

☆ The Tetrahedron Alg & fin. dim irr module

1.7

Thm Let V fin. dim. irr $\boxtimes$ -mod. Then $\exists!$ $\mathcal{O}$ -mod str on V s.t.

$X, Y \curvearrowright V$ as X_{01}, X_{23} resp. Furthermore, V is irr of type $(0,0)$ as $\mathcal{O}$ -mod.

Def Tetrahedron $\boxtimes = \langle X_{rs} \mid X_{rs} + X_{sr} = 0, [X_{rs}, X_{st}] = 2X_{rs} + 2X_{st},$
 $[X_{rs}, [X_{rs}, [X_{rs}, X_{tu}]]] = 4[X_{rs}, X_{tu}] \rangle$

Ons again $\mathcal{O} = \langle X, Y \mid [X, [X, [X, Y]]] = 4[X, Y], \dots \rangle$

rmk We obviously have a map $X \mapsto X_{01}, Y \mapsto X_{23}, \mathcal{O} \hookrightarrow \boxtimes$ [HT]

rmk Thm 3.8 $\Rightarrow X_{rs}$ has e.v. $d, d-2, \dots, -d \Rightarrow V$ has type $(0,0)$

Content: V is irr.

Def Decomposition of V ass. w/ (r,s) e.s. of X_{rs} : $V = V_{rs}(-d) \oplus V_{rs}(2-d) \oplus \dots \oplus V_{rs}(d)$
 $F_i^{rs} := \bigoplus_{j=0}^i V_{rs}(-d+2j)$. Ind. Alg $F^{rs} := (F_0, \dots, F_d) =: V_0 \oplus V_1 \oplus \dots \oplus V_d$

lem $V = \text{f.d. irr. } \boxtimes, r \neq s$, flag ind. by above decomp indep. of s, i i.e. $F_i^{rs} = F_i^{rt} \forall s \neq t$.

Pf: Thm 3.5: $\bigoplus V_{rs}(-d+2i) = \bigoplus V_{rt}(-d+2i) \quad (\lambda = 2i-d)$

rmk We can write F_i^{rs} as F_i^r .

lem $F^1, \dots, F^4$ are mutually opp. Furthermore $V_{rs}(2i-d) = F_i^r \cap F_{d-i}^s$.

Pf. Pick $r \neq s$. Since $X_{rs} = -X_{sr}$, $V_{rs}(2i-d) = V_{sr}(d-2i)$.

By def, $F^r = F^{rs}$ as opp $\Rightarrow F^{sr} = F^s$.

Second statement by def of opp. flag.

Lem $V = \text{f.d. in } \mathbb{A}\text{-mod.}$ r, s, t, u distinct. Let $W \neq 0$ subspace of V s.t.

$X_{rs}W \subseteq W$ and $X_{tu}W \subseteq W$, Then $W = V$.

Pf: WLOG $W = \text{irr } X_{rs}, X_{tu} \text{ mod. } W^\perp$ $W \cap \mathbb{A}\text{-inv}$, i.e. $X_{ab}W \subseteq W \forall ab$.

WLOG $a=s, b=t$.

Let $W' := \{w \mid X_{st}w \in W\}$. By (2), $X_{rs}X_{st} - X_{st}X_{rs} = 2X_{rs} + 2X_{st}$.

$\Rightarrow X_{rs}W' \subseteq W'$, $X_{tu}W' \subseteq W'$

$$X_{st}X_{rs} = \underset{W}{\uparrow} X_{rs}X_{st} - 2\underset{W \uparrow \text{assup}}{X_{rs}} - 2\underset{W' \uparrow}{X_{st}}.$$

$\Rightarrow W' \text{ is } X_{rs}, X_{tu} \text{ mod.} \Rightarrow W' = 0 \text{ or } W$.

claim $W' \neq 0$

Pf: $W \subseteq V$, $W_j := W \cap V_{rs}(2j-d)$ and $W = \bigoplus W_j$. $W \neq 0 \Rightarrow \exists m = \min \{j \mid W_j \neq 0\}$.

Similarly $W_j^* := W \cap V_{tu}(2j-d)$, $n = \min \{j \mid W_j^* \neq 0\}$.

Note $W_n^* \subseteq V_{tu}(2n-d) \subseteq F_n^t$

also $W_n^* \subseteq W = \bigoplus_{j=m}^d W_j \subseteq \bigoplus_{j=m}^d V_{rs}(2j-d) = F_{d-m}^s$.

$\Rightarrow W_n^* \subseteq F_n^t \cap F_{d-m}^s$ By opposite flag, intersection $\neq 0 \Leftrightarrow n+d-m \geq d$.

$W_n^* \neq 0 \Rightarrow n \geq m$. Similarly $m \geq n$.

Hence $n=m$, $W_n^* \subseteq V_{st}(d-2m)$, i.e. $X_{st}W_n^* \subseteq W_n^*$, $W_n^* \subseteq W' \neq 0$.