

More intersection numbers.

Given a DRG $\Gamma = (X, R)$ with diameter D , since the distance matrices $\{A_i\}_{i=0}^D$ form a basis for the Bose-Mesner algebra $\] scalars$

$$p_{ij}^h \in \mathbb{F} \quad (0 \leq h, i, j \leq D)$$

such that

$$A_i A_j = \sum_{h=0}^D p_{ij}^h A_h \quad (*)$$

for $0 \leq i, j \leq D$. Since A_i, A_j commute

$$p_{ij}^h = p_{ji}^h \quad (0 \leq h, i, j \leq D)$$

For $0 \leq h \leq D$ and $x, y \in X$ at $d(x, y) = h$ compute the (x, y) -entry in $(*)$ to find

$$p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$$

So p_{ij}^h is a nonnegative integer. For notational convenience

define

$$p_{ij}^h = 0 \quad \text{unless } 0 \leq i, j \leq D \quad (i, j \in \mathbb{Z})$$

Note that

$$c_i = p_{i,i-1}^i, \quad a_i = p_{i,i}^i, \quad b_i = p_{i,i+1}^i$$

for $0 \leq i \leq D$. We often call any p_{ij}^h an intersection number of Γ .

By simple counting arguments we find

$$p_{0j}^h = \delta_{hj} \quad (0 \leq h, j \leq D)$$

$$p_{i0}^h = \delta_{hi} \quad (0 \leq h, i \leq D)$$

$$p_{ij}^0 = \delta_{ij} k_i \quad (0 \leq i, j \leq D)$$

$$\sum_{i=0}^D p_{ij}^h = k_j \quad (0 \leq h, j \leq D)$$

LEM 44 With above notation

$$k_h p_{ij}^h = k_i p_{hj}^i = k_j p_{ih}^j \quad (0 \leq h, i, j \leq D)$$

pf Each of the three products equals $|X|^{-1}$ times the number

of triples

$$\left| \left\{ x, y, z \mid x, y, z \in X, \partial(x, y) = h, \partial(y, z) = i, \partial(x, z) = j \right\} \right| \quad \square$$

LEM 45 For a DRG Γ with diameter D

and for $0 \leq h, i, j \leq D$

(i) $P_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two

(ii) $P_{ij}^h \neq 0$ if one of h, i, j is equal to the sum of the other two.

pf. The distance function \mathcal{D} satisfies the triangle inequality \square

Next goal: show each P_{ij}^h is determined by $\{c_i\}_{i=1}^D$, $\{b_i\}_{i=0}^{D-1}$.

LEM 46 For a DRG $\Gamma = (X, R)$ with diameter D

and for $0 \leq h, i, j, r \leq D$

$$\sum_{\alpha=0}^D p_{hx}^r p_{ij}^\alpha = \sum_{\beta=0}^D p_{x\beta}^r p_{hi}^\beta$$

pf Expand each side of

$$A_h(A_i A_j) = (A_h A_i) A_j$$

as a linear combination of the distance matrices,

and compare coefficients. \square

LEM 47 For a DRG Γ with diameter D and for

$$0 \leq i, j \leq D$$

$$c_r p_{ij}^{r-1} + a_r p_{ij}^r + b_r p_{ij}^{r+1} = b_{r-1} p_{ij}^{r-1} + a_{r-1} p_{ij}^r + c_{r-1} p_{ij}^r$$

pf Take $h=1$ in LEM 46 □

Note 48 For a DRG Γ with diameter D we can compute

$$p_{ij}^h$$

$$0 \leq h \leq D$$

from $\{c_i\}_{i=0}^D$, $\{b_i\}_{i=0}^D$ using the recursion in LEM 47.

Example 49 For a DRG Γ with diameter D

$$p_{2i-1}^i = c_i \frac{a_i + a_{i-1} - a_1}{c_2} \quad 1 \leq i \leq D$$

$$p_{2i}^i = b_i \frac{a_i + a_{i-1} - a_1}{c_2} \quad 0 \leq i \leq D-1$$

$$p_{2i}^i = \frac{c_i(b_{i-1}) + a_i(a_i - a_{i-1}) + b_i(c_{i-1})}{c_2} \quad 0 \leq i \leq D$$

For a DRG $\Gamma = (X, R)$ with diameter D

recall the polynomials $\{u_i\}_{i=0}^D$ & $\{v_i\}_{i=0}^D$ from defs 23, 30.

We wish to say these are orthogonal polynomials. What does

this mean? what is orthogonal to what?

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

LEM 50 Let X be a finite nonempty set. For $B, C \in \text{Mat}_X(\mathbb{F})$

put

$$\langle B, C \rangle = \text{tr}(B \bar{C}^t)$$

$$\|B\|^2 = \langle B, B \rangle$$

Then for all $B, B', C \in \text{Mat}_X(\mathbb{F})$ and $\alpha \in \mathbb{F}$

$$(i) \quad \langle \alpha B, C \rangle = \alpha \langle B, C \rangle$$

$$(ii) \quad \langle B, C \rangle = \overline{\langle C, B \rangle}$$

$$(iii) \quad \langle B + B', C \rangle = \langle B, C \rangle + \langle B', C \rangle$$

$$(iv) \quad \|B\|^2 \text{ is a nonneg real number}$$

$$(v) \quad \|B\|^2 = 0 \text{ iff } B = 0$$

In other words $\langle \cdot, \cdot \rangle$ is a positive definite Hermitian form on $\text{Mat}_X(\mathbb{F})$

LEM 51 Referring to LEM 50

$$\begin{aligned}\langle AB, C \rangle &= \langle B, \bar{A}^t C \rangle \\ &= \langle A, C \bar{B}^t \rangle\end{aligned}$$

for all $A, B, C \in \text{Mat}_X(\mathbb{F})$

Pf Routine using $\text{tr}(uv) = \text{tr}(vu)$

□

LEM 52 For a DRG $\Gamma = (X, R)$ with diameter D

$$(i) \quad \langle A_i, A_j \rangle = \delta_{ij} k_i |X| \quad (0 \leq i, j \leq D)$$

$$(ii) \quad \langle E_i, E_j \rangle = \delta_{ij} m_i \quad (0 \leq i, j \leq D)$$

Pf (i)

$$\begin{aligned} \langle A_i, A_j \rangle &= \langle A_i, I A_j \rangle \\ &= \langle A_i \bar{A}_j^t, I \rangle \\ &= \langle A_i A_j, I \rangle \\ &= \sum_{h=0}^D p_{ij}^h \underbrace{\langle A_h, I \rangle} \end{aligned}$$

$$\text{tr } A_h = \delta_{h,0} |X|$$

$$= p_{ij}^0 |X|$$

$$= \delta_{ij} k_i |X|$$

$$(ii) \quad \langle E_i, E_j \rangle = \langle E_i, I E_j \rangle$$

$$= \langle E_i \bar{E}_j^t, I \rangle$$

$$= \langle E_i E_j, I \rangle$$

$$= \delta_{ij} \langle E_i, I \rangle$$

$$= \delta_{ij} \text{tr}(E_i) = \delta_{ij} m_i \quad \square$$

thm 53 For a DRG $\Gamma = (X, R)$ with diameter D

(i) For $0 \leq i, j \leq D$

$$\sum_{r=0}^D v_i(\theta_r) v_j(\theta_r) m_r = \delta_{ij} k_i |X| \quad \text{"row orthog"}$$

(ii) For $0 \leq r, s \leq D$

$$\sum_{i=0}^D v_i(\theta_r) v_i(\theta_s) k_i = \delta_{rs} m_r |X| \quad \text{"column orthog"}$$

pf (i) In the equation

$$\langle A_i, A_j \rangle = \delta_{ij} k_i |X|$$

write each of A_i, A_j as a linear combination of the primitive idempotents using LEM 33 (i) and simplify using

LEM 52 (ii).

(ii) In the equation

$$\langle E_r, E_s \rangle = \delta_{rs} m_r$$

write each of E_r, E_s as a linear combination of the

distance matrices and simplify using LEM 52 (i). □

Thm 54 For a DRG $\Gamma = (X, R)$ with diameter D

(i) For $0 \leq i, r \leq D$

$$\sum_{r=0}^D u_i(\theta_r) u_j(\theta_r) m_r = \delta_{i,j} k_i^{-1} / |X|$$

(ii) For $0 \leq r, s \leq D$

$$\sum_{i=0}^D u_i(\theta_r) u_i(\theta_s) k_i = \delta_{r,s} m_r^{-1} / |X|$$

Pf Evaluate thm 53 using $v_k = u_k k_i$ ($0 \leq i \leq D$). \square

Another formula for the p_{ij}^h

LEM 55 For a DRG $\Gamma = (X, R)$ with diameter D

and for $0 \leq h, i, j \leq D$

$$\begin{aligned} p_{ij}^h &= |X|^{-1} k_h^{-1} \langle A_i A_j, A_h \rangle \\ &= |X|^{-1} k_h^{-1} \langle A_h, A_i A_j \rangle \end{aligned}$$

pf Expand $A_i A_j$ using

$$A_i A_j = \sum_{l=0}^D p_{ij}^l A_l$$

and use LEM 52(i) □

Thm 5.6 For a DRG $\Gamma = (X, R)$ with diameter D

and for $0 \leq h, i, j \leq D$

$$P_{ij}^h = |X|^{-h} \sum_{r=0}^D v_i(r) v_j(r) u_h(r) m_r$$

Pf Obs

$$\begin{aligned} \langle A_i A_j, A_h \rangle &= \left\langle \sum_{r=0}^D v_i(r) v_j(r) E_r, \sum_{a=0}^D v_h(a) E_a \right\rangle \\ &= \sum_{r=0}^D v_i(r) v_j(r) v_h(r) m_r \end{aligned}$$

Now evaluate the above equation using LEMSS

□

Thm 57 For a DRG $\Gamma = (X, R)$ with diameter D

the intersection numbers c_i, a_i, b_i are determined by

the spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \dots & \theta_D \\ m_0 & m_1 & \dots & m_D \end{pmatrix}$$

as follows

$$|X| = \sum_{i=0}^D m_i$$

$$b_0 = k = \theta_0$$

$$v_1 = \lambda \quad u_1 = \frac{\lambda}{k}$$

$$c_1 = 1$$

$$a_1 = p_{11}^1 = |X|^{-1} \sum_{r=0}^D v_1(\theta_r) v_1(\theta_r) u_1(\theta_r) m_r$$

$$b_1 = k - a_1 - 1$$

$$u_2 = \frac{\lambda u_1 - a_1 u_1 - c_1 u_0}{b_1}$$

$$c_2 = p_{11}^2 = |X|^{-1} \sum_{r=0}^D v_1(\theta_r) v_2(\theta_r) u_2(\theta_r) m_r$$

$$k_2 = \frac{k b_1}{c_2}$$

$$v_2 = u_2 k_2$$

$$a_2 = p_{21}^2 = |X|^{-1} \sum_{r=0}^D v_2(a_r) v_1(a_r) u_2(a_r) m_r$$

$$b_2 = k - a_2 - c_2$$

$$u_3 = \dots$$

etc.



$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

Given DRG $\Gamma = (X, R)$ with diameter D and standard module

$V = \mathbb{F}^X$. Recall for $0 \leq j \leq D$ the subspace $E_j V$ is the eigenspace

of adjacency matrix A for the eigenvalue θ_j . We now examine the

geometry of the vectors $\{E_j \hat{x} \mid x \in X\}$

LEM 58. Given DRG $\Gamma = (X, R)$ with diameter D .

For $0 \leq i, j \leq D$ and for $x, y \in X$ at $\partial(x, y) = i$

$$(i) \quad \langle E_j \hat{x}, E_j \hat{y} \rangle = |X|^{-1} m_j u_i(\theta_j)$$

$$(ii) \quad \|E_j \hat{x}\|^2 = |X|^{-1} m_j$$

$$(iii) \quad u_i(\theta_j) = \frac{\langle E_j \hat{x}, E_j \hat{y} \rangle}{\|E_j \hat{x}\| \|E_j \hat{y}\|}$$

'cosine of angle between $E_j \hat{x}, E_j \hat{y}$ '

Pf (i) Recall by Lem 33

$$E_j = |X|^{-1}_{m_j} \sum_{h=0}^D u_h(\theta_j) A_h$$

Obs

$$\langle E_j \hat{x}, E_j \hat{y} \rangle = \langle \bar{E}_j^t E_j \hat{x}, \hat{y} \rangle$$

$$= \langle E_j^2 \hat{x}, \hat{y} \rangle$$

$$= \langle E_j \hat{x}, \hat{y} \rangle$$

$$= (E_j)_{xy}$$

$$= |X|^{-1}_{m_j} u_i(\theta_j)$$

(ii) Set $i=0$ $x=y$ in (i)

(iii) Combine (i), (ii)

□

Another view

LEM 59 With ref to LEM 58

$$\det \begin{pmatrix} \|E_2 \hat{x}\|^2 & \langle E_2 \hat{x}, E_2 \hat{y} \rangle \\ \langle E_2 \hat{y}, E_2 \hat{x} \rangle & \|E_2 \hat{y}\|^2 \end{pmatrix} = |X|^{-2} m_j^2 \left(1 - |u_i(\theta_j)|^2 \right)$$

$$= |X|^{-2} m_j^2 \|E_2 \hat{x} - u_i(\theta_j) E_2 \hat{y}\|^2$$

Pf Use LEM 58

□

COR 60 Given DRG Γ with diameter D . Then

$$|u_i(\theta_j)| \leq 1 \quad (0 \leq i, j \leq D)$$

Pf. Clear from LEM 59

□

COR 61. Given a DRG $\Gamma = (X, R)$ with diameter D .

For $0 \leq i, j \leq D$ the following are equiv.

(i) $u_i(\partial_j) = 1$

(ii) $E_j \hat{x} = E_j \hat{y}$ for all $x, y \in X$ at $\partial(x, y) = i$

(iii) $\exists x, y \in X$ with $\partial(x, y) = i$ and $E_j \hat{x} = E_j \hat{y}$

Pf Use LEM 59

□

COR 62 Given DRG $\Gamma = (X, R)$ with diameter D

For $0 \leq i, j \leq D$ the following are equivalent

(i) $u_i(\partial_j) = -1$

(ii) $E_j \hat{x} = -E_j \hat{y}$ for all $x, y \in X$ at $\partial(x, y) = i$

(iii) $\exists x, y \in X$ with $\partial(x, y) = i$ and $E_j \hat{x} = -E_j \hat{y}$

Pf Use LEM 59.

□

EXAMPLE 63 3-cube $\Gamma = H(3,2)$

$$D = 3, \quad a_i = i, \quad b_i = 0 - i$$

$$\text{Spectrum} \quad \begin{pmatrix} 3 & 1 & -1 & -3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

λ	0	1	2	3
u_i	1	$\frac{\lambda}{3}$	$\frac{\lambda^2 - 3}{6}$	$\frac{\lambda(\lambda^2 - 7)}{6}$

table of cosines is

	E_0	E_1	E_2	E_3
A_0	1	1	1	1
A_1	1	$\frac{1}{3}$	$-\frac{1}{3}$	-1
A_2	1	$-\frac{1}{3}$	$\frac{1}{3}$	1
A_3	1	-1	1	-1

(i,j)-entry is $u_i(e_j)$

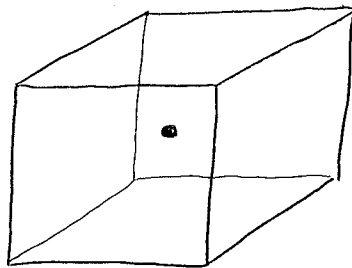
Ex 63, cont Take $\mathbb{F} = \mathbb{R}$

Consider cosines for $E = E_1$

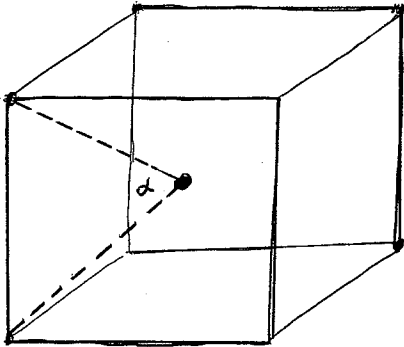
$$\dim EV = 3$$

So view EV as Euclidean 3-space

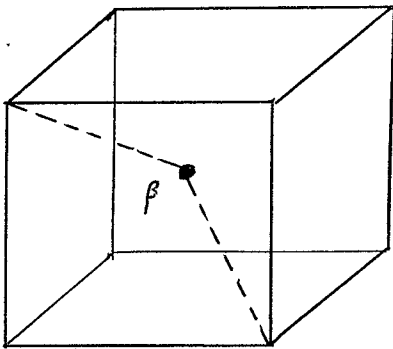
The vectors $\{E\hat{x} \mid x \in X\}$ form the vertices of a
geometric cube centered at the origin



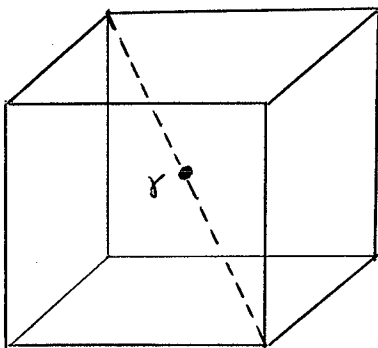
• = origin



$$\cos \alpha = \frac{1}{3}$$



$$\cos \beta = -\frac{1}{3}$$



$$\cos \gamma = -1$$

$F = \mathbb{C} \text{ or } \mathbb{R}$ $\Gamma = (X, R)$ is any DRG diam D .

DEF 64 A representation of Γ is a pair

(ρ, H) where H is a nonzero Hermitian space

(with inner product $\langle \cdot, \cdot \rangle$) and $\rho: X \rightarrow H$ is

a map such that

$$(R1) \quad H = \text{Span} \{ \rho(x) \mid x \in X \}$$

$$(R2) \quad \forall x, y \in X$$

$\langle \rho(x), \rho(y) \rangle$ depends only on $\exists(x, y)$

$$(R3) \quad \forall x \in X$$

$$\sum_{y \in \Gamma(x)} \rho(y) \in \text{Span}(\rho(x))$$

The above representation is nondegenerate whenever

$\{ \rho(x) \mid x \in X \}$ are mutually distinct

Ex 65 For the D -cube $\Gamma = H(0, 2)$ view

$$X = \left\{ \alpha_1 \alpha_2 \dots \alpha_D \mid \alpha_i \in \{1, -1\} \quad 1 \leq i \leq D \right\}$$

Take

$$H = \mathbb{F}^D \quad (\text{column vectors}) \quad \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}$$

$$\langle u, v \rangle = u^t v \quad (u, v \in H)$$

For a vertex $x = \alpha_1 \alpha_2 \dots \alpha_D$ of Γ define

$$\rho(x) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_D \end{pmatrix} \in H$$

Then (ρ, H) is a representation of Γ .

Given a representation (p, H) of a DRG $\Gamma = (X, R)$

the corresponding Gram matrix is the matrix of inner products

$$E = \left(\langle p(x_i), p(x_j) \rangle \right)_{x_i, x_j \in X}$$

$$E \in \text{Mat}_X(\mathbb{F})$$

Representations (p, H) and (p', H') of Γ are called

equivalent whenever their Gram matrices E, E' satisfy

$$E' \in \text{Span}(E)$$

Usually we do not distinguish between equivalent reps of Γ .

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

LEM 66 For a DRG $\Gamma = (X, R)$ with diameter D ,

and given a primitive idempotent E_j of Γ

(i) (ρ, H) is a rep of Γ where

$$H = E_j V \quad \text{with inner product inherited from } V$$

and

$$\rho: \begin{array}{l} X \rightarrow H \\ x \rightarrow E_j \hat{x} \end{array}$$

(ii) For this rep. the Gram matrix is E_j .

(iii) For $0 \leq i \leq D$ and $x \in X$

$$\sum_{y \in \Gamma_i(x)} \rho(y) = v_i(e_j) \rho(x) \quad (*)$$

(iv) the rep is nondegenerate iff $u_i(e_j) \neq 1$ ($1 \leq j \leq D$)

pf (i)-(iii) Check axioms R1-R3 in Def 64

$$R1: \text{Span}(\{p(x) \mid x \in X\}) = \text{col space of } E_2 \\ = H$$

R2: Gram matrix is E_2 by LEM 59 (i)

R3: To verify $*$ consider col x in

$$E_2 A_i = v_i(e_2) E_2$$

Note R3 follows from $*$.

(iv) Use Cor 61

□

Thm 67 Given a DRG $\Gamma = (X, R)$ with diam D

Given a representation (ρ, H) of Γ ,

then this rep is equivalent to a rep of Γ from LEM 67, for

a unique γ .

Pf Consider Gram matrix

$$E = \left(\langle \rho(x), \rho(y) \rangle \right)_{x, y \in X}$$

Show E is a non-scalar multiple of a primitive idempotent of Γ .

Obs $E \neq 0$ by R1

By R2 $\exists \alpha_i \in \mathbb{F}$ ($0 \leq i \leq D$) s.t.

$$E = \sum_{i=0}^D \alpha_i A_i$$

So

$$E \in M$$

Pick $x \in X$. By R3 $\exists \theta \in \mathbb{F}$ s.t.

$$\sum_{y \in \Gamma(x)} \rho(y) = \theta \rho(x) \quad *$$

In (*) take $\langle \cdot, \cdot \rangle$ of each side with $\rho(x)$ to find θ is indep of x

Now (*) implies

$$EA = \theta E$$

So θ is an eigenvalue of Γ and

$$E \in \text{Span}(E_j) \quad \text{where } \theta = \theta_j. \quad \square$$

Given a DRG $\Gamma = (X, R)$ with diameter D

Given a primitive idempotent E_j of Γ

We call E_j nondegenerate whenever

$$u_i(\theta_j) \neq 1 \quad (1 \leq i \leq D)$$

[ie whenever the corresp representation from [66] is nondegenerate]

\square

Recall the Hamming graph $H(0, N)$

$$s_i = i \quad b_i = (N-1)(0-i)$$

$$k_i = m_i = (N-1)^i \binom{p}{i} \quad \theta_i = (0-i)(N-1) - i$$

$$|X| = N^p$$

We now explain how the corresp polynomials u_i are

Krawtchouk polynomials.

Notation

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

For an integer $i \geq 0$ and $a \in \mathbb{F}$

$$(a)_i = \underbrace{a(a+1)(a+2) \cdots (a+i-1)}_{i \text{ terms}}$$

We interpret

$$(a)_0 = 1$$

Obs for integers $j \geq 0$

$$(-j)_i = \begin{cases} \neq 0 & \text{if } 0 \leq i \leq j \\ 0 & \text{if } i \geq j+1 \end{cases}$$

For integers $r, s \geq 0$ and scalars

$$d_1, d_2, \dots, d_r, \beta_1, \beta_2, \dots, \beta_s \in \mathbb{F}$$

The corresponding hypergeometric series in a variable z is

$${}_rF_s \left[\begin{array}{c} d_1, d_2, \dots, d_r \\ \beta_1, \beta_2, \dots, \beta_s \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(d_1)_n (d_2)_n \dots (d_r)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_s)_n} \frac{z^n}{n!}$$

If at least one of d_1, d_2, \dots, d_r is an integer ≤ 0 then

the series has finitely many nonzero terms and questions of convergence

do not arise. In this course we consider this situation only.

For an integer $D \geq 0$ and $p \in \mathbb{F} \setminus \{0, 1\}$,

For $0 \leq i \leq D$ define $k_i = k_i(\lambda, p, D) \in \mathbb{F}[\lambda]$ by

$$k_i = {}_2F_1 \left[\begin{matrix} -i, -\lambda \\ -D \end{matrix} \middle| \frac{1}{p} \right]$$

So

$$k_0 = 1$$

$$k_1 = 1 - \frac{\lambda}{D} \frac{1}{p}$$

$$k_2 = 1 - \frac{2\lambda}{D} \frac{1}{p} + \frac{\lambda(\lambda-1)}{D(D-1)} \frac{1}{p^2}$$

⋮

Call $\{k_i\}_{i=0}^D$ the Krawtchouk polynomials with parameters

D, p

One checks that $\{k_i\}_{i=0}^D$ satisfy the 3-term recurrence

$$-\lambda k_i = p(D-i)k_{i+1} - (p(D-i) + i(1-p))k_i + i(1-p)k_{i-1} \\ (0 \leq i \leq D)$$

Comparing this with our 3-term rec for $\{u_i\}_{i=0}^D$ we find

LEM 68 For the Hamming graph $H(D, N)$

$$u_i(e_j) = K_i(z) \quad (0 \leq i, j \leq D)$$

where

$$K_i = K_i(\lambda, p, D), \quad p = 1 - \frac{1}{N}$$

□

As an aside

We mention a few facts about the Krawtchouk polynomials.

For arb D, p

- the (row) orthogonality is

$$\sum_{r=0}^D K_i(r) K_j(r) \binom{D}{r} p^r (1-p)^{D-r} = \delta_{ij} \binom{D}{i}^{-1} \left(\frac{1-p}{p}\right)^D \quad (0 \leq i, j \leq D)$$

$$\left[\text{For } p = 1 - \frac{1}{N}, N = 2, 3, \dots \text{ this is just thm 54 (i)} \right]$$

- Duality relation:

$$K_i(z) = K_j(i) \quad (0 \leq i, j \leq D)$$

- Difference equation

$$-i K_i(z) = p(D-j) K_i(j+1) - (p(D-j) + j(1-p)) K_i(j) + j(1-p) K_i(j-1) \quad (0 \leq i, j \leq D)$$

• "Forward shift operator"

$$k_i(z+1, p, D) - k_i(z, p, D) = -\frac{i}{Dp} k_{i-1}(z, p, D)$$

• "Backward shift operator"

$$\begin{aligned} (D+1-z) k_i(z, p, D) - z \frac{1-p}{p} k_i(z-1, p, D) \\ = (D+1) k_{i+1}(z, p, D+1) \end{aligned}$$

• Generating function: $F_n \quad 0 \leq i, n \leq D$

$$\left(1 - \frac{1-p}{p} t\right)^z (1+t)^{D-z} = \sum_{i=0}^D \binom{D}{i} k_i(z, p, D) t^i$$

$t = \text{indeterminate}$

The Krawtchouk polynomials are members of

a very general class of polynomials called the Askey-scheme.

We will discuss the Askey scheme a bit later.

Next goal: Krein parameters

$F = \mathbb{C} \text{ or } \mathbb{R}$ Given DRG $\Gamma = (X, R)$ diam D

For $B, C \in \text{Mat}_X(F)$ define

$$B \circ C \in \text{Mat}_X(F)$$

by

$$(B \circ C)_{xy} = B_{xy} C_{xy} \quad \forall x, y \in X \quad \text{"entry-wise mult"}$$

Obs

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq D)$$

So Bose-Mesner alg M is closed under \circ

So $\exists q_{ij}^h \in F \quad (0 \leq h, i, j \leq D)$ s.t.

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D)$$

By cmstr

$$q_{ij}^h = q_{ji}^h \quad (0 \leq h, i, j \leq D)$$

The primitive idempotents are real so

$$q_{ij}^h \in \mathbb{R} \quad (0 \leq h, i, j \leq D)$$

As we will see

$$q_{ij}^h \geq 0 \quad (0 \leq h, i, j \leq D)$$

the q_{ij}^h are called the Krein parameters of Γ .

In order to avoid dealing directly with the entry-wise product we do the following.

Until further notice Fix $x \in X$ We call x the "base vectors"

For $0 \leq i \leq p$ let $E_i^* = E_i^*(x)$ denote the diagonal matrices in

$\text{Mat}_X(\mathbb{F})$ with (y, y) -entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad y \in X$$

Call E_i^* the i th dual idempotent of Γ with respect to x

For $0 \leq i \leq p$ and $y \in X$

$$E_i^* \hat{y} = \begin{cases} \hat{y} & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases}$$

So

$$E_i^* V = \text{Span} \{ \hat{y} \mid y \in \Gamma_i(x) \}$$

We call $E_i^* V$ the i th subconstituent of Γ with respect to x

Obs

$$V = \sum_{i=0}^p E_i^* V \quad (\text{orthog dir sum})$$

By construction

$\{E_i^v\}_{i=0}^p$ are lin indep

$$\begin{aligned} \text{tr}(E_i^v) &= \text{rank}(E_i^v) \\ &= k_i \end{aligned} \quad (0 \leq i \leq p)$$

LEM 69 With above notation

$$(i) \quad I = \sum_{i=0}^p E_i^x$$

$$(ii) \quad E_i^v E_j^x = \delta_{ij} E_i^v \quad (0 \leq i, j \leq p)$$

$$(iii) \quad \overline{E_i^v} = E_i^x \quad (0 \leq i \leq p)$$

$$(iv) \quad (E_i^v)^t = E_i^v \quad (0 \leq i \leq p)$$

pf clear.

DEF 70 With above notation, obs $\{E_i^*\}_{i=0}^D$ is a

basis for a commutative subalgebra of $\text{Mat}_X(\mathbb{F})$ denoted

$M^* = M^*(x)$. We call M^* the dual Bose-Mann algebra

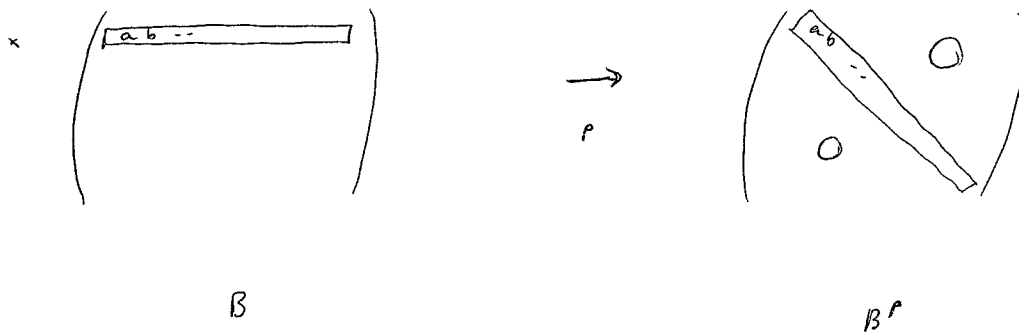
of Γ with respect to x . Obs $\{E_i^*\}_{i=0}^D$ are the primitive

idempotents of M^*

DEF 71 With above notation $\forall B \in \text{Mat}_X(\mathbb{F})$ let

B^p denote the diagonal matrix in $\text{Mat}_X(\mathbb{F})$ with (i,i) -entry

$$(B^p)_{ii} = B_{ii} \quad \forall i \in X$$



LEM 72 With above notation

$$(B \circ C)^p = B^p C^p \quad \forall B, C \in \text{Mat}_X(\mathbb{F})$$

rf clear

□



$\mathbb{F} = \mathbb{R}$ or \mathbb{C} Fix a DRG $\Gamma = (X, R)$ with diameter D and Bose-Mesner algebra M

Fix $x \in X$, let $M^* = M^*(x)$ be the dual Bose-Mesner algebra of Γ

with respect to x . Recall map $p: \text{Mat}_X(\mathbb{F}) \rightarrow \text{Mat}_X(\mathbb{F})$ from Def 71.

LEM 73 With above notation

$$(i) \quad A_i^p = E_i^* \quad (0 \leq i \leq D)$$

$$(ii) \quad I^p = E_0^*$$

$$(iii) \quad J^p = I$$

pf (i) $\forall y \in X$

$$(A_i^p)_{xy} = (A_i)_{xy} = (E_i^*)_{xy}$$

(ii) Set $i=0$ in (i)

(iii) From def of p . □

LEM 74 With above notation

the restriction

$$\rho|_M: M \rightarrow M^*$$

is an isomorphism of vector spaces [Caution: not iso of algebras]

Pf ρ sends the basis $\{A_i\}_{i=0}^p$ to the basis $\{E_i\}_{i=0}^p$ \square

LEM 75 With above notation,

$\forall B, C \in M$ we have

$$\langle B^p, C^p \rangle = |X|^{-1} \langle B, C \rangle$$

Moreover this quantity equals the (x, x) -entry of $B \bar{C}^t$

$$\begin{aligned} \text{Pf } \langle B^p, C^p \rangle &= \text{tr} \left(B^p (\bar{C}^p)^t \right) \\ &= \text{tr} \left(B^p \bar{C}^p \right) \\ &= \sum_{\gamma \in X} (B^p)_{\gamma\gamma} (\bar{C}^p)_{\gamma\gamma} \\ &= \sum_{\gamma \in X} B_{x\gamma} \bar{C}_{\gamma x} \\ &= (B \bar{C}^t)_{xx} \end{aligned}$$

Also

$$\begin{aligned} |X|^{-1} \langle B, C \rangle &= |X|^{-1} \text{tr}(B \bar{C}^t) \\ &= |X|^{-1} \sum_{y \in X} (B \bar{C}^t)_{yy} \\ &= (B \bar{C}^t)_{xx} \end{aligned}$$

Since diagonal entries in $B \bar{C}^t$ are all equal \square

Here is a useful fact about p

LEM 76 With above notation,

$$E_0 E_0^* B = E_0 B^p \quad \forall B \in M$$

Pf Recall $E_0 = |X|^{-1} J$ and

$$E_0^* = \begin{pmatrix} \begin{array}{c|c} 1 & 0 \\ \hline 0 & \bigcirc \end{array} \end{pmatrix}$$

$\forall y, z \in X$

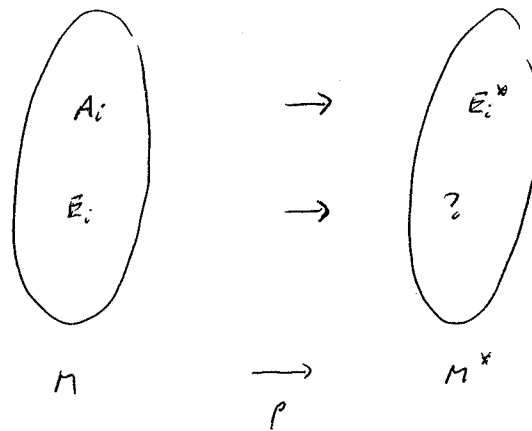
$$\begin{aligned} (E_0 E_0^* B)_{yz} &= \underbrace{(E_0)_{yx}}_{|X|^{-1}} \underbrace{(E_0^*)_{xy}}_1 B_{xz} \\ &= |X|^{-1} B_{xz} \end{aligned}$$

Also since B^p is diagonal,

$$\begin{aligned} (E_0 B^P)_{yz} &= (E_0)_{yz} (B^P)_{zz} \\ &= |X|^{-1} B_{xz} \end{aligned}$$

□

The situation so far:



Def 77 With above notation,

for $0 \leq i \leq 0$ define

$$A_i^* = |X| E_i^P$$

We obs A_i^* is diagonal with $(|X|)^{-1}$ -entry

$$(A_i^*)_{yy} = |X| (E_i)_{xy}$$

$$= m_i u_h(\theta_i)$$

$$h = 2(x, y)$$

Call A_i^* the i th dual distance-matrix

LEM 78 With above notation

(i) $\{A_i^* \}_{i=0}^D$ is a basis for M^*

(ii) For $0 \leq i \leq D$ and $y \in X$

$$A_i^* \hat{y} = \sum_{h=0}^D u_h(e_i) \hat{y} \quad h = \mathcal{J}(x, y)$$

Pf (i) By LEM 74 and since $\{E_i \}_{i=0}^D$ is a basis for M

(ii) By Def 77 □

LEM 79 With above notation

(i) $A_0^* = I$

(ii) $\sum_{i=0}^D A_i^* = |X| E_0^*$

(iii) $\overline{A_i^*} = A_i^* \quad (0 \leq i \leq D)$

(iv) $(A_i^*)^t = A_i^* \quad (0 \leq i \leq D)$

(v) $A_i^* A_j^* = \sum_{h=0}^D \varphi_{ij}^h A_h^* \quad (0 \leq i, j \leq D)$

Pf (i) $A_0^* = |X| E_0^p$ and $E_0 = |X|^{-1} J$

(ii) Apply p to

$$\sum_{i=0}^p E_i = I$$

(iii), (iv) clear

(v) Apply p to

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^p q_{ij}^h E_h$$

□

LEM 80

With above notation

$$(i) \quad A_j^* = m_j \sum_{i=0}^p u_i(\theta_j) E_i^* \quad (0 \leq j \leq p)$$

$$(ii) \quad E_j^* = |X|^{-1} \sum_{i=0}^p v_j(\theta_i) A_i^* \quad (0 \leq j \leq p)$$

Pf Apply p to the equations in LEM 33

□

LEM 81 With above notation

$$(i) \quad \langle E_i^x, E_j^x \rangle = \delta_{ij} k_i \quad (0 \leq i, j \leq D)$$

$$(ii) \quad \langle A_i^x, A_j^x \rangle = \delta_{ij} m_i |X| \quad (0 \leq i, j \leq D)$$

pf Combine LEM 52 and LEM 75 □

Algebraically, the g_{ij}^h, q_{ij}^h are very similar, as next few results show

LEM 82 With above notation, and

for $0 \leq h, i, j \leq D$

$$\begin{aligned} g_{ij}^h &= |X|^{-1} m_h^{-1} \langle A_i^x A_j^x, A_h^x \rangle \\ &= |X|^{-1} m_h^{-1} \langle A_h^x, A_i^x A_j^x \rangle \end{aligned}$$

pf Expand $A_i^x A_j^x$ using

$$A_i^x A_j^x = \sum_{\ell=0}^D g_{i\ell}^j A_\ell^x$$

and use LEM 81 □

LEM 83

With the above notation.

$$m_h z_{it}^h = m_i z_{ht}^i = m_j z_{ih}^j \quad (0 \leq h, i, j \leq D)$$

pf To obtain $m_h z_{it}^h = m_i z_{ht}^i$ observe

$$\begin{aligned} |X| m_h z_{it}^h &= \langle A_i^x A_j^x, A_h^x \rangle \\ &= \langle A_i^x, A_h^x (A_j^x)^{\overline{\leftarrow}} \rangle \\ &= \langle A_i^x, A_h^x A_j^x \rangle \\ &= |X| m_i z_{ht}^i \end{aligned}$$

The rest is similarly obtained. □

LEM 84 With the above notation and for $0 \leq h, i, j, r \leq D$

$$\sum_{\alpha=0}^D q_{h\alpha}^r q_{i\alpha}^s = \sum_{\beta=0}^D q_{j\beta}^r q_{h\beta}^s$$

Pf Expand each side of

$$A_h^* (A_i^* A_j^*) = (A_h^* A_i^*) A_j^*$$

as a linear combination of the dual distance matrices, and

compare coeffs. □

LEM 85 With above notation

$$(i) \quad q_{0j}^h = \delta_{hj} \quad (0 \leq h, j \leq p)$$

$$(ii) \quad q_{i0}^h = \delta_{hi} \quad (0 \leq h, i \leq p)$$

$$(iii) \quad q_{ij}^0 = \delta_{ij} m_i \quad (0 \leq i, j \leq p)$$

$$(iv) \quad \sum_{i=0}^p q_{ij}^h = m_j \quad (0 \leq h, j \leq p)$$

pf (i) obs by L 82

$$q_{0j}^h = |X|^{-1} m_h^{-1} \left\langle \underbrace{A_0^* A_j^*}_I, A_h^* \right\rangle$$

By L 81

$$\langle A_j^*, A_h^* \rangle = \delta_{jh} m_h |X|$$

(ii) Sim

(iii) Use LEM 83

$$(iv) \quad \sum_{i=0}^p q_{ij}^h = |X|^{-1} m_h^{-1} \sum_{i=0}^p \langle A_i^* A_j^*, A_h^* \rangle \quad (\text{by L 82})$$

$$= m_h^{-1} \langle \bar{E}_0^* A_j^*, A_h^* \rangle \quad (\text{by L 79 (ii)})$$

$$= m_h^{-1} m_j \langle \bar{E}_0^*, A_h^* \rangle \quad (\text{by L 80 (i)})$$

$$= m_j \quad (\text{by L 80, L 81})$$

□

The following result shows how to compute the Krein parameters

from the intersection numbers

Theorem 86 For a DRG $\Gamma = (X, R)$ and for $0 \leq h, i, j \leq D$

$$q_{ij}^h = |X|^{-1} m_i^* m_j \sum_{r=0}^D u_r(\theta_i) u_r(\theta_j) v_r(\theta_h)$$

Pf Obs

$$\left\langle A_i^* A_j^v, A_h^* \right\rangle = \left\langle \sum_{r=0}^D m_i m_j u_r(\theta_i) u_r(\theta_j) E_r^*, \sum_{a=0}^D m_h u_a(\theta_h) E_a^* \right\rangle$$

$$= m_h m_i m_j \sum_{r=0}^D u_r(\theta_i) u_r(\theta_j) u_r(\theta_h) k_r$$

$$= m_h m_i m_j \sum_{r=0}^D u_r(\theta_i) u_r(\theta_j) v_r(\theta_h)$$

Now evaluate the above equation using L82

□

Next goal: show Krein parameters are rmreg.

LEM 87 With above notation and for $0 \leq h, i, j, r, s, t \leq 0$

$$(i) \left\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \right\rangle = \delta_{ir} \delta_{js} \delta_{ht} k_h p_{ij}^h$$

$$(ii) \left\langle E_i A_j^* E_h, E_r A_s^* E_t \right\rangle = \delta_{ir} \delta_{js} \delta_{ht} m_h q_{ij}^h$$

$$\begin{aligned} \text{pf (i)} \quad & \left\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \right\rangle \\ &= \text{tr} \left(E_i^* A_j E_h^* \overline{\left(E_r^* A_s E_t^* \right)^t} \right) \\ &= \text{tr} \left(E_i^* A_j E_h^* E_t^* A_s E_r^* \right) \\ &= \delta_{ir} \delta_{ht} \text{tr} \left(E_i^* A_j E_h^* A_s \right) \end{aligned}$$

$$\begin{aligned} \text{tr} \left(E_i^* A_j E_h^* A_s \right) &= \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} (A_j)_{yz} (E_h^*)_{zz} (A_s)_{zy} \\ &= \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} \underbrace{(A_j \circ A_s)_{yz}}_{\delta_{js} A_j} (E_h^*)_{zz} \\ &= \delta_{js} \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} (A_j)_{yz} (E_h^*)_{zz} \\ &= \delta_{js} \sum_{\substack{y \in P_i(X) \\ z \in P_h(X) \\ \mathcal{C}(y, z) = \emptyset}} 1 \\ &= \delta_{js} k_h p_{ij}^h \end{aligned}$$

$$(ii) \quad \langle E_i A_j^* E_h, E_r A_d^* E_t \rangle$$

$$= \text{tr} \left(E_i A_j^* E_h \overline{(E_r A_d^* E_t)^t} \right)$$

$$= \text{tr} \left(E_i A_j^* E_h E_t A_d^* E_r \right)$$

$$\text{tr}(uv) = \text{tr}(vu)$$

$$= \delta_{ir} \delta_{ht} \text{tr} \left(E_i A_j^* E_h A_d^* \right)$$

$$\text{tr} \left(E_i A_j^* E_h A_d^* \right) = \sum_{y \in X} \sum_{z \in X} (E_i)_{yz} \underbrace{(A_j^*)_{zz}}_{|X| (E_j)_{xx}} (E_h)_{zy} \underbrace{(A_d^*)_{yy}}_{|X| (E_d)_{xx}}$$

$$= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_d)_{xy} (E_i \circ E_h)_{yz} (E_j)_{zx}$$

$$= |X|^2 \left((x,x)\text{-entry of } E_d (E_i \circ E_h) E_j \right)$$

$$= |X| \text{trace} \left(E_d (E_i \circ E_h) E_j \right)$$

$$= |X| \text{tr} \left((E_i \circ E_h) E_j E_d \right)$$

$$= \delta_{jd} |X| \text{tr} \left((E_i \circ E_h) E_j \right)$$

$$|X|^2 \sum_{\ell=0}^p \varphi_{ih}^{\ell} E_{\ell}$$

$$= \delta_{jd} \varphi_{ih}^j \text{tr} (E_j)$$

$$= \delta_{jd} \varphi_{ih}^j m_j$$

$$= \delta_{jd} m_h \varphi_{ij}^h$$

□

Ref to LEM 87 setting $r=i, s=j, t=h$ gives

$$\|E_i^* A_j E_h\|^2 = \lambda_{ih} p_{ij}^h \quad (0 \leq h, i, j \leq n)$$

$$\|E_i A_j^* E_h\|^2 = \mu_{ih} q_{ij}^h \quad (0 \leq h, i, j \leq n)$$

Thm 88 (Krein condition) We have

$$q_{ij}^h \geq 0 \quad (0 \leq h, i, j \leq n)$$

Pf Recall $\|B\|^2 \geq 0 \quad \forall B \in \text{Mat}_X(\mathbb{F})$ □

Thm 89 (triple product relations) For $0 \leq h, i, j \leq n$

$$(i) \quad E_i^* A_j E_h = 0 \quad \text{iff} \quad p_{ij}^h = 0$$

$$(ii) \quad E_i A_j^* E_h = 0 \quad \text{iff} \quad q_{ij}^h = 0$$

Pf Recall $\|B\|^2 = 0 \iff B = 0, \quad \forall B \in \text{Mat}_X(\mathbb{F})$ □

Notation For subspaces Y, Z of $\text{Mat}_X(\mathbb{F})$ define

$$YZ = \text{Span} \{ yz \mid y \in Y, z \in Z \}$$

Thm 90 With above notation

(i) the space $M^* M M^*$ has an orthogonal basis

$$\{ E_i^* A_j E_h \mid 0 \leq h, i, j \leq 0, 1 \leq h \neq 0 \}$$

(ii) the space $M M^* M$ has an orthogonal basis

$$\{ E_i A_j^* E_h \mid 0 \leq h, i, j \leq 0, 1 \leq h \neq 0 \}$$

Pf Routine using LEM 87, Thm 89

□



$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

Fix a DRG $\Gamma = (X, R)$ with diameter D and Bose-Mannan algebra M

Fix $x \in X$ and let $M^x = M^x(x)$ be the dual Bose-Mannan algebra of Γ with x .

Next goal: an interpretation of the Krein parameters.

Consider standard module $V = \mathbb{F}^X$

Put $v \in V$

$$v = \sum_{y \in X} \alpha_y \hat{y} \quad \alpha_y \in \mathbb{F}$$

View v as function

$$v : \begin{array}{l} X \rightarrow \mathbb{F} \\ y \rightarrow \alpha_y \end{array}$$

View V as set of all functions $X \rightarrow \mathbb{F}$

Vector space V , together with product of functions is

\mathbb{F} -algebra. F_n

$$v = \sum_{y \in X} \alpha_y \hat{y}$$

$$w = \sum_{y \in X} \beta_y \hat{y}$$

in V , write

$$v \circ w = \sum_{y \in X} \alpha_y \beta_y \hat{y}$$

to represent the product of v, w viewed as functions.

LEM 91 With above notation

for $0 \leq i \leq 0$ and $v \in V$

$$A_i^* v = |X| E_i \hat{x} \circ v$$

Pf For $y \in X$ coordinate y of $A_i^* v$ is

$$\begin{aligned} (A_i^* v)_y &= (A_i^*)_yy v_y \\ &= |X| (E_i)_{xy} v_y \end{aligned}$$

Also

$$\begin{aligned} (E_i \hat{x} \circ v)_y &= (E_i \hat{x})_y v_y \\ &= (E_i)_{yx} v_y \\ &= (E_i)_{xy} v_y \end{aligned}$$

Result follows. □

For subspaces $Y, Z \subseteq V$ define

$$Y \circ Z = \text{Span} \{ y \circ z \mid y \in Y, z \in Z \}$$

Thm 9.2 With above notation and for $0 \leq i, j \leq D$

$$E_i V \circ E_j V = \sum_{\substack{0 \leq h \leq D \\ q_{ij}^h \neq 0}} E_h V$$

Pf \Leftarrow : Given h ($0 \leq h \leq D$) such that $q_{ij}^h = 0$ show

$$E_h (E_i V \circ E_j V) = 0$$

Suffices to show: $\forall y, z \in X$

$$E_h (E_i \hat{y} \circ E_j \hat{z}) = 0$$

Since our base vertex x was arbitrary, wlog $x=y$

Show

$$E_h (E_i \hat{x} \circ E_j \hat{z}) = 0 \quad (*)$$

But

$$\begin{aligned} E_h (E_i \hat{x} \circ E_j \hat{z}) &= E_h (|\hat{x}|^{-1} A_i^x E_j \hat{z}) \\ &= |\hat{x}|^{-1} (E_h A_i^x E_j) \hat{z} \end{aligned}$$

But $q_{ij}^h = 0$ so $E_h A_i^x E_j = 0$ which gives $(*)$.

\geq : For $0 \leq h \leq D$ such that $q_{i,j}^h \neq 0$ show

$$E_i V \circ E_j V \geq E_h V.$$

Obs

$$E_i V \circ E_j V = \text{Span} \left\{ E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X \right\}$$

$$\geq \text{Span} \left\{ \underbrace{E_i \hat{y} \circ E_j \hat{y}}_{(E_i \circ E_j) \hat{y}} \mid y \in X \right\}$$

$$(E_i \circ E_j) \hat{y}$$

$$= (E_i \circ E_j) \text{Span} \{ \hat{y} \mid y \in X \}$$

$$= (E_i \circ E_j) V$$

$$\geq (E_i \circ E_j) E_h V$$

$$= \left(|X|^{-1} \sum_{l=0}^D q_{i,j}^l E_l \right) E_h V$$

$$= |X|^{-1} \underset{\neq 0}{q_{i,j}^h} E_h V$$

$$= E_h V$$

□

COR 93 With above notation and for $0 \leq i, j \leq D$

$$m_i m_j \geq \sum_{\substack{0 \leq h \leq D \\ i \neq h}} m_h$$

pf

Recall

$$m_r = \dim E_r V \quad (0 \leq r \leq D)$$

obs

$$\dim(E_i V \cap E_j V) \leq (\dim E_i V)(\dim E_j V)$$

Result follows.

□

COR 93 Gives another "Feasibility condition" on the intersection numbers.

LEM 94 With above notation and for $0 \leq j \leq D$,

define a binary operation

$$\begin{array}{l} * : E_j V \times E_j V \rightarrow E_j V \\ \quad u \quad v \quad \rightarrow u * v \end{array}$$

where $u * v = E_j(u \circ v)$

Then

$$(i) \quad v * u = u * v \quad \forall u, v \in E_j V$$

$$(ii) \quad u * (v + v') = u * v + u * v' \quad \forall u, v, v' \in E_j V$$

$$(iii) \quad (\alpha u) * v = \alpha(u * v) \quad \forall \alpha \in \mathbb{F} \quad \forall u, v \in E_j V$$

$$(iv) \quad u * v = 0 \quad \forall u, v \in E_j V \quad \text{iff} \quad q_{jj}^2 = 0$$

pf (i) - (iii) clear

(iv) By th 92

□

Note 95 With reference to LEM 94 We call the vector space $E_j V$

together with $*$ the Norton algebra on $E_j V$. We denote

this algebra by N_j . N_j is commutative, nonassociative, no multiplicative identity.

By an automorphism of N_3 we mean an iso of vector

spaces $\sigma: E_3 V \rightarrow E_3 V$ such that

$$\sigma(u * v) = \sigma(u) * \sigma(v) \quad \forall u, v \in E_3 V$$

The set of automorphisms of N_3 forms a group under

composition, denoted $\text{Aut}(N_3)$.

Next goal: How is $\text{Aut}(N_3)$ related to $\text{Aut}(\Gamma)$?

LEM 96 Given a DRG $\Gamma = (X|R)$ with diameter D

then for $0 \leq j \leq D$

$$(i) \quad \sigma|_{E_j V} \in \text{Aut}(N_j) \quad \forall \sigma \in \text{Aut}(\Gamma)$$

$$(ii) \quad \begin{array}{ccc} \text{Aut}(\Gamma) & \rightarrow & \text{Aut}(N_j) \\ \sigma & \rightarrow & \sigma|_{E_j V} \end{array} \quad \text{is hom of gps.}$$

(iii) Suppose $u_i(\sigma_j) \neq 1$ for $1 \leq i \leq D$. Then the hom in (ii) is injective.

Pf (i) σ is invertible on V and $\sigma|_{E_j V} \subseteq E_j V$ so

$$\sigma|_{E_j V} : E_j V \rightarrow E_j V \quad \text{is iso of v.s.}$$

Also $\forall u, v \in E_j V$

$$\begin{array}{ccc} \sigma(u * v) & \stackrel{?}{=} & \sigma(u) * \sigma(v) \\ \parallel & & \parallel \\ \sigma(E_j(u \circ v)) & & E_j(\sigma(u) \circ \sigma(v)) \\ \parallel & & \parallel \end{array}$$

$$E_j \sigma(u \circ v)$$

Obs $\sigma(u \circ v) = \sigma(u) \circ \sigma(v)$ since σ permutes

the coordinates in V

(ii) By constr.

(iii) Each $\sigma \in \text{Aut}(\Gamma)$ permutes the vectors

$$\{ E_{\alpha} \mid \alpha \in X \}$$

and these vectors are mutually distinct by L 66 (iv)

Result follows.

□

• The monster finite simple group (Mon) was first constructed around 1980
by Robert Griess.

• At that time the character table of Mon was known, but it was not known if that table corresponded with an actual group.

• Here is a summary of the construction of Mon, due to Griess.

• There is a mild generalization of a PRG called a commutative association scheme.

• Any finite gp G gives an example, called the group scheme Γ_G

• There is a natural injection of groups $G \rightarrow \text{Aut}(\Gamma_G)$.

• Using the character table of Mon, compute the intersection numbers p_{ij}^h and krewin parameters q_{ij}^h of Γ_{Mon} .

• Find j where m_j is small and $q_{jj}^j \neq 0$

• Guess abstract structure of Norton algebra N_j using p_{ij}^h, q_{ij}^h

• Compute $\text{Aut}(N_j)$

• Using the injection $\text{Mon} \rightarrow \text{Aut}(\Gamma_{\text{Mon}}) \rightarrow \text{Aut}(N_j)$ find Mon as a subgp of $\text{Aut}(N_j)$ \square

Given a DRG $\Gamma = (X|R)$ with diameter D and primitive idempotents $\{E_i\}_{i=0}^D$. So far we have assumed the E_i are in the natural order:

$$k = e_0 > e_1 > \dots > e_D$$

This was for notational convenience only.

We could have used any order such that $e_0 = k$.

For the moment let $\{E_i\}_{i=0}^D$ be any ordering such that $e_0 = k$.

Def 77 With the above notation the ordering $\{E_i\}_{i=0}^D$ is called

Q-polynomial whenever the following (i), (ii) hold for $0 \leq h, i, j \leq D$

(i) $q_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two

(ii) $q_{ij}^h \neq 0$ if one of h, i, j is equal to the sum of the other two.

Γ might have 0, 1, or ≥ 2 Q-poly orderings

of its primitive idempotents.

Until further notice fix a \mathbb{Q} -polynomial ordering

$\{E_i\}_{i=0}^D$ of the primitive idempotents of Γ .

Define

$$c_i^* = q_{i,i-1}, \quad a_i^* = q_{i,i}, \quad b_i^* = q_{i,i+1}$$

for $0 \leq i \leq D$. So

$$c_0^* = 0, \quad a_0^* = 0, \quad b_0^* = 0, \quad q^* = 1,$$

$$c_i^* > 0 \quad 1 \leq i \leq D,$$

$$b_i^* > 0 \quad 0 \leq i \leq D-1$$

LEM 98 With above notation

$$(i) \quad m_i c_i^* = m_{i+1} b_i^* \quad (1 \leq i \leq D)$$

$$(ii) \quad m_i = \frac{b_0^* b_i^* - b_{i-1}^*}{q^* c_i^* - c_i^*} \quad (0 \leq i \leq D)$$

Pf (i) This is

$$m_h q_{h,i}^* = m_i q_{i,h}^*$$

with $j=1$ and $h=i+1$

(i) Use (i) above and $m_0 = 1$ □

Fix $x \in X$ and recall $A_i^* = A_i^*(x)$ $E_i^* = E_i^*(x)$ ($0 \leq i \leq 0$)

We abbreviate $A^* = A^* \mathbf{1}$ and call this the dual adjacency matrix wrt x (and the given Q -poly structure)

LEM 99 With above notation

$$A^* A_i^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^* \quad (0 \leq i \leq 0)$$

where $A_{-1}^* = 0$, $A_{0+1}^* = 0$, $b_{-1}^* = 1$, $c_{0+1}^* = 1$.

pf This is just

$$A_i^* A_j^* = \sum_{h=0}^0 q_{ij}^h A_h^*$$

with $j=1$ □

$$F = \mathbb{R} \text{ or } \mathbb{C}$$

DEF 100 With above notation define polynomials

$$\{v_i^*\}_{i=0}^{D+1} \text{ in } F[\lambda] \text{ by}$$

$$v_0^* = 1, \quad v_1^* = \lambda,$$

$$\lambda v_i^* = c_{i+1}^* v_{i+1}^* + a_i^* v_i^* + b_{i+1}^* v_{i-1}^* \quad (1 \leq i \leq D)$$

where $c_{D+1}^* = 1$.

LEM 101 With above notation

$$(i) \quad \deg v_i^* = i \quad (0 \leq i \leq D+1)$$

$$(ii) \quad \text{the coeff of } \lambda^i \text{ in } v_i^* \text{ is } (c_1^* c_2^* \dots c_i^*)^{-1} \quad (0 \leq i \leq D+1)$$

$$(iii) \quad v_i^*(A^*) = A_i^* \quad (0 \leq i \leq D)$$

$$(iv) \quad v_{D+1}^*(A^*) = 0$$

PF (i), (ii) Clear from Def 100

(iii), (iv) Compare Lem 99, def 100 □

In LEM 80 we saw

$$A^* = m_1 \sum_{i=0}^p u_i(\theta_1) E_i^*$$

Abbrev

$$\theta_i^* = m_i u_i(\theta_1) \quad (0 \leq i \leq p)$$

so that

$$A^* = \sum_{i=0}^p \theta_i^* E_i^*$$

We will see in a moment that $\{\theta_i^*\}_{i=0}^p$ are mutually distinct.

LEM 102 With above notation

- (i) A^* generates M^*
- (ii) $\{\theta_i^*\}_{i=0}^p$ are mutually distinct
- (iii) $\{\theta_i^*\}_{i=0}^p$ are the zeros of v_{011}^*

Pf (i) By LEM 101 (iii) and since $\{A_i^*\}_{i=0}^p$ is a basis for M^* .

(ii) By (i)

(iii) By L101 (iv) v_{011}^* is a scalar mult of the min poly of A^* \square

COR 105 With above notation

The eigenvalues of B^* are $\{\theta_i^*\}_{i=0}^p$.

LEM 106 With above notation for any eigenvalue

$\theta^* = \theta_i^*$ of B^* define a row vector

$$v^* = (v_0^*(\theta^*), v_1^*(\theta^*), \dots, v_p^*(\theta^*))$$

where the v_i^* are from Def 100. Then

$$v^* B^* = \theta^* v^*$$

PF Sim to L29

□

DEF 107 With above notation define poly $\{u_i\}_{i=0}^p$

in $\mathbb{F}[\lambda]$ by

$$u_0^* = 1, \quad u_i^* = \frac{\lambda}{m_i}$$

$$\lambda u_i^* = c_i^* u_{i-1}^* + a_i^* u_i^* + b_i^* u_{i+1}^* \quad (1 \leq i \leq p-1)$$

LEM 108 With above notation

$$u_i^* = \frac{v_i^*}{m_i} \quad (0 \leq i \leq p)$$

Pf Sim to L31.

□

LEM 109 With above notation for an eigenvalue

θ^* $\neq \theta_j^*$ & B^* define a col vector

$$u^* = \begin{pmatrix} u_0^*(\theta^*) \\ u_1^*(\theta^*) \\ \vdots \\ u_p^*(\theta^*) \end{pmatrix}$$

Then

$$B^* u^* = \theta^* u^*$$

Pf Sim to L32

Thm 110 (Askey-Wilson duality) With above notation

$$u_i^*(\theta_j) = u_j^*(\theta_i^*) \quad (0 \leq i, j \leq D)$$

Pf obs

$$\begin{aligned} A_j^* &= A_j^* I \\ &= A_j^* \sum_{i=0}^p E_i^* \\ &= v_j^*(A^*) \sum_{i=0}^D E_i^* \\ &= \sum_{i=0}^p v_j^*(\theta_i^*) E_i^* \\ &= m_j \sum_{i=0}^D u_j^*(\theta_i^*) E_i^* \end{aligned}$$

Also by L80

$$A_j^* = m_j \sum_{i=0}^D u_i(\theta_j) E_i^*$$

Result follows. □

LEM 111 With above notation

$$(i) \quad u_i^*(\theta_0^*) = 1 \quad (0 \leq i \leq n)$$

$$(ii) \quad v_i^*(\theta_0^*) = m_i \quad (0 \leq i \leq n)$$

pf (i) $u_i^*(\theta_0^*) = u_0(\theta_i)$
 $= 1$

(ii) By (i) and since $v_i^* = m_i u_i^*$

□

With above notation

earlier we obtained some "row" and "column" orthogonality relations for the polynomials u_i and v_i

Using the same methods we could get similar relations for the polynomials u_i^* and v_i^* . We could also use Askey-Wilson duality

Thm 112 With above notation

(i) for $0 \leq i, j \leq 0$

$$\sum_{r=0}^0 u_i^*(\theta_r^*) u_j^*(\theta_r^*) k_r = \delta_{ij} m_i^{-1} |X|$$

(ii) for $0 \leq r, s \leq 0$

$$\sum_{i=0}^0 u_i^*(\theta_r^*) u_i^*(\theta_s^*) m_i = \delta_{rs} k_r^{-1} |X|$$

Pf Combine Th 54 and Th 110

□

Thm 113 With above notation

(i) $\forall a \ 0 \leq i, j \leq p$

$$\sum_{r=0}^p v_i^*(\theta_r^*) v_j^*(\theta_r^*) k_r = \delta_{ij} m_i / |X|$$

(ii) $\forall a \ 0 \leq r, s \leq p$

$$\sum_{i=0}^p v_i^*(\theta_r^*) v_i^*(\theta_s^*) m_i^{-1} = \delta_{rs} k_r / |X|$$

pf Eval Thm 112 using $v_i^* = m_i u_i^* \ (0 \leq i \leq p) \quad \square$

We will see shortly that for the Hamming graph $H(0, N)$

and for the ordering $\theta_0 > \theta_1 > \dots > \theta_p$

$$q_{ij}^h = p_{ij}^h \quad (0 \leq h, i, j \leq p)$$

$$a_i^* = \theta_i \quad (0 \leq i \leq p)$$

So $\{E_i\}_{i=0}^p$ is \mathbb{Q} -polynomial and

$$u_i^* = u_i, \quad v_i^* = v_i \quad 0 \leq i \leq p$$

\square

$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$

Given a DRG $\Gamma = (X, R)$ with diameter D and standard

module $V = \mathbb{F}^X$. Primitive idempotents $\{E_i\}_{i=0}^D$ in any

order except $E_0 = |X|^{-1} J$. Fix $x \in X$ into $M^* = M^*(x)$ etc.

Next goal: A characterization of the Q -poly property

in terms of the function algebra V, \circ . We start with a handy fact

LEM 114 With above notation TFAE

(i) the ordering $\{E_i\}_{i=0}^D$ is Q -polynomial

(ii) For $0 \leq h, j \leq D$

$$q_{ij}^h = \begin{cases} 0 & \text{if } h > j+1 \\ \neq 0 & \text{if } h = j+1 \end{cases}$$

(iii) \exists polynomials $\{v_i^*\}_{i=0}^D$ in $\mathbb{F}[\lambda]$ such that

$$\deg v_i^* = i \quad (0 \leq i \leq D)$$

$$A_i^* = v_i^*(A_1^*) \quad (0 \leq i \leq D)$$

Pf ex using the def of Q -poly

□

Referring to the function algebra $V, 0$

for a subspace $W \subseteq V$ consider the subalgebra of V

generated by W . This subalgebra contains \mathbb{I} by the

definition of subalgebra. To see what else

is in the subalgebra, define a binary relation

$\sim = \sim_w$ on X by

$y \sim z$ whenever $\forall w \in W$

y -coord of $w = z$ -coord of w .

Obs \sim is an equivalence relation

For a subset $Y \subseteq X$ let

$$\hat{Y} = \sum_{y \in Y} \hat{y}$$

"characteristic vector of Y "

LEM 115 With above notation the following are equal

(i) the subalgebra of V generated by W

(ii) $\text{Span} \{ \hat{Y} \mid Y \text{ an equiv class of } \sim \}$

Pf \subseteq : $\text{Span} \{ \hat{Y} \mid Y \text{ an equiv class of } \sim \}$

is a subalgebra of V that contains W

\supseteq List the equiv classes of \sim

$$Y_1, Y_2, \dots, Y_t$$

For $w \in W$ write

$$w = \sum_{i=1}^t \alpha_i(w) \hat{Y}_i \quad \alpha_i(w) \in \mathbb{F}$$

For dist i, j ($1 \leq i, j \leq t$) $\exists w_{ij} \in W$

s.t

$$\alpha_i(w_{ij}) \neq \alpha_j(w_{ij})$$

Obs for $1 \leq i \leq t$

$$\hat{Y}_i = \prod_{\substack{1 \leq j \leq t \\ j \neq i}} \frac{w_{ij} - \alpha_j(w_{ij}) \hat{Y}_j}{\alpha_i(w_{ij}) - \alpha_j(w_{ij})}$$

where the product is with respect to \cdot . Result follows \square

COR 117 For a primitive idempotent E of Γ the following

are equivalent:

(i) EV generates V in the function algebra $V, 0$

(ii) E is nondegenerate

Pf EV generates V

$$\Leftrightarrow V = \text{Span} \{ \psi^x \mid x \text{ an equiv class for } \sim_{EV} \}$$

$$\Leftrightarrow \text{Each equiv class of } \sim_{EV} \text{ has single element}$$

$$\Leftrightarrow \{ E\psi^x \mid x \in X \} \text{ muti distinct}$$

$$\Leftrightarrow E \text{ nondegenerate.} \quad \square$$

Thm 118 With above notation TFAE

(i) the ordering $\{E_i\}_{i=0}^D$ is \mathbb{Q} -polynomial

(ii) E_i is nondegenerate and

$$E_i V \circ E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \quad (0 \leq i \leq D)$$

where $E_{-1} = 0$, $E_{D+1} = 0$

(iii) For $0 \leq i \leq D$

$$E_0 V + E_1 V + \dots + E_i V = \underbrace{(E_0 V + E_1 V) \circ (E_0 V + E_1 V) \circ \dots \circ (E_0 V + E_1 V)}_{i \text{ factors}}$$

Pf (i) \rightarrow (ii) Recall dual eigenvalues

$$\theta_i^* = m_i u_i(\sigma) \quad 0 \leq i \leq D.$$

$\{\theta_i^*\}_{i=0}^D$ are mut distinct

so

$$\theta_i^* \neq \theta_h^* \quad 1 \leq i \leq D$$

so E_i non deg.

Recall

$$E_i V \circ E_i V = \sum_{\substack{0 \leq h \leq D \\ g_{i,h} \neq 0}} E_h V$$

and

$$f_{ij}^h = 0 \quad \text{if } |h-i| > 1$$

(ii) \rightarrow (iii) Recall $E_0 V$ is spanned by the all 1's vector in V

$$\text{so } E_0 V \circ E_j V = E_j V \quad \text{for } 0 \leq j \leq 0.$$

$$\text{So } \underbrace{(E_0 V + E_1 V) \circ (\quad) \circ \dots \circ (\quad)}_i = E_0 V + E_1 V + E_1 V \circ E_1 V + \dots + \underbrace{E_1 V \circ \dots \circ E_1 V}_i \\ \subseteq E_0 V + \dots + E_i V$$

by induction

Using Cor 117 we get = in above line.

(iii) \rightarrow (i) For $0 \leq j \leq n-1$

$$(E_0 V + \dots + E_j V) \circ (E_0 V + E_1 V) = E_0 V + \dots + E_{j+1} V$$

So by Thm 92

$$f_{ij}^h = \begin{cases} 0 & \text{if } h > j+1 \\ \neq 0 & \text{if } h = j+1 \end{cases}$$

Now $\{E_i\}_{i=0}^n$ is \mathbb{Q} -polynomial by LEM 114. □

