Lecture 1 Wednesday Jan 21

MATH 846 - Algebraic Graph Theory
B211 VV 11:00 MWF
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References

- Bannai and Ito. Algebraic Combinatorics I: Association Schemes (Benjamin Cummings 1984)
- Brouwer, Cohen, Neumaier, Distance-Regular Graphs (Springer-Verlag 1989)
- Biggs. Algebraic Graph theory 2nd Ed (Cambridge 1993)

1. Introduction

Fix a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

Let X = finite nonempty set

$n = |X|$

$\mathrm{Mat}_X(\mathbb{F})$ = $\mathbb{F}$-algebra of all $n \times n$ matrices with entries in $\mathbb{F}$. View rows/columns as indexed by X

$\mathbb{F}^X$ = vector space $\mathbb{F}^n$ (column vectors) with coordinates indexed by X.

$\mathrm{Mat}_X(\mathbb{F})$ acts on $\mathbb{F}^X$ by left multiplication

Endow $\mathbb{F}^X$ with a Hermitean dot product

$$\langle u, v \rangle = u^t \bar{v} \qquad u, v \in \mathbb{F}^X$$

From now on abbrev

$$V = \mathbb{F}^X$$

For $x \in X$ define $\hat{x} \in V$ by

$$\hat{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow x \text{ coord}$$

Obs

$$\{\hat{x} \mid x \in X\} \quad \text{is ortho-normal basis for } V$$

Write

$$\mathbb{1} = \sum_{x \in X} \hat{x}$$

A matrix $B \in \mathrm{Mat}_X(\mathbb{F})$ is <u>Hermitean</u> whenever

$$\overline{B}^t = B$$

In this case B is diagonalizable over $\mathbb{F}$ and the eigenvalues of B are real.

Let $\theta_0 > \theta_1 > \cdots > \theta_r$ denote the distinct eigenvalues of B (roots of min polynomial of B)

For $0 \le i \le r$ let

$$V_i = \text{eigenspace of } B \text{ for } \theta_i$$

Then

$$V = V_0 + V_1 + \cdots + V_r \qquad \text{(orthog direct sum)}$$

For $0 \le i \le r$ define $E_i \in \mathrm{Mat}_X(\mathbb{F})$ by

$$(E_i - I)V_i = 0$$

$$E_i V_j = 0 \qquad \forall\ i \ne j \quad (0 \le j \le r)$$

Obs

$$E_i E_j = \delta_{ij} E_i \qquad (0 \le i, j \le r)$$

$$\sum_{i=0}^{r} E_i = I$$

$$B = \sum_{i=0}^{r} \theta_i E_i$$

$$E_i = \prod_{\substack{0 \le j \le r \\ j \ne i}} \frac{B - \theta_j I}{\theta_i - \theta_j} \qquad 0 \le i \le r$$

$$E_i^t = E_i \qquad 0 \le i \le r$$

Let M = subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ gen by B.

Then $\{B^i\}_{i=0}^{r}$ is a basis for $\mathbb{F}$-vectorspace M

Also $\{E_i\}_{i=0}^{r}$ is a basis for $\mathbb{F}$-vector space M.

Call $\{E_i\}_{i=0}^{r}$ the <u>primitive</u> <u>idempotents</u> for B

For $0 \le i \le r$ Define $m_i = \dim V_i$

$$= \operatorname{rank} E_i = \operatorname{trace} E_i$$

Call m_i the <u>multiplicity</u> of θ_i

the <u>spectrum</u> of B is the array

$$\mathrm{Spec}(B) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_0 & m_1 & \cdots & m_r \end{pmatrix}$$

A *graph* is a pair $\Gamma = (X, R)$

where X = nonempty finite set (the *vertices*)

and R = set of distinct 2-element subsets of X

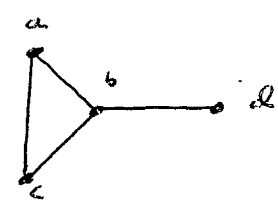
(the *edges*)

Vertices $x, y \in X$ called *adjacent* whenever $xy \in R$

[Note our graphs are undirected, without loops or multiple edges]

Ex

Γ :



$X = \{a, b, c, d\}$ $\qquad$ $R = \{ab, bc, ca, bd\}$

For a graph $\Gamma = (X, R)$ the *adjacency matrix* $A \in \mathrm{Mat}_X(\mathbb{F})$

satisfies

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in R \\ 0 & \text{if } xy \notin R \end{cases} \qquad x, y \in X$$

Ex

$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 0 \end{array}$$

Note $\bar{A} = A$, $A^t = A$ so A Hermitean.

Let M = subalgebra of $Mat_X(\mathbb{F})$ gen by A.

M called <u>adjacency algebra</u> or <u>Bose-Mesner algebra</u> of Γ

By the <u>spectrum</u> of Γ we mean Spec(A)

<u>Classical problem</u> What combinatorial info about a graph Γ can we deduce from its spectrum?

The next few lemmas give examples.

Fix a graph $\Gamma = (X, R)$

For $x \in X$ define

$$\Gamma(x) = \{ y \in X \mid xy \in R \}$$

$$k(x) = |\Gamma(x)| \qquad \text{"valency of } x\text{" or "degree of } x\text{"}$$

Call Γ <u>regular with valency k</u> whenever $k(x) = k$ $\forall x \in X$

LEM 1. With above notation

(i) $\theta_0 \le \max\{k(x) \mid x \in X\}$ $(= k^{max})$

(ii) Suppose Γ is regular val k. Then $\theta_0 = k$

Pf (i) Let v denote an eigenvector for θ_0 So

$$Av = \theta_0 v, \qquad v \neq 0$$

Write

$$v = \sum_{x \in X} \alpha_x \hat{x} \qquad \alpha_x \in \mathbb{F}$$

Define

$$\alpha = \max\{|\alpha_x| \;/\; x \in X\}$$

Obs $\alpha > 0$

Pick $x \in X$ s.t

$$|\alpha_x| = \alpha$$

From coord x of $Av = \theta_0 v$ get

$$\theta_0 \alpha_x = \sum_{\substack{y \in X \\ xy \in R}} \alpha_y$$

So

$$\theta_0 \alpha = |\theta_0 \alpha_x|$$

$$\leq \sum_{\substack{y \in X \\ xy \in R}} |\alpha_y|$$

$$\leq k(x)\, \alpha$$

$$\leq k^{max}\, \alpha$$

So

$$\theta_0 \leq k^{max}$$

(ii) The "all 1's vector" $\mathbb{1} = \sum_{x \in X} \hat{x}$ satisfies

$$A \mathbb{1} = k \mathbb{1}$$ $\square$

Given a graph $\Gamma = (X, R)$

For $x, y \in X$

For an integer $l \geq 0$

A <u>path of length l from x to y</u> is a sequence of vertices

$$x_0, x_1, \ldots, x_l$$

where $x_0 = x$, $x_l = y$,

x_{i-1}, x_i adjacent for $1 \leq i \leq l$.

Let $A =$ adj matrix of Γ.

One checks

$$(A^l)_{xy} = \text{the number of paths of length } l \text{ from } x \text{ to } y$$

A path $x_0, x_1, \ldots, x_l$ is <u>closed</u> whenever $x_0 = x_l$

LEM 2 Given a graph $\Gamma = (X, R)$ with spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_0 & m_1 & \cdots & m_r \end{pmatrix}$$

Given an integer $\ell \geq 0$

$$\sum_{i=0}^{r} m_i \theta_i^{\ell} = \text{the number of closed paths in } \Gamma \text{ that have length } \ell$$

Pf Recall

$$A = \sum_{i=0}^{r} \theta_i E_i$$

so

$$A^{\ell} = \sum_{i=0}^{r} \theta_i^{\ell} E_i$$

so

$$\operatorname{tr}(A^{\ell}) = \sum_{i=0}^{\ell} \theta_i^{\ell} \underbrace{\operatorname{tr}(E_i)}_{m_i}$$

tr = trace

Also

$$\operatorname{tr}(A^{\ell}) = \sum_{x \in X} \underbrace{(A^{\ell})_{xx}}_{\#\text{ paths length } \ell \text{ from } x \text{ to } x}$$

$$= \#\text{ closed paths of length } \ell$$

COR 3 With the notation of Lem 2

(i) $\sum_{i=0}^{r} m_i = |X|$

(ii) $\sum_{i=0}^{r} m_i \theta_i = 0$

(iii) $\sum_{i=0}^{r} m_i \theta_i^2 = 2|R|$

pf set $l = 0, 1, 2$ in LEM 2. $\square$

Fix a graph $\Gamma = (X, R)$

For $x, y \in X$ the <u>distance</u>

$$\partial(x,y) = \min\{ \ell \mid \exists \text{ path of length } \ell \text{ from } x \text{ to } y \}$$

$$\in \mathbb{Z}^{\geq 0} \cup \infty$$

Γ is <u>connected</u> whenever $\partial(x,y) < \infty \quad \forall x, y \in X$

· Until further notice assume Γ is connected

For $x \in X$ def

$$d(x) = \max\{ \partial(x,y) \mid y \in X \} \qquad \text{"diameter wrt } x\text{"}$$

$$D = \max\{ d(x) \mid x \in X \} \qquad \text{"diameter of } \Gamma\text{"}$$

LEM 4 A connected graph Γ with diameter D has at least $D+1$ distinct eigenvalues

Pf. The Bose-Mesner algebra M has basis $\{A^i\}_{i=0}^{r}$ where $r+1 = \#$ dist eigenvalues of Γ.

Show $r \geq D$ Suppose $r < D$.

$\exists\ x, y \in X$ with $\partial(x,y) = r+1$

For $0 \leq i \leq r+1$

$$(A^i)_{xy} = \#\text{ paths of length } i \text{ from } x \text{ to } y$$

$$= \begin{cases} 0 & \text{if } 0 \leq i \leq r \\ \neq 0 & \text{if } i = r+1 \end{cases}$$

Now $B_{xy} = 0 \quad \forall\ B \in M$

and $A^{r+1}_{xy} \neq 0$

contradicting the fact that $A^{r+1} \in M$. So $r \geq D$ $\square$

$\square$

Lecture 2 Friday January 23

Next goal: Use Perron / Frobenius theory of nonnegative matrices to get information about the eigenvalues of a graph.

Until further notice

X = nonempty finite set $\qquad \mathbb{F} = \mathbb{R} \qquad V = \mathbb{R}^X$

LEM 5 Given a symmetric $B \in \mathrm{Mat}_X(\mathbb{R})$ assume all eigenvalues of B are nonnegative (ie B is positive semi-definite). Then $\exists$ vectors $\{v_x\}_{x \in X}$ in V such that

$$\langle v_x, v_y \rangle = B_{xy} \qquad \forall x, y \in X$$

pf By elem linear algebra V has an orthonormal basis $\{w_x\}_{x \in X}$ consisting of eigenvectors for B. Define $P \in \mathrm{Mat}_X(\mathbb{R})$ s.t.

$$w_x = \text{column } x \text{ of } P \qquad \forall x \in X$$

Observe

$$P^t P = I$$

For $x \in X$ let $\theta_x =$ eigenvalue of B for w_x

and define

$$H = \operatorname{diag}(\theta_x)_{x \in X}$$

So

$$BP = PH$$

Now

$$B = PHP^{-1}$$

$$= PHP^t$$

$$= QQ^t$$

where

$$Q = P \operatorname{diag}\left(\sqrt{\theta_x}\right)_{x \in X}$$

Note entries of Q are real since $\theta_x \geq 0$ $\forall x \in X$

For $x \in X$ define $v_x =$ col x of Q^t

For $x, y \in X$

$$\langle v_x, v_y \rangle = (QQ^t)_{xy}$$

$$= B_{xy}$$

$\square$

A graph is <u>complete</u> whenever each pair of distinct vertices are adjacent.

K_n denotes the complete graph on n vertices

LEM 6 The eigenvalues of K_n are

$n-1$ (with mult 1)

-1 (with mult $n-1$)

pf ex.

LEM 7 For a graph $\Gamma = (X, R)$ the following are equiv:

(i) The minimal eigenvalue of Γ is at least -1

(ii) Γ is a disjoint union of complete graphs

pf (i) $\to$ (ii) Define a binary relation $\sim$ on X by

$$x \sim y \quad \text{iff} \quad x = y \text{ or } xy \in R \qquad (\forall x, y \in X)$$

Suffices to show $\sim$ is an equivalence relation.

Obs $A + I$ is pos semi def (A = adj matrix of Γ)

By Lem 5 $\exists$ vectors $\{v_x\}_{x \in X}$ in V such that

$$\langle v_x, v_y \rangle = (A + I)_{xy} = \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{if } x \not\sim y \end{cases} \qquad (\forall x, y \in X)$$

For $x, y \in X$

$$\|v_x - v_y\|^2 = \langle v_x - v_y, v_x - v_y \rangle$$

$$= 2 - 2\langle v_x, v_y \rangle$$

$$= \begin{cases} 0 & \text{if } x \sim y \\ 2 & \text{if } x \not\sim y \end{cases}$$

So

$$x \sim y \quad \text{iff} \quad v_x = v_y$$

So $\sim$ is equiv relation. (ii) $\to$ (i) By LEM 6 $\square$

A graph $\Gamma = (X, R)$ is called <u>bipartite</u> whenever there exists a bipartition

$$X = X^+ \cup X^- \qquad \text{(disjoint union)}$$

such that neither of $X^\pm$ contains an edge.

We will present a few spectral results about bipartite graphs. These results are special cases of the following results on real symmetric matrices.

Given a symmetric $C \in \mathrm{Mat}_X(\mathbb{R})$ call C <u>bipartite</u> whenever $\exists$ bipartition

$$X = X^+ \cup X^- \qquad \text{(disj union)}$$

such that $\forall x, y \in X$

$$C_{xy} = 0 \qquad \text{if} \quad x, y \in X^+ \quad \text{or} \quad x, y \in X^-$$

LEM 8. Given a bipartite symmetric $C \in \mathrm{Mat}_X(\mathbb{R})$.

For all eigenvalues θ of C, $-\theta$ is also an eigenvalue of C. Moreover $\theta, -\theta$ have the same multiplicity.

pf. Let $X = X^+ \cup X^-$ be the bipartition for C.

Define a diagonal matrix $H \in \mathrm{Mat}_X(\mathbb{R})$ by

$$H_{xx} = \begin{cases} 1 & \text{if } x \in X^+ \\ -1 & \text{if } x \in X^- \end{cases} \qquad (\forall x \in X)$$

One checks

$$CH = -HC$$

Now for $v \in V$

v is an eigenvector for C with eigenvalue θ

$\leftrightarrow$

Hv is an eigenvector for C with eigenvalue $-\theta$.

Result follows. $\square$

COR 9 For a bipartite graph Γ.

For all eigenvalues θ of Γ, $-\theta$ is also an eigenvalue of Γ. Moreover $\theta, -\theta$ have the same multiplicity.

Pf. Immed from LEM 8. $\square$

Given a symmetric $C \in Mat_X(\mathbb{R})$ call C reducible whenever there exists a bipartition

$$X = X^+ \cup X^- \quad (\text{disjoint union of nonempty sets})$$

such that

$$C_{xy} = 0 \qquad \forall x \in X^+ \qquad \forall y \in X^-$$

LEM 10 For a symmetric $C \in Mat_X(\mathbb{R})$

(i) Suppose C is bipartite and $|X| \geq 2$. Then C^2 is reducible

(ii) Suppose C^2 is reducible, C is irreducible, $C_{xy} \geq 0$ $\forall x, y \in X$. Then C is bipartite.

Pf ex $\square$

thm 11 (Perron - Frobenius)

Given an irreducible symmetric $C \in Mat_X(\mathbb{R})$ such that $C_{xy} \geq 0$ $\forall x, y \in X$. let θ_0 = max'l eigval of C

(i) let $v \in V$ denote a nonzero eigenvector for θ_0.

Then the coordinates of v are all pos or all negative.

(ii) $\theta_0 \geq 0$

(iii) The multiplicity of θ_0 is 1

(iv) $\theta \geq -\theta_0$ for all eigenvalues θ of C.

(v) C has eigenvalue $-\theta_0$ if and only if C is bipartite.

Pf (i) Write

$$v = \sum_{x \in X} \alpha_x \hat{x} \qquad \alpha_x \in \mathbb{R}$$

Define

$$X^+ = \{ x \in X \mid \alpha_x > 0 \}$$

$$X^- = \{ x \in X \mid \alpha_x \le 0 \}$$

Replacing v by $-v$ if necessary, WLOG $X^+ \neq \phi$.

Show $X^- = \phi$.

Define

$$B = \theta_0 I - C$$

B symmetric, pos semi def.

By Lem 5 $\exists$ vectors $\{v_x\}_{x \in X}$ in V s.t $\forall x, y \in X$.

$$\langle v_x, v_y \rangle = B_{xy}$$

$$= \begin{cases} \theta_0 - C_{xx} & \text{if } x = y \\ -C_{xy} & \text{if } x \neq y. \end{cases}$$

Claim $$0 = \sum_{x \in X} \alpha_x v_x$$

To see this recall $Cv = \theta_0 v$ so $Bv = 0$

Now

$$\left\| \sum_{x \in X} \alpha_x v_x \right\|^2 = \sum_{x \in X} \sum_{y \in X} \alpha_x \alpha_y B_{xy}$$

$$= v^t \underbrace{B v}_{0}$$

$$= 0$$

and claim is proved.

Set

$$p = \sum_{x \in X^+} \alpha_x v_x$$

$$= - \sum_{x \in X^-} \alpha_x v_x$$

Obs

$$\| p \|^2 = - \left\langle \sum_{x \in X^+} \alpha_x v_x \, , \sum_{y \in X^-} \alpha_y v_y \right\rangle$$

$$= \sum_{x \in X^+} \sum_{y \in X^-} \underbrace{\alpha_x}_{\geq 0} \underbrace{\alpha_y}_{\leq 0} \underbrace{C_{xy}}_{\geq 0}$$

$$\leq 0$$

So $p = 0$

Claim

$$C_{xy} = 0 \qquad \forall\ x \in X^+ \quad \forall_y \in X^-$$

Indeed for $y \in X^-$

$$0 = \langle p, v_y \rangle$$

$$= \sum_{x \in X^+} \alpha_x \langle v_x, v_y \rangle$$

$$= - \sum_{x \in X^+} \underset{\overset{\vee}{0}}{\alpha_x} \underset{\overset{\Downarrow}{0}}{C_{xy}}$$

So

$$C_{xy} = 0 \qquad \forall\ x \in X^+$$

and claim proved.

Now $X^- = \phi$ since C is irreducible ✓

(ii) For $v = \sum_{x \in X} \alpha_x \hat{x}$ as in (i)

For $x \in X$ compute coord x in $Cv = \theta_0 v$ to find

$$\theta_0 \alpha_x = \sum_{y \in X} C_{xy} \alpha_y$$

So

$$\theta_0 = \sum_{y \in X} \underbrace{C_{xy}}_{\geq 0} \underbrace{\frac{\alpha_y}{\alpha_x}}_{> 0}$$

≥ 0

(iii) Suppose θ_0 has mult ≥ 2. Then $\forall x \in X$,

$\exists$ nonzero eigenvector of C for θ_0 that has x-coord 0

contradicting (i)

(iv) We suppose not and get contradiction.

Let $\theta =$ min eigval of C

Let $w \in V$ denote a nonzero eigenvector for θ

Obs w is orthog to vector v in (i)

so coords of w not all same sign.

Consider C^2

C^2 symmetric

$(C^2)_{xy} \geq 0 \qquad \forall x, y \in X$

C^2 is irreducible, else C is bip by L10 (ii)

forcing $\theta = -\theta_0$ cont.

Obs θ^2 is max'l eigval of C^2, w is assoc eigvector.

By part (i) coords of w must be same sign, cont.

(v) Suppose C has eigval $-\theta_0$ Show C bip.

Assume $\theta_0 \neq 0$, else $C = 0$ which is bip.

C^2 has max'l eigenvalue θ_0^2

Corresp eigenspace is

$$W = W^+ + W^-$$

where W^+ = eigenspace of C for θ_0

W^- = ... $-\theta_0$

So $\dim W \geq 2$

So C^2 is reducible by (iii) above

So C is bip by LEM 10. Result follows $\square$

Lecture 3 Monday Jan 26

Cor 12 Given a connected graph $\Gamma = (X, R)$

Let θ_0 = max'l eigval of Γ

(i) Let $v \in V$ denote a nonzero eigenvector for θ_0.

Then the coordinates of v are all pos or all neg

(ii) $\theta_0 \geq 0$

(iii) The mult of θ_0 is 1

(iv) $\theta \geq -\theta_0$ for all eigenvalues θ of Γ

(v) Γ has eigenvalue $-\theta_0$ iff Γ is bipartite.

One more bound for later use ...

LEM 13 Given a connected regular graph

$\Gamma = (X, R)$ with $|X| \geq 2$. TFAE

(i) the second largest eigenvalue of Γ is at most -1

(ii) Γ is complete

pf (i) → (ii) Suf to show Γ has valency $k \geq |X| - 1$

Consider spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 \ldots & \theta_r \\ m_0 & m_1 & \ldots m_r \end{pmatrix}$$

Since Γ is regular $\theta_0 = k$

By Cor 12 $m_0 = 1$

By Cor 3

$$\sum_{i=1}^{r} m_i = |X| - 1$$

$$\sum_{i=1}^{r} m_i \theta_i = -k$$

Now

$$|X| - 1 - k = \sum_{i=1}^{r} m_i \underbrace{(1+\theta_i)}_{\leq 0}$$

$$\leq 0$$

and result follows

(ii) → (i) By Lem 6 $\square$

For the time being $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

We now consider a class of graphs whose spectrum is well behaved.

Let $\Gamma = (X, R)$ denote a connected graph with diameter D. For $0 \le i \le D$ let $A_i \in \mathrm{Mat}_X(\mathbb{F})$ have (x,y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad \forall x, y \in X$$

Call A_i the ith <u>distance</u> <u>matrix</u> for Γ. For notational conv put $A_{-1} = 0$, $A_{D+1} = 0$

LEM 14 With above notation

(i) $\{A_i\}_{i=0}^{D}$ are linearly independent

(ii) For $x \in X$ and $0 \le i \le D$

$$A_i \hat{x} = \sum_{\substack{y \in X \\ \partial(x,y) = i}} \hat{y}$$

Pf (i) Each A_i is non 0. Also for $i \neq j$ A_i and A_j have non0 entries in disjoint locations.

(ii) Routine $\square$

LEM 15 With above notation

(i) $A_0 = I$

(ii) $A_1 = A$ (the adjacency matrix)

(iii) $\sum_{i=0}^{D} A_i = J$ (the all 1's matrix)

(iv) $A_i^t = A_i$ $(0 \le i \le D)$

Pf Routine

For a connected graph $\Gamma = (X, R)$ with diameter D

For $x \in X$ and $i \in \mathbb{Z}$ define

$$\Gamma_i(x) = \{ y \in X \mid \partial(x,y) = i \}$$

So $\Gamma_0(x) = \{x\}$, $\Gamma_1(x) = \Gamma(x)$, $\Gamma_i(x) = \phi$ unless $0 \le i \le D$.

We call Γ distance-regular (or DRG) whenever for $0 \le i \le D$

and all $x, y \in X$ at $\partial(x,y) = i$ the scalars

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|, \qquad a_i = |\Gamma_i(x) \cap \Gamma(y)|$$

$$b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|$$

are independent of x, y and depend only on i

We call c_i, a_i, b_i the <u>intersection</u> <u>numbers</u> of Γ.

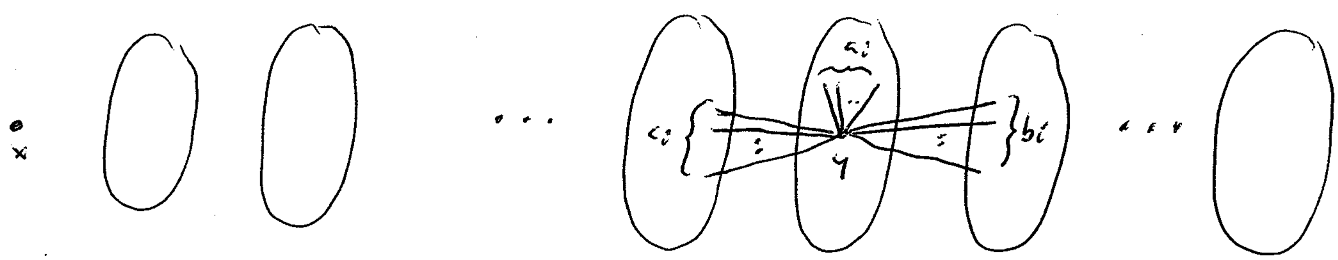
Observe $\quad c_0 = 0, \quad a_0 = 0, \quad c_1 = 1, \quad b_D = 0$

$$c_i > 0 \qquad 1 \le i \le D$$

$$b_i > 0 \qquad 0 \le i \le D-1$$

Illustration $\quad$ Fix $x, y \in X$ at $\partial(x,y) = i$

Dist from x	0	1	2	...	$i-1$	i	$i+1$	...	D



LEM 16 $\quad$ Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D

(i) $\quad \Gamma$ is regular with valency $k = b_0$

(ii) $\quad k = c_i + a_i + b_i \qquad (0 \le i \le D)$.

Pf (i) $\quad$ Set $i=0$ in the def of b_i

(ii) $\quad$ See the above diagram. $\square$

there are many known infinite families of distance-regular graphs

We mention three families

EXAMPLE 17 Let D denote a positive integer. The following are distance-regular graphs with diameter D

(i) the Hamming graphs $H(D,N)$ $(N \geq 2)$

Fix a set S with $|S| = N$

X = set of D-tuples of elements from S

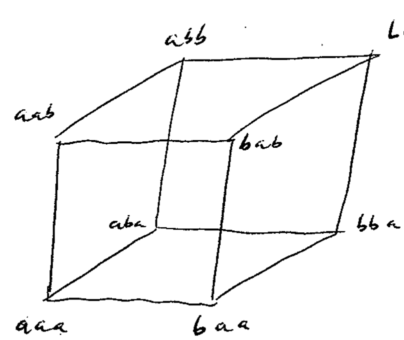
$R = \{xy \mid x,y \in X,$ x,y differ in exactly one coordinate$\}$

Here

$$c_i = i \qquad b_i = (D-i)(N-1) \qquad 0 \leq i \leq D$$

We call $H(D,2)$ the D-cube or hypercube

3-cube



$S = \{a, b\}$

(ii) The Johnson graph $J(D,N)$ $(N \geq 2D)$

Fix set S with $|S| = N$

X = set of all D-element subsets of S

$$R = \{ xy \mid x,y \in X, \quad |x \cap y| = D-1 \}$$

Here

$$c_i = i^2 \qquad b_i = (D-i)(N-D-i) \qquad (0 \leq i \leq D)$$

(iii) The q-Johnson graph $J_q(D,N)$ $(N \geq 2D)$

Fix a finite field $\mathbb{F}_q$

Let W denote an N-dimensional vector space over $\mathbb{F}_q$

X = set of all D-dimensional subspaces of W

$$R = \{ xy \mid x,y \in X, \quad \dim(x \cap y) = D-1 \}$$

Here

$$c_i = \left(\frac{q^i - 1}{q-1} \right)^2 \qquad (0 \leq i \leq D)$$

$$b_i = \frac{q^{2i+1}(q^{D-i} - 1)(q^{N-D-i} - 1)}{(q-1)^2} \qquad (0 \leq i \leq D)$$

$\square$

Lecture 4 Wednesday January 28

For a DRG $\Gamma = (X, R)$ with diameter D

Fix $x \in X$ and write

$$k_i = |\Gamma_i(x)| \qquad 0 \le i \le D$$

Obs $k_0 = 1$, $k_1 = k$ the valency.

LEM 18 With above notation

(i) $k_i c_i = k_{i-1} b_{i-1} \qquad 1 \le i \le D$

(ii) $k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad 0 \le i \le D$

(iii) k_i is independent of x.

Pf (i) Each of $k_i c_i$, $k_{i-1} b_{i-1}$ equal the number of edges between $\Gamma_{i-1}(x)$, $\Gamma_i(x)$

(ii) From (i)

(iii) From (ii) $\square$

We call k_i the <u>ith valency</u> of Γ.

Last time we gave three families of DRGs.

We now give another kind of example.

For the moment let $\Gamma = (X, R)$ be any graph

An <u>automorphism</u> of Γ is a bijection $\sigma : X \to X$ that leaves R invariant, so $\forall\, x, y \in X$

x, y adjacent $\iff$ $\sigma(x), \sigma(y)$ adjacent.

The set $\mathrm{Aut}(\Gamma)$ of automorphisms of Γ forms a group under composition.

EXAMPLE 19 For the complete graph $\Gamma = K_n$

$\mathrm{Aut}(\Gamma)$ is the symmetric group S_n

Note. For a graph $\Gamma = (X, R)$ and for $\sigma \in \mathrm{Aut}(\Gamma)$, for notational convenience we often view σ as a permutation matrix in $\mathrm{Mat}_X(\mathbb{F})$. From this point of view

$$\mathrm{Aut}(\Gamma) = \{ P \in \mathrm{Mat}_X(\mathbb{F}) \mid P \text{ a perm matrix and } PA = AP \}$$

Given a graph $\Gamma = (X, R)$

Any subgroup $G \subseteq Aut(\Gamma)$ acts on set X.

type of action	meaning
transitive	$\forall x, y \in X \ \exists \ \sigma \in G$ s.t. $\sigma(x) = y$
regular	$\forall x, y \in X \ \exists$ unique $\sigma \in G$ s.t $\sigma(x) = y$
generously transitive	$\forall x, y \in X \ \exists \ \sigma \in G$ s.t $\sigma(x) = y$ and $\sigma(y) = x$
Distance-transitive	$\forall x, y, x', y' \in X$ s.t. $\partial(x, y) = \partial(x', y')$ $\exists \ \sigma \in G$ s.t $\sigma(x) = x'$ and $\sigma(y) = y'$

LEM 20 For a connected graph $\Gamma = (X, R)$

assume $Aut(\Gamma)$ is distance-transitive on X. Then

Γ is distance-regular.

Pf ex.

Note All the graphs in Example 17 are distance-trans.

Next goal: show that any DRG with diameter D

has exactly $D+1$ distinct eigenvalues.

For the time being $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

LEM 21 For a DRG $\Gamma = (X, R)$ with diameter D

$$AA_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \qquad (0 \le i \le D)$$

where $b_{-1} = 1$, $c_{D+1} = 1$ and we recall $A_{-1} = 0$, $A_{D+1} = 0$

pf For $x, y \in X$

$$(AA_i)_{xy} = \sum_{z \in X} A_{xz} (A_i)_{zy}$$

$$= \sum_{z \in \Gamma(x) \cap \Gamma_i(y)} 1$$

$$= |\Gamma(x) \cap \Gamma_i(y)|$$

$$= \begin{cases} b_{i-1} & \text{if } \partial(x,y) = i-1 \\ a_i & \text{if } \partial(x,y) = i \\ c_{i+1} & \text{if } \partial(x,y) = i+1 \end{cases}$$

$= (x,y)$-entry of

$$b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$$

LEM 22 With above notation

(i) $\{A_i\}_{i=0}^{D}$ form a basis for the Bose-Mesner algebra M

(ii) The $\mathbb{F}$-vector space M has dim $D+1$

(iii) Γ has exactly $D+1$ distinct eigenvalues.

Pf (i) $\{A_i\}_{i=0}^{D}$ are lin indep by L14. Set

$M' = \text{Span}\{A_i\}_{i=0}^{D}$

$AM' \subseteq M'$ by L2 and $I \in M'$ so

$$M \subseteq M'$$

$\dim M' = D+1$ and $\dim M \geq D+1$ so $M = M'$

(ii), (iii) follows from (i) $\square$

EX Given a graph $\Gamma = (X, R)$ and given a subgroup

$G \subseteq \text{Aut}(\Gamma)$ that acts distance-transitively on X,

the Bose-Mesner algebra

$$M = \left\{ B \in \text{Mat}_X(\mathbb{F}) \;\middle|\; BP = PB \quad \forall P \in G \right\}$$

EX Given a DRG $\Gamma = (X, R)$ with diameter D

then for $0 \le j \le D$

(i) $E_j V$ is a module for $Aut(\Gamma)$

(ii) Suppose $G \subseteq Aut(\Gamma)$ is distance-transitive on X. Then

the $E_j V$ is irreducible as a G-module.

pf (i) $\forall$ $\rho \in Aut(\Gamma)$

$\rho B = B \rho$ $\forall B \in M$

so

$\rho E_j = E_j \rho$

so

$\rho E_j V \subseteq E_j V$

(ii) Suppose $\exists$ $0 \neq W \subseteq E_j V$ such that $GW \subseteq W$.

Show $W = E_j V$

$\forall x \in G$ let G_x denote the stabilizer of x in G

By the distance-transitivity assumption G_x is trans on $\Gamma_i(x)$

Define

$$S_x = |G_x|^{-1} \sum_{\rho \in G_x} \rho$$

$$\in \mathrm{Mat}_X(\mathbb{F})$$

S_x acts on V as follows: Given $v \in V$ write

$$v = \sum_{y \in X} \alpha_y \hat{y}$$

Then

$$S_x v = \sum_{i=0}^{D} \alpha_i A_i \hat{x} \qquad (*)$$

where

$$\alpha_i = k_i^{-1} \sum_{y \in \Gamma_i(x)} \alpha_y \qquad \text{"average value of } \{\alpha_y \mid y \in \Gamma_i(x)\}\text{"}$$

Note $\alpha_0 = \alpha_x = \langle v, \hat{x} \rangle$ so

$$S_x v \neq 0 \qquad \text{if} \qquad \langle v, \hat{x} \rangle \neq 0 \qquad (**)$$

By $(*)$ and since $\{A_i \hat{x}\}_{i=0}^{D}$ is a basis for $M\hat{x}$

$$S_x V = M\hat{x}$$

By construction

$$S_x W \subseteq W \subseteq E_j V$$

so

$$S_x W \subseteq M\hat{x} \cap E_j V$$
$$= \operatorname{Span}(E_j \hat{x})$$

Claim $\quad S_x W = \operatorname{Span}(E_j \hat{x})$

pf cl $\quad$ Since $\operatorname{Span}(E_j \hat{x})$ has dim 1, suffices to show $S_x W \neq 0$

Suppose $S_x W = 0$. Then $\langle W, \hat{x} \rangle = 0$ by $(**)$

But W is G-inv and G is trans on X so $\langle W, \hat{y} \rangle = 0$

$\forall y \in X$, forcing $W = 0$ for a contradiction. The claim is proved.

Now
$$E_j \hat{x} \in S_x W$$
$$\subseteq W$$

Now $\quad E_j V = \operatorname{Span}(E_j \hat{x} \mid x \in X)$
$$\subseteq W$$

so $E_j V = W$ $\quad \square$

How to compute the eigenvalues of a DRG from the intersection numbers

Notation let λ = indeterminate

$\mathbb{F}[\lambda]$ denotes the $\mathbb{F}$-algebra of all polynomials in λ that have all coeffs in $\mathbb{F}$.

DEF 23 Given a DRG $\Gamma = (X, R)$ with diameter D

Define polynomials $\{v_i\}_{i=0}^{D+1}$ in $\mathbb{F}[\lambda]$ by

$$v_0 = 1 \qquad v_1 = \lambda$$

$$\lambda v_i = c_{i+1} v_{i+1} + a_i v_i + b_{i-1} v_{i-1} \qquad 1 \le i \le D$$

where $c_{D+1} := 1$.

LEM 24 With above notation

(i) $\deg v_i = i$ $\qquad (0 \le i \le D+1)$

(ii) the coeff of λ^i in v_i is $(c_1 c_2 \cdots c_i)^{-1}$ $\qquad (0 \le i \le D+1)$

(iii) $v_i(A) = A_i$ $\qquad (0 \le i \le D)$

(iv) $v_{D+1}(A) = 0$

(v) The distinct eigenvalues of Γ are precisely the zeros of v_{D+1}

pf (i), (ii) From Def 23

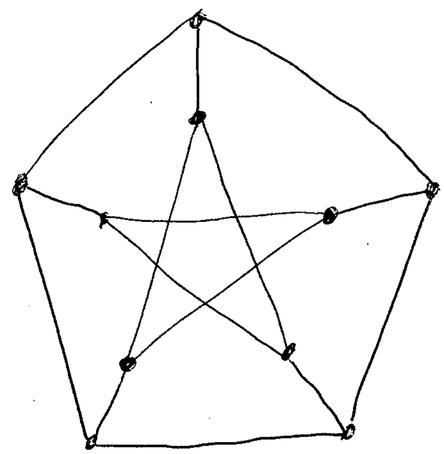
(iii), (iv) Compare LEM 21, Def 23

(v) Let $m \in \mathbb{F}[\lambda]$ denote the minimal polynomial of A

ie monic poly of least degree s.t $m(A) = 0$

Since A is diagonalizable, the distinct eigenvalues of Γ are precisely the zeros of m. Each of m, v_{D+1} has degree $D+1$ and $v_{D+1}(A) = 0$ so v_{D+1} is a scalar multiple of m. Result follows $\square$

EXAMPLE 25 Peterson's graph



$|X| = 10$

IS DRG with $D = 2$

$c_1 = 1 \quad c_2 = 1$

$a_0 = 0 \quad a_1 = 0 \quad a_2 = 2$

$k = 3 \quad b_1 = 2$

Here

$v_0 = 1 \qquad v_1 = \lambda, \qquad v_2 = \lambda^2 - 3$

$$v_3 = \lambda^3 - 2\lambda^2 - 5\lambda + 6$$

$$= (\lambda - 3)(\lambda - 1)(\lambda + 2)$$

Dist eigenvalues are $\theta_0 = 3, \quad \theta_1 = 1, \quad \theta_2 = -2$

□

A DRG with diameter $D=2$ is called <u>Strongly-regular</u> (SRG)

The eigenvalues of any SRG can be easily computed as in EX2

Next goal For any DRG find the multiplicities m_i

in terms of the intersection numbers

DEF 26. For a DRG $\Gamma=(X,R)$ with diameter D def

$$B = \begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & \ddots & & b_{D-1} \\ & & & & c_D & a_D \end{pmatrix}$$

$D+1 \times D+1$

Call B the <u>intersection matrix</u> of Γ.

LEM 27 With above notation the following coincide

(i) the min poly of A

(ii) the min poly of B

(iii) the char poly of B

Pf (i)=(ii) B represents the action of A on the basis $\{A_i\}_{i=0}^{D}$, and this action is faithful

(ii) = (iii) min poly of B has degree $D+1$ $\square$

COR 28. With above notation

The eigenvalues of B are mutually distinct and are precisely the dist eigenvalues of A.

LEM 29 With above notation For an eigenvalue θ of B define a row vector

$$v = \left(v_0(\theta), v_1(\theta), \ldots, v_D(\theta) \right)$$

where the v_i are from Def 23 Then

$$v B = \theta v$$

Pf Use Def 23 and Def 26 $\square$

DEF 30 Given a DRG $\Gamma = (X, R)$ of diameter D define polynomials $\{u_i\}_{i=0}^{D}$ in $\mathbb{F}[\lambda]$ by

$$u_0 = 1, \qquad u_1 = \frac{\lambda}{k}$$

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad (1 \le i \le D-1)$$

LEM 31 With above notation

$$u_i = \frac{v_i}{k_i} \qquad (0 \le i \le D)$$

pf Define $v_i' = u_i k_i$ for $0 \le i \le D$.

Obs $v_0' = 1 = v_0$ $\quad v_1' = \lambda = v_1$

Using LEM 18 and Def 30 one finds

$$\lambda v_i' = c_{i+1} v_{i+1}' + a_i v_i' + b_{i-1} v_{i-1}' \qquad 1 \le i \le D-1.$$

Comparing this and Def 23 we find $v_i = v_i'$ for $0 \le i \le D$. $\Box$

LEM 32 With above notation,

for an eigenvalue θ of B define a column vector

$$u = \begin{pmatrix} u_0(\theta) \\ u_1(\theta) \\ \vdots \\ u_D(\theta) \end{pmatrix}$$

Then

$$Bu = \theta u$$

pf Use Def 26 and Def 30

$\square$

Lecture 5 Friday January 30

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

For a DRG $\Gamma = (X, R)$ with diameter D

the Bose-Mesner algebra M has basis $\{A_i\}_{i=0}^{D}$ and $\{E_i\}_{i=0}^{D}$

We now compute the transition matrices in terms of the polynomials u_j, v_j.

LEM 33 With above notation

(i) $A_j = \sum_{i=0}^{D} v_j(\theta_i) E_i \qquad (0 \le j \le D)$

(ii) $E_j = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) A_i \qquad (0 \le j \le D)$

pf (i)

$$
\begin{aligned}
A_j &= A_j I \\
&= A_j \sum_{i=0}^{D} E_i \\
&= v_j(A) \sum_{i=0}^{D} E_i \\
&= \sum_{i=0}^{D} v_j(\theta_i) E_i \qquad \text{Since } A E_i = \theta_i E_i
\end{aligned}
$$

(ii) Write $E_j' =$ sum on RHS

show $E_j = E_j'$

Obs $E_j' \in M$

Show $A E_j' = \theta_j E_j'$

$$A E_j' = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) \, A A_i$$

$$= |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) \left(c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1} \right)$$

$$= |X|^{-1} m_j \sum_{\ell=0}^{D} A_\ell \left(c_\ell u_{\ell-1}(\theta_j) + a_\ell u_\ell(\theta_j) + b_\ell u_{\ell+1}(\theta_j) \right)$$

$$= |X|^{-1} m_j \sum_{\ell=0}^{D} A_\ell \, u_\ell(\theta_j) \, \theta_j$$

$$= \theta_j E_j'$$

Now $E_j' = \alpha E_j$ some $\alpha \in \mathbb{F}$

to show $\alpha = 1$ show E_j, E_j' have same trace

$\operatorname{tr} E_j = m_j$

$\operatorname{tr} A_i = \delta_{i0} |X|$ $(0 \le i \le D)$

so $\operatorname{tr} E_j' = m_j$ ✓

$\square$

The following theorem is due to Norman Biggs 1974.

thm 34. For a DRG $\Gamma = (X, R)$ with diameter D

for an eigenvalue θ of Γ the corresponding multiplicity is

$$\frac{|X|}{\sum_{i=0}^{D} u_i(\theta)\, v_i(\theta)}$$

Pf. Write $\theta = \theta_j$. Obs

$$E_j = E_j^2$$

$$= \left(|X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j)\, A_i \right) E_j$$

$$= |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j)\, v_i(A)\, E_j$$

$$= |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j)\, v_i(\theta_j)\, E_j$$

So

$$1 = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j)\, v_i(\theta_j)$$

and result follows. $\square$

COR 35 The spectrum of a DRG is determined by its intersection numbers.

Pf Let

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ m_0 & m_1 & \cdots & m_D \end{pmatrix}$$

be the spectrum in question.

Given the intersection numbers c_i, a_i, b_i

we get $\{v_i\}_{i=0}^{D+1}$ by Def 23

we get $\{\theta_i\}_{i=0}^{D}$ as zeros of V_{D+1}

we get $\{u_i\}_{i=0}^{D}$ by Def 30

we get $\{k_i\}_{i=0}^{D}$ by $k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$

we get $|X|$ by $|X| = \sum_{i=0}^{D} k_i$

we get $\{m_i\}_{i=0}^{D}$ by Thm 34. $\square$

EX 36 Peterson's graph revisited.

Using COR 35 the spectrum is

$$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

Note Given pos integers $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$

does there exist a corresponding DRG Γ?

Compute spectrum of Γ as in COR 35

If m_i fail to be positive integers then Γ does not exist.

this is a "feasibility condition" on the

intersection numbers.

EXAMPLE 37 Show that there does not exist a DRG with diameter $D=3$ and intersection numbers

$$c_1 = 1 \qquad c_2 = 1 \qquad c_3 = 3$$

$$k = 3 \qquad b_1 = 2 \qquad b_2 = 1$$

Pf ex. $\square$

We have a few comments

LEM 38 For a DRG $\Gamma = (X, R)$

(i) $E_0 = |X|^{-1} J$ $\quad (J = \text{all } 1\text{'s matrix})$

(ii) $E_0 V = \text{Span}(\mathbb{1})$ $\quad (\mathbb{1} = \sum_{x \in X} \hat{x})$

Pf (i) Write $E = |X|^{-1} J$

Obs $E = |X|^{-1} \sum_{i=0}^{D} A_i \in M$

and

$$E^2 = E$$

and

$$\text{rank } E = 1$$

so E is a primitive idempotent of Γ

Also $AE = kE$ (k = valency of Γ) and $\theta_0 = k$ so $E = E_0$ $\square$

LEM 39 For a DRG $\Gamma = (X, R)$ with diameter D

(i) $u_i(\theta_0) = 1 \qquad (0 \le i \le D)$

(ii) $v_i(\theta_0) = k_i \qquad (0 \le i \le D)$

pf (i) By LEM 33

$$E_0 = |X|^{-1} \underbrace{m_0}_{1} \sum_{i=0}^{D} u_i(\theta_0) A_i$$

By LEM 38

$$E_0 = |X|^{-1} J$$

$$= |X|^{-1} \sum_{i=0}^{D} A_i$$

the result follows since $\{A_i\}_{i=0}^{D}$ are linearly undep.

(ii) By LEM 31 and (i) above □

Next goal: find the spectrum of the Hamming graphs $H(D,r)$

Clarification: Given any graph $\Gamma = (X, R)$ with adjacency matrix A.

- By <u>the eigenvalues of Γ</u> we mean the multiset of zeros for the characteristic polynomial of A
- By <u>the distinct eigenvalues of Γ</u> we mean the set of zeros for the minimal polynomial of A.

Given finite nonempty sets X, X' the <u>Cartesian product</u>

$$X \times X' = \{ xx' \mid x \in X, \ x' \in X' \}.$$

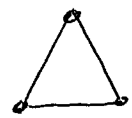
Given graphs $\Gamma = (X, R)$ and $\Gamma' = (X', R')$

their <u>Cartesian product</u> is the graph $\Gamma \times \Gamma'$ with vertex set $X \times X'$. Vertices xx' and yy' are adjacent in $\Gamma \times \Gamma'$ iff

$x = y$ and $x'y' \in R'$ or $xy \in R$ and $x' = y'$

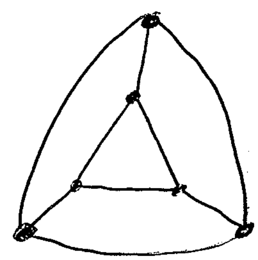
EX 40 take $\Gamma = K_3$ $\Gamma' = K_2$

Γ:



Γ':

$\Gamma \times \Gamma'$:



LEM 41 Given graphs $\Gamma = (X, R)$ and $\Gamma' = (X', R')$

Denote the eigenvalues of Γ by

$$\alpha_1, \alpha_2, \ldots, \alpha_n \qquad n = |X|$$

Denote the eigenvalues of Γ' by

$$\beta_1, \beta_2, \ldots, \beta_m \qquad m = |X'|$$

Then the eigenvalues of $\Gamma \times \Gamma'$ are

$$\alpha_i + \beta_j \qquad 1 \le i \le n \qquad 1 \le j \le m$$

Pf let $V = \mathbb{F}^X$ be the standard module for Γ

let $V' = \mathbb{F}^{X'}$ .. Γ'

Interpret standard module for $\Gamma \times \Gamma'$ as $V \otimes V'$

Interpret

$$Mat_{X \times X'}(\mathbb{F}) = Mat_X(\mathbb{F}) \otimes Mat_{X'}(\mathbb{F})$$

For $B \in Mat_X(\mathbb{F})$ and $B' \in Mat_{X'}(\mathbb{F})$

$B \otimes B'$ acts on $V \otimes V'$ as

$$(B \otimes B')(u \otimes u') = (Bu) \otimes (B'u') \qquad u \in V \quad u' \in V'$$

let $A \in Mat_X(\mathbb{F})$ be adj matrix for Γ

.. $A' \in Mat_{X'}(\mathbb{F})$ ... Γ'

Adj matrix for $\Gamma \times \Gamma'$ is

$A \otimes I' + I \otimes A'$ where I = identity in $Mat_X(\mathbb{F})$
I' = ... $Mat_{X'}(\mathbb{F})$

let $\{u_i\}_{i=1}^n$ be a basis for V s.t.

$A u_i = \alpha_i u_i \qquad 1 \le i \le n.$

let $\{v_i\}_{i=1}^m$ be a basis for V' s.t

$$A' v_i = \beta_i v_i \qquad 1 \le i \le m$$

Obs

$$u_i \otimes v_j \qquad 1 \le i \le n, \qquad 1 \le j \le m$$

is a basis for $V \otimes V'$. Also

$$\left(A \otimes I' + I \otimes A'\right) u_i \otimes v_j = (\alpha_i + \beta_j)\, u_i \otimes v_j$$

$$1 \le i \le n, \qquad 1 \le j \le m$$

Result follows. $\square$

COR 42 The Hamming graph $H(D,r)$ has spectrum

$$\theta_i = (D-i)(r-1) - i \qquad (0 \le i \le D)$$

$$m_i = \binom{D}{i}(r-1)^i \qquad (0 \le i \le D)$$

Pf Obs

$$H(D,r) = \underbrace{K_r \times K_r \times \cdots \times K_r}_{D}$$

where K_r = complete graph on r vertices.

Recall spectrum of K_r is

$$\begin{pmatrix} r-1 & -1 \\ 1 & r-1 \end{pmatrix}$$

By LEM41 the eigenvalues of $H(D,r)$ are the sums

$$y_1 + y_2 + \cdots + y_D$$

where

$$y_i = r-1 \text{ or } y_i = -1 \quad \text{for } 1 \le i \le D.$$

Result follows. $\square$

COR 43 The hypercube $H(D,2)$ has spectrum

$$\theta_i = D - 2i \qquad (0 \le i \le D)$$

$$m_i = \binom{D}{i} \qquad (0 \le i \le D)$$

$\square$