_h^* \end{aligned}$$

□

We also have the dual equations

LEM 16 The following hold for  $0 \leq i, j \leq 0$ .

$$(i) \quad E_0^* A_i E_j^* = \delta_{ij} E_0^* A_i$$

$$(ii) \quad E_0^* E_i E_j^* = |X|^{-1} m_i u_j(0) E_0^* A_j$$

$$(iii) \quad E_0^* A_i A_j^* = m_j u_i(0) E_0^* A_i$$

$$(iv) \quad E_0^* E_i A_j^* = \sum_{h=0}^p q_{ij}^h E_0^* E_h$$

Pf Similar to the proof of L15

□

We mention some special cases of LEM 15, 16

LEM 17 For  $0 \leq j \leq n$

$$(i) \quad E_0 E_0^x E_j = |X|^{-1} E_0 A_j^x$$

$$(ii) \quad E_0 E_0^x A_j = E_0 E_j^x$$

$$(iii) \quad E_0^x E_0 E_j^x = |X|^{-1} E_0^x A_j$$

$$(iv) \quad E_0^x E_0 A_j^x = E_0^x E_j$$

Pf Use L15, L16

□

LEM 18

$$(i) \quad E_0 E_i^x E_0 = |X|^{-1} k_i E_0 \quad (0 \leq i \leq n)$$

$$(ii) \quad E_0 E_0^x E_0 = |X|^{-1} E_0$$

$$(iii) \quad E_0^x E_i E_0^x = |X|^{-1} m_i E_0^x \quad (0 \leq i \leq n)$$

$$(iv) \quad E_0^x E_0 E_0^x = |X|^{-1} E_0^x$$

Pf use L15, 16

□

Note more reduction rules come by taking the transpose

in L15-17

The following reduction rules are of a slightly different nature

LEM 19 For  $0 \leq i, j \leq n$

$$(i) \quad A_i E_0^x A_j = |X| E_i^x E_0 E_j^x$$

$$(ii) \quad E_i E_0^x A_j = A_i^x E_0 E_j^x$$

$$(iii) \quad A_i E_0^x E_j = E_i^x E_0 A_j^x$$

$$(iv) \quad E_i E_0^x E_j = |X|^{-1} A_i^x E_0 A_j^x$$

$$\begin{aligned} \text{PF } (i) \quad A_i E_0^x A_j &= |X| A_i E_0^x E_0 E_j^x \quad \text{by L17 (iii)} \\ &= |X| E_i^x E_0 E_j^x \quad \text{by L17 (ii) transpose} \end{aligned}$$

(ii) - (iv) Similar

□

LEM 20 We have

$$\sum_{i=0}^p k_i^{-1} E_i^* E_0 E_i^* = \sum_{j=0}^p m_j^{-1} E_j E_0^* E_j \quad (*)$$

$$\text{Pf LHS} = |X|^{-1} \sum_{i=0}^p k_i^{-1} A_i E_0^* A_i$$

$$= |X|^{-1} \sum_{i=0}^p k_i^{-1} \left( \sum_{r=0}^p v_i(e_r) E_r \right) E_0^* \left( \sum_{a=0}^p v_i(e_a) E_a \right)$$

$$= |X|^{-1} \sum_{r=0}^p \sum_{a=0}^p E_r E_0^* E_a \left( \underbrace{\sum_{i=0}^p k_i^{-1} v_i(e_r) v_i(e_a)}_{\delta_{rs} m_r^{-1} |X|} \right)$$

$$= \sum_{j=0}^p m_j^{-1} E_j E_0^* E_j$$

□

DEF 21 Define  $e_0 = e_0(x)$  to be  $|X|$  times

the common value in  $(*)$ . We obs  $e_0 \in T$  and

$$\bar{e}_0 = e_0, \quad e_0^t = e_0$$

For the moment let  $W$  denote the primary  $T$ -module.

Obs by LEM 4

$$V = W + W^\perp \quad (\text{orthog. dir. sum of } T\text{-modules})$$

where  $W^\perp =$  orthog. complement of  $W$  in  $V$ .

LEM 22 With above notation

$$(i) \quad (e_0 - I)W = 0,$$

$$(ii) \quad e_0 W^\perp = 0,$$

In other words  $e_0$  acts on  $V$  as the orthogonal

projection  $V \rightarrow W$ .

pf (i)  $W$  has basis  $\{\mathbb{1}_i\}_{i=0}^D$

For  $0 \leq h \leq D$

$$\begin{aligned}
 e_0 \mathbb{1}_h &= |X| \sum_{i=0}^D k_i^{-1} E_i^* E_0 E_i^* \mathbb{1}_h \\
 &= |X| k_h^{-1} E_h^* E_0 \mathbb{1}_h \\
 &= |X| k_h^{-1} E_h^* \underbrace{v_h(\theta_0)}_{k_h} E_0 \hat{x} \\
 &= |X| E_h^* E_0 \hat{x} \\
 &= E_h^* J \hat{x} \\
 &= E_h^* \mathbb{1} \\
 &= \mathbb{1}_h
 \end{aligned}$$

(ii) Given  $v \in W^\perp$  show  $e_0 v = 0$

$e_0 \in T$  and  $W^\perp$  is  $T$ -module so  $e_0 v \in W^\perp$

By construction  $e_0 \in M E_0^* M$  so

$$\begin{aligned}
 e_0 v &\in M E_0^* M v \\
 &\subseteq M E_0^* v \\
 &\subseteq M \hat{x} \\
 &= W
 \end{aligned}$$

$$E_0^* v = \text{span}(\hat{x})$$

Now  $e_0 v \in W \cap W^\perp = 0$

□

Recall the center

$$Z(T) = \left\{ B \in T \mid Bt = tB \ \forall t \in T \right\}$$

is a subalgebra of  $T$ .

COR 23 We have

(i)  $e_0 \in Z(T)$

(ii)  $e_0^2 = e_0$

(iii)  $e_0V = \text{primary module}$

(iv)  $\text{rank } e_0 = d+1$

PF (i)  $\forall B \in T$  the expression  $Be_0 - e_0B$

vanishes on  $W$  and  $W^\perp$ , where  $W = \text{prim module}$ .

But  $V = W + W^\perp$  so  $Be_0 - e_0B$  vanishes on  $V$ .

(ii), (iii) By L22

(iv) Prim module has dim  $d+1$ . □

From now on we denote the primary module by  $e_0V$



Recall  $e_0V$  has a basis  $\{\Pi_i\}_{i=0}^0$  and a basis  $\{\Pi_i^x\}_{i=0}^0$

LEM 24 For  $0 \leq i, j \leq 0$ ,

$$(i) \quad \langle \Pi_i, \Pi_j \rangle = \delta_{ij} k_i$$

$$(ii) \quad \langle \Pi_i^x, \Pi_j^x \rangle = \delta_{ij} |X| m_i$$

$$(iii) \quad \langle \Pi_i, \Pi_j^x \rangle = k_i m_j u_i(e_j)$$

pf (i) clear

$$\begin{aligned} (ii) \quad \text{LHS} &= |X|^2 \langle E_i \hat{x}, E_j \hat{x} \rangle \\ &= |X|^2 \langle \hat{x}, E_i E_j \hat{x} \rangle \\ &= \delta_{ij} |X|^2 \langle x_i, E_i \hat{x} \rangle \\ &= \delta_{ij} |X|^2 \left( \underbrace{x_i x_i}_{m_i |X|^{-2}} \text{entry of } E_i \right) \end{aligned}$$

$$\begin{aligned} (iii) \quad \text{LHS} &= |X| \langle A_i \hat{x}, E_j \hat{x} \rangle \\ &= |X| \langle \hat{x}, A_i E_j \hat{x} \rangle \\ &= |X| v_i(e_j) \langle \hat{x}, E_j \hat{x} \rangle \\ &= |X| v_i(e_j) |X|^{-2} m_j \\ &= m_j v_i(e_j) = k_i m_j u_i(e_j) \end{aligned}$$

□

We now give the action of the T-generators on the basis  $\{\Pi_i\}_{i=0}^p$

LEMMA For  $0 \leq i, j \leq p$

$$(i) \quad E_i^* \Pi_j = \delta_{ij} \Pi_j$$

$$(ii) \quad A_i^* \Pi_j = m_i u_j(\theta_i) \Pi_j$$

$$(iii) \quad E_i \Pi_j = |X|^{-1} m_i k_j u_j(\theta_i) \sum_{h=0}^p u_h(\theta_i) \Pi_h$$

$$(iv) \quad A_i \Pi_j = \sum_{h=0}^p p_{ij}^h \Pi_h$$

Pf (i) clear

$$(ii) \quad \text{Use (i) and} \quad A_i^* = m_i \sum_{h=0}^p u_h(\theta_i) E_h^*$$

$$(iii) \quad \text{Use} \quad E_i \Pi_j = E_i A_j \hat{x} = v_j(\theta_i) E_i \hat{x}$$

and

$$v_j(\theta_i) = k_j u_j(\theta_i), \quad E_i \hat{x} = |X|^{-1} m_i \sum_{h=0}^p u_h(\theta_i) A_h \hat{x} = |X|^{-1} m_i \sum_{h=0}^p u_h(\theta_i) \Pi_h$$

$$(iv) \quad A_i \Pi_j = A_i A_j \hat{x} = \sum_{h=0}^p p_{ij}^h A_h \hat{x} = \sum_{h=0}^p p_{ij}^h \Pi_h \quad \square$$

We now give the action of the T-generators on the basis

$$\{ \Pi_i^* \}_{i=0}^p$$

LEMMA For  $0 \leq i, j \leq p$

$$(i) \quad E_i \Pi_j^* = \delta_{ij} \Pi_j^*$$

$$(ii) \quad A_i \Pi_j^* = k_i u_i(\theta_j) \Pi_j^*$$

$$(iii) \quad E_i^* \Pi_j^* = |X|^{-1} k_i m_j u_i(\theta_j) \sum_{h=0}^p u_i(\theta_h) \Pi_h^*$$

$$(iv) \quad A_i^* \Pi_j^* = \sum_{h=0}^p q_{ij}^h \Pi_h^*$$

Pf Similar to L25

□

We now show how to transition between the bases

$$\{\Pi_i\}_{i=0}^p \quad \text{and} \quad \{\Pi_i^*\}_{i=0}^p$$

LEM 27 For  $0 \leq j \leq p$

$$(i) \quad \Pi_j = |X|^{-1} k_j \sum_{i=0}^p u_j(\theta_i) \Pi_i^*$$

$$(ii) \quad \Pi_j^* = m_j \sum_{i=0}^p u_i(\theta_j) \Pi_i$$

Pf (i)  $\Pi_j = A_j \hat{x}$

Recall  $A_j = k_j \sum_{i=0}^p u_j(\theta_i) E_i$

and  $E_i \hat{x} = |X|^{-1} \Pi_i^* \quad \forall i$

(ii) Similar

□

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Referring to LEM 27

as special cases we mention

$$\begin{aligned}\hat{x} &= \mathbb{1}_0 \\ &= |\lambda|^{-1} \sum_{i=0}^p \mathbb{1}_i^*\end{aligned}$$

and

$$\begin{aligned}\mathbb{1} &= \mathbb{1}_0^* \\ &= \sum_{i=0}^p \mathbb{1}_i\end{aligned}$$

We emphasize the role of the polynomials

LEM 28

(i) We have

$$\mathbb{1}_i = v_i(A) \mathbb{1}_0 \quad (0 \leq i \leq n)$$

(ii) Assume  $\{E_i\}_{i=0}^n$  is  $\mathbb{Q}$ -poly. then

$$\mathbb{1}_i^* = v_i^*(A^*) \mathbb{1}_0^* \quad (0 \leq i \leq n)$$

Pf (i)  $\mathbb{1}_i = A_i x^n$

Recall  $A_i = v_i(A)$

and  $x^n = \mathbb{1}_0$

(iii)  $\mathbb{1}_i^* = A_i^* \mathbb{1}$

Recall  $A_i^* = v_i^*(A^*)$

and

$$\mathbb{1} = \mathbb{1}_0^*$$

□

We mention a special case of LEM 25, 26

LEM 29

(i) With respect to the bases  $\{\Pi_i\}_{i=0}^D$  and  $\{\Pi_i^*\}_{i=0}^D$

the matrices representing  $A$  are

$$\begin{pmatrix} a_0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & & \\ & c_2 & & \ddots & & \\ & & & & \ddots & \\ 0 & & & & & & b_{D-1} \\ & & & & & & c_D & a_D \end{pmatrix} \quad \begin{pmatrix} \theta_0 & & & & & & \\ & \theta_1 & & & & & \\ & & \ddots & & & & \\ & & & \theta_0 & & & \\ & & & & \ddots & & \\ & & & & & \theta_0 & \end{pmatrix}$$

respectively.

(ii) Assume  $\{E_i\}_{i=0}^D$  is  $Q$ -polynomial, then with respect to the bases  $\{\Pi_i\}_{i=0}^D$  and  $\{\Pi_i^*\}_{i=0}^D$  the matrices rep  $A^*$  are

$$\begin{pmatrix} \theta_0^* & & & & & \\ & \theta_1^* & & & & \\ & & \ddots & & & \\ & & & \theta_0^* & & \\ & & & & \ddots & \\ & & & & & \theta_0^* \end{pmatrix} \quad \begin{pmatrix} a_0^* & b_0^* & & & & \\ c_1^* & a_1^* & b_1^* & & & \\ & c_2^* & & \ddots & & \\ & & & & \ddots & \\ 0 & & & & & & b_{D-1}^* \\ & & & & & & c_D^* & a_D^* \end{pmatrix}$$

Pf clear

□

Aside. In the  $\mathbb{Q}$ -polynomial case the actions of

$A$  and  $A^*$  on  $e_0V$  give an example of a linear-algebraic object called a Leonard pair. A Leonard pair is defined as follows.

For the moment let  $\mathbb{F}$  be any field.

Let  $m$  denote a square matrix over  $\mathbb{F}$ .

Call  $m$  tridiagonal whenever each non 0 entry of  $m$  is on the diagonal, the subdiagonal, or the superdiagonal.

Assume  $m$  is tridiagonal. Call  $m$  irreducible whenever each entry on the subdiagonal is non 0 and each entry on the superdiagonal is non 0.

DEF 30 Given a finite-dimensional non 0 vector space  $V$  over  $\mathbb{F}$

A Leonard pair in  $V$  is an ordered pair of linear trans

$A: V \rightarrow V$  and  $A^*: V \rightarrow V$  s.t.



(i)  $\exists$  basis for  $V$  wrt which the matrix representing  $A$  is upper triangular and the matrix representing  $A^*$  is diag.

(ii)  $\exists$  basis for  $V$  wrt which the matrix representing  $A^*$  is upper triangular and the matrix representing  $A$  is diag.

Back to DRG's  $\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$

Given DRG  $\Gamma = (X, R)$  diam  $D$

Fix  $x \in X$ , write  $T = T(x)$  etc.

COR 31 Referring to LEM 29 assume  $\{E_i\}_{i=0}^D$

is  $Q$ -polynomial. Then the pair  $A, A^*$  acts on

the primary module  $e_0 V$  as a Leonard pair.

Pf Immed from L29, Def 30 □

We clasped the LP's in 2000. We will

return to LP's shortly. □

We have some comments in general irred  $T$ -modules.

Prop 32 Let  $W, U$  denote nonorthogonal irreducible  $T$ -modules. Then  $W, U$  are isomorphic as  $T$ -modules.

Pf.  $\forall w \in W$  let  $\sigma(w)$  denote the orthogonal projection of  $w$  into  $U$ . So  $\sigma(w)$  is the unique vector in  $U$  s.t.

$$\sigma(w) \in U \quad \text{and} \quad w - \sigma(w) \in U^\perp$$

Show

$$\begin{aligned} \sigma : W &\rightarrow U \\ w &\rightarrow \sigma(w) \end{aligned}$$

is a  $T$ -module iso.

claim 1:

$$(B\sigma - \sigma B)W = 0 \quad \forall B \in T$$

pf 2:  $\forall w \in W$

$$B\sigma(w) \in U \quad \text{since } U \text{ is } T\text{-module}$$

and

$$Bw - B\sigma(w) \in U^\perp \quad \text{since } U^\perp \text{ is } T\text{-module}$$

so

$$B\sigma(w) = \sigma(Bw)$$

claim 1 is proved.

claim 2  $\sigma: W \rightarrow U$  is injective.

Pf claim 2 Let  $K$  denote the kernel of  $\sigma$  on  $W$ .

Using claim 1  $K$  is a  $T$ -submodule of  $W$

so by the unrad of  $W$

$$K = 0 \text{ or } K = W$$

But  $K \neq W$  since  $\langle W, U \rangle \neq 0$  so  $K = 0$

claim 3  $\sigma: W \rightarrow U$  is surjective

pf claim 3 Let  $\text{Im}$  denote the image of  $\sigma$  on  $U$ .

Using claim 1  $\text{Im}$  is a  $T$ -submodule of  $U$ .

By the unrad of  $U$ ,

$$\text{Im} = 0 \text{ or } \text{Im} = U.$$

But  $\text{Im} \neq 0$  since  $\langle W, U \rangle \neq 0$  so  $\text{Im} = U$ .  $\square$

Notation Let  $\Psi = \Psi(x)$  denote the set of isomorphism classes of irreducible  $T$ -modules.

The elements of  $\Psi$  are called types

For  $\phi \in \Psi$  define

$V_\phi =$  subspace of  $V$  spanned by the irreducible  $T$ -modules of type  $\phi$ .

Call  $V_\phi$  the  $\phi$ -homogeneous component of  $V$

Observe  $V_\phi$  is a  $T$ -module.

By Prop 32

$$V = \sum_{\phi \in \Psi} V_\phi \quad (\text{orthog dir sum of } T\text{-modules})$$

Given  $\phi \in \Psi$  and an irreducible  $T$ -module  $W \subseteq V_\phi$

the dimension, diameter, endpt etc of  $W$  depends only

on  $\phi$ . So we often denote these by

$\dim \phi, d(\phi), r(\phi)$  etc.

Write  $V_\phi$  as an orthog direct sum of irreducible  $T$ -modules

$$V_\phi = W_1 + W_2 + \dots + W_m \quad (*)$$

[ this decomp is not unique ]

For  $1 \leq i \leq m$   $W_i$  has type  $\phi$  so

$$\dim(V_\phi) = m \dim(\phi)$$

ic

$$m = \frac{\dim(V_\phi)}{\dim(\phi)}$$

In particular  $m$  is independent of which decomposition is used in  $(*)$

We call  $m$  the multiplicity of  $\phi$  in  $V$

For example the primary module has multiplicity 1.

---

Another view of  $T$

For any subgroup  $G \subseteq \text{Aut}(\Gamma)$

Recall the centralizer algebra

$$C_G = \left\{ B \in \text{Mat}_X(\mathbb{F}) \mid B\sigma = \sigma B \forall \sigma \in G \right\}$$

Put

$$G = \text{Stabilizer of } x \text{ in } \text{Aut}(\Gamma)$$

Then

$$T \subseteq C_G$$

Since

$$A \in C_{\text{Aut}(\Gamma)} \subseteq C_G$$

$$E_i^* \in C_G \quad \text{OZIED}$$

and  $T$  gen by  $A, \{E_i^*\}_{i \in V}$

As we will see, in some cases

$$T = C_G$$

In any case we view  $T$  as a combinatorial analog  
of  $C_G$ .

Extended example

Until further notice  $\Gamma = (X, R)$  will be the complete graph  $K_n$

Fix  $x \in X$  write  $T = T(x)$  etc.

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

We will describe  $T$ .

For  $n=2$   $\Gamma$  consists of single edge and  $T$  is  $\text{Mat}_{2 \times 2}(\mathbb{F})$

Now assume  $n \geq 3$ .

---

We have

$$E_0 = n^{-1} J$$

$$E_1 = I - E_0$$

$$= I - n^{-1} J$$

$$A_0 = I$$

$$A = J - I$$

$$A_0 = E_0 + E_1$$

$$E_0 = \frac{1}{n} A_0 + \frac{1}{n} A$$

$$A = (n-1) E_0 - E_1$$

$$E_1 = \frac{n-1}{n} A_0 - \frac{1}{n} A$$

$$E_0^* = \begin{array}{c} \times \\ \begin{array}{c} \begin{array}{c} \text{r(x)} \\ \begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \end{array} \end{array} \end{array}$$

$$E_1^* = I - E_0^* = \begin{array}{c} \begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \dots 1 \end{array} \end{array}$$

$$A_0^* = I$$

$$A^* = n E_0^* - A_0^* \quad (A^* = A I^*)$$

$$= \begin{array}{c} \begin{array}{c|c} n-1 & 0 \\ \hline 0 & -1 \dots -1 \end{array} \end{array}$$

$$A_0^* = E_0^* + E_1^*$$

$$E_0^* = \frac{1}{n} A_0^* + \frac{1}{n} A^*$$

$$A^* = (n-1) E_0^* - E_1^*$$

$$E_1^* = \frac{n-1}{n} A_0^* - \frac{1}{n} A^*$$



By L18

$$n E_0^* E_0 E_0^* = E_0^*,$$

$$n E_0 E_0^* E_0 = E_0$$

The idempotent  $e_0$  from Def 21 is

$$e_0 = n \left( E_0^* E_0 E_0^* + k^{-1} E_1^* E_0 E_1^* \right) \quad k = n-1$$

$$= \left( \begin{array}{c|c} 1 & 0 \\ \hline 0 & \begin{array}{c} \vdots \\ \frac{1}{n-1} \\ \vdots \end{array} \end{array} \right)$$

$$= \frac{n}{n-1} \left( E_0 + E_0^* - E_0 E_0^* - E_0^* E_0 \right)$$

LEM 33 We have

$$(i) (e_0 V)^\perp = E_1 V \cap E_1^* V$$

(ii) Each 1-diml subspace of  $E_1 V \cap E_1^* V$  is  
an irred  $T$ -module on which  $A, A^*$  act as  $-I$

Pf (i)  $\subseteq$ :

$$E_0 V \subseteq e_0 V$$

so

$$(e_0 V)^\perp \subseteq (E_0 V)^\perp = E_1 V$$

Sim

$$(e_0 V)^\perp \subseteq E_1^* V$$

$\supseteq$ : obs  $E_1 V \cap E_1^* V$  is a  $T$ -module

so it is a dsum of irred  $T$ -modules.

These modules have endpt  $I$  so are not primary.

These modules are attch to  $e_0 V$  by Prop 32

(iii)  $E_1 V$  is eigspace for  $A$  with eigenval  $-1$ ,  $A, A^*$  gen  $T$ ,  
 $E_1^* V$  is eigspace for  $A^*$  with eigenval  $-1$ ,  $\square$

COR 34

(i) Up to iso  $\exists$  unique non primary mod  $T$ -module  $W$ .(ii)  $W$  has dim 1,

endpt 1, dual endpt 1,

diam 0, dual diam 0

(iii) Each of  $A, A^*$  acts on  $W$  as  $-I$ ,(iv)  $W$  appears in  $V$  with mult  $n-2$ (v) The corresp homogenous component of  $V$  is  $E_V \cap E^*V$ 

Pf Routine using L33.

□

Notation  $F_n$   $K_n$  denote set of iso classes of  $T$ -modules by

$$\Psi = \left\{ \begin{array}{l} \circ, \pm \end{array} \right\}$$

|
\

primary
non primary

obv  $e_i := I - e_0$  is the projectem onto the

non prim homog component.

LEM 35 The matrices

$$E_0^x, E_1^x, E_0^x J E_1^x, E_1^x J E_0^x, E_1^x J E_1^x - E_1^x \quad (*)$$

form a basis for  $T$  that is orthogonal w.r.t. inner product  $\langle \cdot, \cdot \rangle$  from  $L^2 \Omega$ . Moreover  $\dim(T) = 5$ .

Pf Obs

$$E_0^x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_1^x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E_0^x J E_1^x = \begin{pmatrix} 0 & 1 \dots 1 \\ 0 & 0 \end{pmatrix}$$

$$E_1^x J E_0^x = \begin{pmatrix} 0 & 0 \\ 1 \dots 1 & 0 \end{pmatrix}$$

$$E_1^x J E_1^x - E_1^x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \dots 1 \\ 0 & 1 \dots 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

Above five matrices have non 0 entries in mutually disjoint positions, so they are mutually orthogonal. One checks

the span of  $*$  is closed under mult and contains  $A, A^x$ , so it is  $T$ .  $\square$

LEM 36 Each of the following is a basis for  $T$

(i)  $I, E_0, E_0^\vee, E_0 E_0^\vee, E_0^\vee E_0$

(ii)  $I, A, A^\vee, AA^\vee, A^\vee A$

Pf Ex

---

LEM 37 the  $F$ -algebra  $T$  is iso to

$$\text{Mat}_{2 \times 2}(F) \oplus F$$

Pf We display an algebra iso

$$\sigma: \text{Mat}_{2 \times 2}(F) \oplus F \rightarrow T$$

4	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right\}$	$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \right\}$	$\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0 \right\}$	$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right\}$	$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right\}$
$\sigma(4)$	$E_0^2$	$\frac{E_0^2 J E_1^2}{n-1}$	$\frac{E_1^2 J E_0^2}{n-1}$	$\frac{E_1^2 J E_1^2}{n-1}$	$e_1$

□

Recall  $\text{Aut}(K_n)$  is symmetric group  $S_n$

Let  $G$  denote the stabilizer of  $x$  in  $\text{Aut}(K_n)$

and obs  $G \cong S_{n-1}$

LEM 38 With above notation

$$T = C_G \quad (\text{centralizer algebra of } G)$$

Pf One checks  $C_G$  has basis consisting of the matrices

(\*) in L35. □

We now show that for  $K_n$ ,  $T$  has an inner automorphism

that swaps  $A, A^v$

We define  $\Delta = \Delta(x) \in \text{Mat}_x(\mathbb{F})$  by

$$\Delta = \left( \begin{array}{c|cccc} & 1 & & & \\ \hline & & 1 & \dots & \\ & 1 & 1-n & & \\ & & & 1 & \\ \vdots & & & & \ddots \\ & & & & & 1-n \\ & & & & & & 1 \end{array} \right)$$

Obs

$$\Delta^t = \Delta, \quad \bar{\Delta} = \Delta$$

One checks  $\Delta$  is invertible with

$$\Delta^{-1} = n^{-1} \left( \begin{array}{c|cccc} & 1 & 1 & \dots & 1 \\ \hline & & & & \\ & 1 & -1 & & \\ & & & -1 & \\ \vdots & & & & \ddots \\ & & & & & -1 \\ & & & & & & 1 \end{array} \right)$$

One checks

$$\begin{aligned} \Delta &= n(E_0 - E_1^v) = n(E_0^* - E_1) \\ &= n(E_0 + E_0^* - I) = A + A^v + (2-n)I \end{aligned}$$



LEM 39 We have:

$$(i) \quad A \Delta = \Delta A^*$$

$$(ii) \quad A^{\circ} \Delta = \Delta A$$

Pf (i) Simplify each side using

$$\Delta = n(\bar{E}_0 + \bar{E}_0^* - I),$$

$$A = n\bar{E}_0 - I, \quad A^* = n\bar{E}_0^* - I.$$

(ii) By (i) and since each of  $\Delta, A, A^*$  is symmetric.  $\square$

COR 40 For  $i=0,1$

$$(i) \quad \Delta E_i^* V = E_i V$$

$$(ii) \quad \Delta E_i V = E_i^* V$$

Pf By L39 and since  $\Delta^{-1}$  exists  $\square$



By L39 we see  $\Delta^2 \in \mathbb{Z}(T)$

One finds

$$\Delta^2 = n \left( \begin{array}{c|cccc} & & & & 0 \\ \hline & 1 & & & \\ \hline & n-1 & -1 & -1 & \\ & -1 & n-1 & & \dots \\ & -1 & -1 & n-1 & \\ & & \vdots & \ddots & \\ & & & & n-1 \end{array} \right)$$

$$= n^2 \left( I - E_0 - E_0^* + E_0 E_0^* + E_0^* E_0 \right)$$

$$= ne_0 + n^2 e_1$$