We continue to discuss the Hamming graph $H(D,n)$ $n \geq 3$.

Let $W \subseteq V$ denote an irred T-module with diameter d. Until further notice identify $[\mathcal{L},\mathcal{L}]$ with $sl_2(\mathbb{C})$ via the iso in LEM 81 and let $\{v_i\}_{i=0}^{d}$ denote the corresponding basis for W from LEM 79. In th 83 we saw

$$v_i \in E^*_{r+i} W \qquad 0 \leq i \leq d. \qquad \left(\begin{matrix} r = \text{endpt} \\ \text{of } W \end{matrix}\right)$$

So the matrix representing A^* rel $\{v_i\}_{i=0}^{d}$ is

$$\mathrm{diag}(\theta^*_r, \theta^*_{r+1}, \dots, \theta^*_{r+d})$$

We now consider the matrix representing A

Thm 84 With the above notation the matrix representing A rel $\{v_i\}_{i=0}^{d}$ is

$$A : \begin{pmatrix} a_0(W) & b_0(W) & & & 0 \\ c_1(W) & a_1(W) & b_1(W) & & \\ & c_2(W) & \ddots & \ddots & \\ & 0 & \ddots & \ddots & b_{d-1}(W) \\ & & & c_d(W) & a_d(W) \end{pmatrix}$$

where

$$c_i(W) = i \qquad 1 \le i \le d$$

$$b_i(W) = (n-1)(d-i) \qquad 0 \le i \le d-1$$

$$a_i(W) = \frac{2-n}{2}(d-2i) + \frac{n}{2}(D-\gamma) - D \qquad 0 \le i \le d$$

(γ = displacement of W)

pf We are identifying

$$A - \Phi \equiv (n-1)e + f + \frac{2-n}{2} h$$

The actions of e, f, h on $\{v_i\}_{i=0}^{d}$ are given in LEM 79.

Also Φ acts on W as

$$\frac{n}{2}(D-\gamma) - D$$

times the identity I. Result follows. (ex) $\square$

Let $W \subseteq V$ denote an irred T-module with diameter d.

Until further notice identify $[\mathcal{L}, \mathcal{L}]$ with $sl_2(\mathbb{C})$ via the isomorphism in LEM 82, and let $\{v_i^*\}_{i=0}^d$ denote the corresp basis from Lem 79.

Just as in Th 83 we find

$$v_i^* \in \mathbb{E}_{t+i} W \qquad 0 \le i \le d$$

(t = dual endpt of W)

Moreover

$$t = \frac{3 + D - d}{2} \qquad (ex)$$

$$= r$$

Also for $0 \le i \le D$

$$\mathbb{E}_i W \ne 0 \quad \text{iff} \quad t \le i \le t+d \qquad (ex)$$

and for $0 \le i \le d$

$$\dim \mathbb{E}_{t+i} W = 1 \qquad (ex)$$

Relative the basis $\{v_i^*\}_{i=0}^d$ the matrix representing A is

$$\mathrm{diag}(\theta_t, \theta_{t+1}, \ldots, \theta_{t+d})$$

thm 85 With above notation the matrix representing A^* rel $\{v_i^*\}_{i=0}^d$ is

$$
A^*: \begin{pmatrix}
a_0^*(W) & b_0^*(W) & & & 0 \\
c_1^*(W) & a_1^*(W) & b_1^*(W) & & \\
 & c_2^*(W) & \ddots & \ddots & \\
 & & \ddots & \ddots & b_{d-1}^*(W) \\
0 & & & c_d^*(W) & a_d^*(W)
\end{pmatrix}
$$

where

$$c_i^*(W) = i \qquad 1 \le i \le d$$

$$b_i^*(W) = (n-1)(d-i) \qquad 0 \le i \le d-1$$

$$a_i^*(W) = \frac{2-n}{2}(d-2i) + \frac{n}{2}(D-\zeta) - D \qquad 0 \le i \le d$$

$$(\zeta = \text{displacement of } W)$$

pf We are identifying

$$A^* - \Phi \equiv (n-1)e + f + \frac{2-n}{2}h$$

the actions of e, f, h on $\{v_i^*\}_{i=0}^d$ are given in L79

Also Φ acts on W as

$$\frac{n}{2}(D-\zeta) - D$$

times I. Result follows. $\square$

6/4/10

LEM 86 let $W \subseteq V$ denote an irred T-module with diameter d, endpt r, dual endpt t

(i) For $0 \le i \le d$

$$c_i(W) + a_i(W) + b_i(W) = \theta_t$$

where $c_0(W) = 0$, $b_d(W) = 0$

(ii) For $0 \le i \le d$

$$c_i^*(W) + a_i^*(W) + b_i^*(W) = \theta_r^*$$

where $c_0^*(W) = 0$, $b_d^*(W) = 0$

pf (ex) $\square$

6/4/10

LEM 87 Let $W \subseteq V$ denote an irred T-module with diam d, endpt r, dual endpt t.

(i) the basis $\{v_i\}_{i=0}^d$ for W from lem 84 satisfies

$$\sum_{i=0}^d v_i \in E_t W$$

(ii) the basis $\{v_i^*\}_{i=0}^d$ for W from lem 85 satisfies

$$\sum_{i=0}^d v_i^* \in E_r^* W$$

pf (i) Write

$$v = \sum_{i=0}^d v_i$$

Suf to show

$$Av = \theta_t v$$

Obs

$$Av = A \sum_{i=0}^d v_i$$

$$= \sum_{i=0}^d \Big(c_{i+1}(W) v_{i+1} + a_i(W) v_i + b_{i-1}(W) v_{i-1} \Big)$$

$$= \sum_{i=0}^d v_i \Big(\underbrace{c_i(W) + a_i(W) + b_i(W)}_{\theta_t} \Big)$$

$$= \theta_t \sum_{i=0}^d v_i$$

$$= \theta_t v \quad \checkmark$$

(ii) Sim $\square$

6/4/10

LEM 88 Let $W \subseteq V$ denote an irred $\mathbb{T}$-module. Then each of

$$A - \Phi, \qquad A^* - \Phi$$

has trace 0 on W.

pf #1 View W as an $[\mathcal{L}, \mathcal{L}]$ module

Recall $[\mathcal{L}, \mathcal{L}]$ contains
$$A - \Phi, \qquad A^* - \Phi$$

Each element of $[\mathcal{L}, \mathcal{L}]$ has trace 0 on W,

since $\forall\ y, z \in \mathcal{L}$

$$\operatorname{tr} [y, z]|_W = \operatorname{tr}\left(y|_W\, z|_W - z|_W\, y|_W\right)$$

$$= \operatorname{tr} y|_W\, z|_W - \operatorname{tr} z|_W\, y|_W$$

$$= 0.$$

pf #2

Φ acts on W as

$$\left(\frac{n}{2}(D - \eta) - D\right) I$$

η = displacement of W

so on W

$$\operatorname{tr} \Phi = (d+1)\left(\frac{n}{2}(D - \eta) - D\right)$$

d = diam of W

trace of A on W is

$$\sum_{i=0}^{d} \theta_{t+i} = \sum_{i=0}^{d} \left((n-1)D - n(t+i) \right)$$

$$= (d+1)\Big((n-1)D - n\underbrace{t}_{\frac{3+D-d}{2}}\Big) - n\underbrace{(0+1+\cdots+d)}_{\frac{d(d+1)}{2}}$$

$$= (d+1)\left(\frac{n}{2}(D-3) - D \right)$$

So

$A - \Phi$ has trace 0 on W.

Sim $A^* - \Phi$ has trace 0 on W. $\square$

6/4/10

Let $W \subseteq \underline{V}$ denote an irred T-module.

Then the actions of A, A^* on W give an

example of a <u>Leonard</u> <u>pair</u>

This is a pair of linear transformations that act in a tridiagonal fashion on each others eigenspaces

See

Ter: Two linear trans each tridiagonal wrt an eigenbasis for the other (arXiv)

The action of $\tilde{R}, \tilde{L}, \tilde{H}$ on $\mathbb{V} = V^{\otimes D}$

In Def 49 we defined a basis

$$\tilde{R}, \tilde{L}, \tilde{H}$$

for $[\mathcal{L}, \mathcal{L}]$. By Lem 54

$$A - \Phi = \tilde{R} - \frac{n}{2} \tilde{H}$$

$$A^* - \Phi = \tilde{L} + \frac{n}{2} \tilde{H}$$

LEM 89 $\exists$ Lie algebra iso $[\mathcal{L}, \mathcal{L}] \to sl_2(\mathbb{C})$

that sends

$$\tilde{R} \to f$$

$$\tilde{L} \to ne$$

$$\tilde{H} \to h$$

pf Routine using L50 (ex) $\square$

COR 90 the Lie algebra iso $[\mathcal{L},\mathcal{L}] \to sl_2(\mathbb{C})$ from Lem 89 reads

$$A - \Phi \longrightarrow f - \frac{n}{2} h$$

$$A^* - \Phi \longrightarrow ne + \frac{n}{2} h$$

pf By L89 and comment above it. $\square$

Let $W \subseteq V$ denote an irred T-module with diam d

Until further notice identify $[\mathcal{L},\mathcal{L}]$ with $sl_2(\mathbb{C})$ via the iso in Lem 89, and let $\{\tilde{v}_i\}_{i=0}^{d}$ denote the corresp basis for W from Lem 79.

6/4/10

Th 91 With above notation the matrices rep A, A^* rel $\{\tilde{v}_i\}_{i=0}^d$ are

$$A: \begin{pmatrix} \theta_{t+d} & & & & & 0 \\ 1 & \theta_{t+d-1} & & & & \\ & 2 & \ddots & & & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & \theta_{t+1} & \\ & & & & d & \theta_t \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta^*_r & nd & & & & 0 \\ & \theta^*_{r+1} & n(d-1) & & & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & \ddots & \\ & & & & \ddots & n \\ & & & & & \theta^*_{r+d} \end{pmatrix}$$

pf We are identifying

$$A - \Phi \equiv f - \frac{n}{2} h$$

$$A^* - \Phi \equiv ne + \frac{n}{2} h$$

The actions of e, f, h on $\{\tilde{v}_i\}_{i=0}^d$ are given in L79

Φ acts on W as

$$\frac{n}{2}(D-3) - D$$

times I. Result follows. $\square$

The inner product on $\mathbb{V} = V^{\otimes D}$

Recall our Hermitean inner product $\langle\,,\rangle$

on V

$$\langle u, v\rangle = u^t \bar{v} \qquad u, v \in V$$

Extend $\langle\,,\rangle$ to Hermitean inner product on $\mathbb{V} = V^{\otimes D}$

as follows:

For

$$u = u_1 \otimes u_2 \otimes \cdots \otimes u_D \quad \in \mathbb{V}$$

$$v = v_1 \otimes v_2 \otimes \cdots \otimes v_D \quad \in \mathbb{V}$$

$$\langle u, v\rangle = \langle u_1, v_1\rangle\langle u_2, v_2\rangle \cdots \langle u_D, v_D\rangle$$

For $y, z \in X$

$$\langle \hat{y}, \hat{z}\rangle = \begin{cases} 1 & \text{if } y = z \\ 0 & \text{if } y \neq z \end{cases}$$

So the basis

$$\{\hat{y} \mid y \in X\}$$

for $\mathbb{V}$ is orthonormal wrt $\langle\,,\rangle$

Recall the displacement decomp of $\mathbb{V}$:

$$\mathbb{V} = \sum_{\eta=0}^{D} \mathbb{V}_\eta \qquad (\text{ds of } \mathcal{L}\text{-modules})$$

LEM 92 The subspaces $\{V_z\}_{z=0}^{p}$ are mutually orthogonal rel $\langle . \rangle$

pf We saw earlier that

$$V = e_0 V + e_1 V \qquad (\text{orthog ds})$$

Result follows by the def of V_z and $\langle . \rangle$ (ex) $\square$

Recall for K_n

$$V = E_0^* V + E_1^* V \qquad \text{(orthog ds)}$$

So

$$\mathbb{V} = V^{\otimes D}$$
$$= \left(E_0^* V + E_1^* V \right)^{\otimes D}$$
$$= \sum E_{i_1}^* V \otimes E_{i_2}^* V \otimes \cdots \otimes E_{i_D}^* V \qquad (*)$$

where sum over all sequences $(i_1, i_2, \ldots, i_D)$ of elements taken from $\{0, 1\}$

Sum $(*)$ is direct and orthog.

For each summand in $(*)$ define

$$\text{weight} = \left| \left\{ j \mid 1 \le j \le D, \ i_j = 1 \right\} \right|$$

LEM 93 For $H(D,n)$ and $0 \le i \le D$ the ith subconstituent

$$\mathbb{E}_i^* \mathbb{V} = \text{sum of terms in } (*) \text{ that have weight } i$$

$\square$

pf (ex)

Recall for K_n

$$V = E_0V + E_1V \qquad \text{(orthog ds)}$$

So

$$\mathbb{V} = V^{\otimes D}$$

$$= (E_0V + E_1V)^{\otimes D}$$

$$= \sum E_{i_1}V \otimes E_{i_2}V \otimes \cdots \otimes E_{i_D}V \qquad (*)$$

where sum over all sequences $(i_1, i_2, \ldots, i_D)$ of elements taken from $\{0,1\}$

Sum $(*)$ is direct and orthog.

For each summand in $(*)$ define

$$\text{dual weight} = \left|\{ j \mid 1 \le j \le D,\ i_j = 1 \}\right|$$

LEM 94 For $H(D,n)$ and $0 \le i \le D$ the ith eigenspace

$$\mathbb{E}_i \mathbb{V} = \text{sum of terms in } (*) \text{ that have dual weight } i$$

pf Given a term in $(*)$:

$$U = E_{i_1}V \otimes E_{i_2}V \otimes \cdots \otimes E_{i_D}V$$

Find the action of $\mathbb{A}$ on this term

Recall $\mathbb{A}$, A agree on $\mathbb{V}$

Given $u = u_1 \otimes u_2 \otimes \cdots \otimes u_D \in U$

$$
\begin{aligned}
Au &= A\, u_1 \otimes u_2 \otimes \cdots \otimes u_p \\
&= \sum_{j=1}^{p} u_1 \otimes u_2 \otimes \cdots \otimes u_{j-1} \otimes \underbrace{(A u_j)}_{= \begin{cases} (n-1) u_j & \text{if } i_j = 0 \\ -u_j & \text{if } i_j = 1 \end{cases}} \otimes u_{j+1} \otimes \cdots \otimes u_p \\
&= \Big(\underbrace{(p - i)(n-1) - i}_{\theta_i} \Big)\, u_1 \otimes u_2 \otimes \cdots \otimes u_p \\
&= \theta_i\, u
\end{aligned}
$$

$\square$

Result follows

The eigenspaces of $\tilde{H}$ on $\mathbb{V} = V^{\otimes D}$

Recall for K_n

eigenspaces of $\tilde{H}$	E_0^*V	E_0V	e_1V
eigval	1	-1	0

Obs

$$\mathbb{V} = V^{\otimes D}$$

$$= \left(E_0^*V + E_0V + e_1V \right)^{\otimes D}$$

$$= \sum V_1 \otimes V_2 \otimes \cdots \otimes V_D \qquad (*)$$

sum over all sequences $(V_1, V_2, \ldots, V_D)$ of elements taken from E_0^*V, E_0V, e_1V

Sum $(*)$ is direct but not orthog in gen.

For each summand in $(*)$ define

$$\alpha = \left| \{ j \mid 1 \le j \le D, \; V_j = E_0^*V \} \right|$$

$$\beta = \left| \{ j \mid 1 \le j \le D, \; V_j = E_0V \} \right|$$

Recall displacement

$$\eta = \left| \{ j \mid 1 \le j \le D, \; V_j = e_1V \} \right|$$

For $\cancel{-1} \le i,j \le D$ define

$$V_{ij} = (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV)$$

obs for $0 \le i,j \le D$

$$V_{i-1,j} \subseteq V_{ij}$$

$$V_{i,j-1} \subseteq V_{ij}$$

So

$$V_{i-1,j} + V_{i,j-1} \subseteq V_{ij}$$

Define

$\tilde{V}_{ij}$ = orthog complement of

$V_{i-1,j} + V_{i,j-1}$ in V_{ij}.

Then

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij} \qquad \text{(dsum)} \qquad \text{"split dec"}$$

(ex) or see Ter: the split decomp of a Q-poly DRG (arxiv)

We have

$$\tilde{V}_{ij} = V_{ij} \cap V_z \qquad z = i+j-D$$

from L 75

(ex)

LEM 95 For $0 \le i,j \le D$

$\tilde{V}_{ij}$ = sum of terms in (*) that have

$$\alpha = j - z \qquad \beta = i - z$$

pf Given term in (*) with

$$\alpha = j - \eta, \qquad \beta = i - \eta$$

show this term is in $\tilde{V}_{i\eta}$

wlog term is

$$u = \underbrace{E_0^* V \otimes \cdots \otimes E_0^* V}_{\alpha} \otimes \underbrace{E_0 V \otimes \cdots \otimes E_0 V}_{\beta} \otimes \underbrace{e_1 V \otimes \cdots \otimes e_1 V}_{\eta}$$

Obs $u \subseteq V_\eta$.

Also

$$u \subseteq \underbrace{E_0^* V \otimes \cdots \otimes E_0^* V}_{\alpha} \otimes \underbrace{V \otimes \cdots \otimes V}_{\beta + \eta = i}$$

$$= \underbrace{E_0^* V \otimes \cdots \otimes E_0^* V}_{\alpha} \otimes (E_0^* V + E_1^* V)^{\otimes i}$$

$$\subseteq \mathbb{E}_0^* \mathbb{V} + \cdots + \mathbb{E}_i^* \mathbb{V}$$

Similarly

$$u \subseteq \mathbb{E}_0 \mathbb{V} + \cdots + \mathbb{E}_j \mathbb{V}$$

Now

$$u \subseteq \mathbb{V}_{i\eta} \cap V_\eta$$

$$= \tilde{\mathbb{V}}_{i\eta}$$

$\square$

LEM 96 For $0 \le i, j \le D$

$\tilde{H}$ acts on $\tilde{V}_{ij}$ as

$$(j-i)I$$

pf By L95 and since

$$E_0^* V, \quad E_0 V, \quad e_\phi V$$

are eigenspaces for $\tilde{H}$ with eigvals $1, -1, 0$ (ex) $\square$

Thm 97

(i) $\tilde{H}$ is diagonalizable on $\mathbb{V}$

(ii) the eigenvalues for $\tilde{H}$ on $\mathbb{V}$ are

$$-D, 1-D, \ldots, D$$

(iii) For $-D \le \lambda \le D$ the λ-eigenspace for $\tilde{H}$ on $\mathbb{V}$ is

$$\sum_{\substack{0 \le i, j \le D \\ \lambda = j-i}} \tilde{V}_{ij}$$

$\square$

pf clear from L96.