MANILA Lecture 5 Wednesday June 2 6/2/10

Tensor products

Aside on general Lie algebras

Let L = any Lie algebra over $\mathbb{C}$.

Let V = vector space over $\mathbb{C}$

An L-module structure on V is nothing but a Lie algebra hom

$$L \to gl(V)$$

Given L-module V.

Given $x, y \in L$ Given $v \in V$

$[x, y]$ acts on v as

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

LEM 70 Given L-modules U, V.

Then the tensor product $U \otimes V$ has an L-module structure s.t.

$$x \cdot (u \otimes v) = (x \cdot u) \otimes v + u \otimes (x \cdot v)$$

$x \in L$, $u \in U$, $v \in V$.

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pf Need to verify

$$[x,y] \cdot w = x \cdot (y \cdot w) - y \cdot (x \cdot w) \qquad *$$

$$x, y \in L \qquad w \in U \otimes V$$

WLOG $\quad w = u \otimes v \qquad u \in U \qquad v \in V$

obs:

$$\begin{aligned}
[x,y] \cdot w &= [x,y] \cdot (u \otimes v) \\
&= ([x,y] \cdot u) \otimes v + u \otimes ([x,y] \cdot v) \\
&= x \cdot (y \cdot u) \otimes v - y \cdot (x \cdot u) \otimes v \\
&\quad + u \otimes x \cdot (y \cdot v) - u \otimes y \cdot (x \cdot v)
\end{aligned}$$

Also

$$\begin{aligned}
x \cdot (y \cdot w) &= x \cdot y \cdot (u \otimes v) \\
&= x \cdot \big((y \cdot u) \otimes v + u \otimes (y \cdot v) \big) \\
&= x \cdot (y \cdot u) \otimes v + (y \cdot u) \otimes (x \cdot v) \\
&\quad + (x \cdot u) \otimes (y \cdot v) + u \otimes x \cdot (y \cdot v)
\end{aligned}$$

$$\begin{aligned}
y \cdot (x \cdot w) &= y \cdot x (u \otimes v) \\
&= y \cdot \big(x \cdot u \otimes v + u \otimes x \cdot v \big) \\
&= y \cdot (x \cdot u) \otimes v + x \cdot u \otimes y \cdot v \\
&\quad + y \cdot u \otimes x \cdot v + u \otimes y \cdot (x \cdot v)
\end{aligned}$$

By these comments we obtain $*$ $\square$

Let V denote an L-module

Write
$$V^{\otimes p} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{p} \qquad p \geq 1$$

By L70 $V^{\otimes p}$ is L-module with action

$$x \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_p) = \sum_{i=1}^{p} v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \otimes (x \cdot v_i) \otimes v_{i+1} \otimes \cdots \otimes v_p$$

$$x \in L, \qquad v_i \in V \qquad 1 \leq i \leq p$$

The Hamming graph $H(D,n)$

Back to DRGs

In previous Lectures we considered the complete graph K_n

Which is the Hamming graph $H(D,n)$ for $D=1$

We now consider $H(D,n)$ for general D

Continue to assume $n \geq 3$

Notation symbols for $H(D,n)$ are in bold / caps

Concept	$H(D,n)$	K_n
st. module	$\mathbb{V}$	V
adj matrix	$\mathbb{A}$	A
subconst alg	$\mathbb{T}$	T

View $H(D,n) = \underbrace{K_n \times K_n \times \cdots \times K_n}_{D}$ Cartesian product

So vertex set

$$\mathbb{X} = \underbrace{S \times S \times \cdots \times S}_{D}$$

S = vertex set for K_n

WLOG $S = \{1, 2, \ldots n\}$

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Base vertex for K_n is $\mathbb{1}$

Base vertex for $H(D,n)$ is $x = \underbrace{11\cdots1}_{D}$

View

$$\underline{V} = V^{\otimes D}$$

Given $y \in \underline{X}$

write

$$y = (y_1, y_2, \ldots y_D) \qquad y_i \in S \qquad 1 \le i \le D.$$

View

$$\hat{y} = \hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_D$$

Recall $\mathcal{L}$ = Lie subalgebra of $gl(V)$ gen by A, A^*

Obs V is $\mathcal{L}$-module

So

$$\underline{V} = V^{\otimes D} \quad \text{is } \mathcal{L}\text{-module.}$$

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LEM 71

The action of A (resp. A^*) on $\mathbb{V}$ coincides with the action of $\mathbb{A}$ (resp. $\mathbb{A}^*$) on $\mathbb{V}$.

pf

Given $y \in \mathbb{X}$ show

$$A\hat{y} = \mathbb{A}\hat{y}$$

Write $y = (y_1, y_2, \dots, y_D)$ $\qquad y_i \in S \quad 1 \le i \le D$

$$A\hat{y} = A(\hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_D)$$

$$= \sum_{i=1}^{D} \hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_{i-1} \otimes \underbrace{A\hat{y}_i}_{= \sum_{y_i' \in S \setminus y_i} \hat{y}_i'} \otimes \hat{y}_{i+1} \otimes \cdots \otimes \hat{y}_D$$

Also

$$\mathbb{A}\hat{y} = \sum_{\substack{z \in \mathbb{X} \\ y, z \text{ adj}}} \hat{z}$$

$$= \sum_{i=1}^{D} \sum_{\substack{z \in \mathbb{X} \\ y, z \text{ differ in} \\ \text{coord } i \text{ only}}} \hat{z}$$

$$= \sum_{i=1}^{D} \sum_{y_i' \in S \setminus y_i} \hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_{i-1} \otimes \hat{y}_i' \otimes \hat{y}_{i+1} \otimes \cdots \otimes \hat{y}_D$$

$$= A\hat{y} \quad \checkmark$$

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Show $$A^* \hat{y} = \mathbb{A}^* \hat{y}$$

Write
$$\begin{aligned} j &= \partial(x,y) \\ &= \left| \{ i \mid 1 \le i \le D,\ y_i \ne 1 \} \right| \end{aligned}$$

$$\begin{aligned} A^* \hat{y} &= A^* \left(\hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_D \right) \\ &= \sum_{i=1}^{D} \hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_{i-1} \otimes \underbrace{A^* \hat{y}_i}_{\shortparallel} \otimes \hat{y}_{i+1} \otimes \cdots \otimes \hat{y}_D \end{aligned}$$

$$\begin{cases} (n-1)\hat{y}_i & \text{if } y_i = 1 \\ -\hat{y}_i & \text{if } y_i \ne 1 \end{cases}$$

$$= \Big(\underbrace{(n-1)(D-j) - j}_{\theta^*_j} \Big) \hat{y}$$

$$= \mathbb{A}^* \hat{y}$$ ✓

□

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COR 72 For a subspace $W \subseteq V$ TFAE

(i) W is a Π-module

(ii) W is an $\mathcal{L}$-module

(iii) $AW \subseteq W$ and $A^* W \subseteq W$

pf (i) $\Leftrightarrow$ (iii) By L 71 and the def of Π-module

(ii) $\Leftrightarrow$ (iii) Since $\mathcal{L}$ is gen by A, A^* $\square$

COR 73 With ref to COR 72 suppose (i) - (iii) hold. Then:

W is irred as Π-module iff W is irred as $\mathcal{L}$-module. $\square$

pf (ex)

Action of Φ on $\mathbb{V} = V^{\otimes D}$

Recall $\Phi \in \mathcal{L}$.

We consider the action of Φ on $\mathbb{V} = V^{\otimes D}$

Recall

$$V = e_0 V + e_1 V$$ ds of $\mathcal{L}$-modules

Φ acts on $e_0 V$ as $\frac{n-2}{2} I$

Φ acts on $e_1 V$ as $-I$

Obs

$$\mathbb{V} = V^{\otimes D}$$
$$= (e_0 V + e_1 V)^{\otimes D}$$
$$= \sum e_{i_1} V \otimes e_{i_2} V \otimes \cdots \otimes e_{i_D} V \qquad (*)$$

sum over all sequences $(i_1, i_2, \ldots, i_D)$ of elements taken from $\{0, 1\}$

Sum $(*)$ is dsum of $\mathcal{L}$-modules

For each summand in $(*)$ define

$$\text{Displacement} = \left| \{ j \mid 1 \le j \le D, \ i_j = 1 \} \right|$$

Ex 74 For $n=3$

$\mathcal{L}$-module	displacement
$e_0V \otimes e_0V \otimes e_0V$	0
$e_1V \otimes e_0V \otimes e_0V$ $e_0V \otimes e_1V \otimes e_0V$ $e_0V \otimes e_0V \otimes e_1V$	1
$e_1V \otimes e_1V \otimes e_0V$ $e_1V \otimes e_0V \otimes e_1V$ $e_0V \otimes e_1V \otimes e_1V$	2
$e_1V \otimes e_1V \otimes e_1V$	3

Def 75 For $0 \le z \le D$ define

$\underline{\mathbb{V}}_z$ = sum of those terms in ✗ that have displacement z.

Obs

$$\underline{\mathbb{V}} = \sum_{z=0}^{D} \underline{\mathbb{V}}_z \qquad \text{ds of } \mathcal{L}\text{-modules}$$

LEM 76

(i) Φ is diagonalizable on $\mathbb{V}$

(ii) The eigenspaces for Φ on $\mathbb{V}$ are

$$\mathbb{V}_j \qquad 0 \le j \le D$$

(iii) For $0 \le j \le D$ the eigenvalue of Φ for $\mathbb{V}_j$ is

$$\frac{n}{2}(D-j) - D$$

pf Given $\mathcal{L}$-module

$$W = e_{i_1}V \otimes e_{i_2}V \otimes \cdots \otimes e_{i_D}V \qquad i_j \in \{0,1\} \quad 1 \le j \le D$$

Find action of Φ on W

Given $$w = w_1 \otimes \cdots \otimes w_D \in W$$

$$\Phi w = \Phi(w_1 \otimes \cdots \otimes w_D)$$

$$= \sum_{j=1}^{D} w_1 \otimes w_2 \otimes \cdots \otimes w_{j-1} \otimes \underbrace{\Phi w_j}_{= \begin{cases} \frac{n-2}{2} w_j & \text{if } i_j = 0 \\ -w_j & \text{if } i_j = 1 \end{cases}} \otimes w_{j+1} \otimes \cdots \otimes w_D$$

$$= \Big(\underbrace{(D-j)\frac{n-2}{2} - j}_{= \frac{n}{2}(D-j) - D} \Big) w_1 \otimes w_2 \otimes \cdots \otimes w_D$$

Result follows. $\square$

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LEM 77 Let $W \subseteq V$ denote an irred $\mathcal{L}$-module.

Then $\exists$ unique η $(0 \le \eta \le d)$ s.t

$$W \subseteq V_\eta$$

Call η the displacement of W

pf $\quad \Phi W \subseteq W$ since $\Phi \in \mathcal{L}$

Since $\mathbb{C}$ is alg closed $\exists$ nonzero vector w in W that is an eigenvector for Φ. Let λ = eigenvalue

Let $W_\lambda = \{ v \in W \mid \Phi v = \lambda v \}$

So $\quad w \in W_\lambda \qquad W_\lambda \neq 0$

Claim W_λ is $\mathcal{L}$-submodule of W:

$\forall y \in \mathcal{L} \quad \forall v \in W_\lambda$ show $yv \in W_\lambda$

$$\begin{aligned} \Phi y v &= y \Phi v && \text{since } \Phi \in Z(\mathcal{L}) \\ &= y \lambda v \\ &= \lambda y v \end{aligned}$$

so $\quad yv \in W_\lambda$ ✓

claim proved

Now $\quad W_\lambda = W$ since $\mathcal{L}$-module W is irred

So $\quad \Phi$ acts on W as λI

So $\quad \lambda$ is an eigval of Φ and W is in the corresp eigenspace. Result follows via L76

$\square$

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Action of R, L, F on $\mathbb{V} = V^{\otimes\infty}$

In Lem 25 we defined elements

$$R, L, F$$

in $\mathcal{L}$.

$$R = \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & \\ 1 & O \\ \vdots & \\ 1 & \end{array}\right) \qquad L = \left(\begin{array}{c|c} 0 & 1\ 1\ \cdots\ 1 \\ \hline 0 & O \end{array}\right)$$

$$F = \left(\begin{array}{c|c} 0 & O \\ \hline O & \begin{matrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \\ 1 & 1 & \ddots & \\ \vdots & & & 0 \end{matrix} \end{array}\right)$$

Consider action of R, L, F on $\mathbb{V} = V^{\otimes\infty}$

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Recall base vertex $x = 11\cdots1$

LEM 78 $\forall y \in X$

(i) $$R\hat{y} = \sum_{\substack{z \in X \\ y,z \text{ adj} \\ \partial(x,z) = \partial(x,y)+1}} \hat{z}$$

(ii) $$L\hat{y} = \sum_{\substack{z \in X \\ y,z \text{ adj} \\ \partial(x,z) = \partial(x,y)-1}} \hat{z}$$

(iii) $$F\hat{y} = \sum_{\substack{z \in X \\ y,z \text{ adj} \\ \partial(x,z) = \partial(x,y)}} \hat{z}$$

pf (i) Write $y = (y_1, y_2, \ldots, y_D)$

$y_i \in S \quad 1 \le i \le D$

$$R\hat{y} = R(\hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_D)$$

$$= \sum_{i=1}^{D} \hat{y}_1 \otimes \hat{y}_2 \otimes \cdots \otimes \hat{y}_{i-1} \otimes \underbrace{R\hat{y}_i}_{\shortparallel} \otimes \hat{y}_{i+1} \otimes \cdots \otimes \hat{y}_D$$

$$\begin{cases} 0 & \text{if } y_i \ne 1 \\ \sum_{y_i' \in S \setminus 1} \hat{y}_i' & \text{if } y_i = 1 \end{cases}$$

$$= \sum_{\substack{z \in X,\ y,z \text{ adj} \\ \partial(x,z) = \partial(x,y)+1}} \hat{z} \quad \checkmark$$

(ii), (iii) exer $\square$

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the irred Π-modules

Aside on $sl_2(\mathbb{C})$-modules

LEM 79 $\exists$ a family of finite-diml irred $sl_2(\mathbb{C})$ modules

$$V_d \qquad\qquad d = 0, 1, 2, \ldots \qquad\qquad (*)$$

with following properties.

V_d has a basis $\{v_i\}_{i=0}^{d}$ s.t.

$$h.v_i = (d-2i)\,v_i \qquad 0 \le i \le d$$

$$f.v_i = (i+1)\,v_{i+1} \qquad 0 \le i \le d-1, \qquad f.v_d = 0$$

$$e.v_i = (d-i+1)\,v_{i-1} \qquad 1 \le i \le d, \qquad e.v_0 = 0$$

Each f.d. irred $sl_2(\mathbb{C})$-module is iso to exactly one of the modules $(*)$.

pf (ex) See Humphreys
Intro to Lie algebra

Let $W \subseteq V$ denote an irred T-module

(= irred $\mathcal{L}$-module)

Since $[\mathcal{L},\mathcal{L}] \subseteq \mathcal{L}$ we may view W as

an $[\mathcal{L},\mathcal{L}]$ module.

LEM 80 With the above notation

the $[\mathcal{L},\mathcal{L}]$ module W is irreducible.

pf Suppose U is a nonzero subspace of W

that is $[\mathcal{L},\mathcal{L}]$-invariant. Show $U = W$.

Recall

$$\mathcal{L} = [\mathcal{L},\mathcal{L}] + Z(\mathcal{L}) \qquad \text{ds}$$

$$Z(\mathcal{L}) = \mathbb{C}\,\Phi$$

By L77 Φ acts on W as a scalar mult of I

So $\Phi U \subseteq U$.

Now $\mathcal{L}U = \underbrace{[\mathcal{L},\mathcal{L}]U}_{\subseteq U} + \underbrace{\Phi U}_{\subseteq U}$

$\subseteq U$.

$\mathcal{L}$-module W is irred so $U = W$. $\square$

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Let $W \subseteq V$ denote an irred T-module.

Via Lem 80 view W as irred $[\mathcal{L}, \mathcal{L}]$ module

Recall $[\mathcal{L}, \mathcal{L}]$ iso $sl_2(\mathbb{C})$

So W has str of irred $sl_2(\mathbb{C})$-module.

this module is iso $sl_2(\mathbb{C})$-module V_d

where
$$d+1 = \dim(W)$$

Call d the diameter of W

Let us clarify the iso $[\mathcal{L}, \mathcal{L}] \to sl_2(\mathbb{C})$

L 81. The Lie alg iso $[\mathcal{L}, \mathcal{L}] \to sl_2(\mathbb{C})$ from L 47 sends

$$A - \Phi \to (n-1)e + f + \frac{2-n}{2} h$$

$$A^* - \Phi \to \frac{n}{2} h$$

pf Use L 44 and L 47 (ex) $\square$

LEM 82 $\exists$ Lie alg iso $[\mathcal{L}, \mathcal{L}] \to sl_2(\mathbb{C})$

that sends

$$A - \Phi \to \frac{n}{2} h$$

$$A^* - \Phi \to (n-1)e + f + \frac{2-n}{2} h$$

pf Use the matrix representations of $A - \Phi$, $A^* - \Phi$ wrt dual std basis for EoV

(above L 43) (ex) $\square$

Let $W \subseteq V$ denote an irred T-module.

Define

$$r = \min\{ i \mid 0 \le i \le D,\ E_i^* W \neq 0 \}$$

"endpoint of W"

$$t = \min\{ i \mid 0 \le i \le D,\ E_i W \neq 0 \}$$

"dual endpoint of W"

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Th 83 Let $W \subseteq V$ denote an irred T-module with diameter d, endpt r, and displacement η.

(i) $\eta + D - d$ is even

(ii) $r = \frac{\eta + D - d}{2}$

(iii) $r + d \leq D$, $\quad 2r + d - D \geq 0$

(iv) For $0 \leq i \leq D$

$$E_i^* W \neq 0 \quad \text{iff} \quad r \leq i \leq r + d$$

(v) For $0 \leq i \leq d$

$$\dim E_{r+i}^* W = 1$$

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pf: Identify $[\mathcal{L},\mathcal{L}]$ with $sl_2(\mathbb{C})$ via the isomorphism in LEM 81.

Let $\{v_i\}_{i=0}^{d}$ denote the basis for W from LEM 79.

So for $0 \le i \le d$

$$h\, v_i = (d-2i)\, v_i$$

By LEM 76, LEM 77

$$\Phi\, v_i = \left(\tfrac{n}{2}(D-\jmath) - D\right) v_i$$

We identified

$$A^* - \Phi = \tfrac{n}{2} h$$

so

$$A^* v_i = \left(\Phi + \tfrac{n}{2} h\right) v_i$$

$$\in \mathbb{C} v_i$$

So v_i is an eigenvector for A^*

So $\exists\, \jmath$ $(0 \le \jmath \le D)$ such that

$$v_i \in E_\jmath^* W$$

How are $i, \jmath$ related:

Require

$$\theta_\jmath^* = \tfrac{n}{2}(D-\jmath) - D + \tfrac{n}{2}(d-2i)$$

Recall

$$\theta_\jmath^* = \theta_\jmath$$

$$= (n-1)D - n\jmath$$

This gives

$$\eta - i = \frac{\eta + D - d}{2}$$

So

$$\eta + D - d \quad \text{even}$$

Min value of η occurs at $i = 0$. So

$$r = \frac{\eta + D - d}{2}$$

Recall $\eta \geq 0$ so

$$2r + d - D \geq 0$$

So far

$$v_i \in E^*_{r+i} W \qquad 0 \leq i \leq d \qquad (*)$$

Taking $i = d$ get

$$r + d \leq D$$

Assertions (iv), (v) follow from $(*)$ and since $\{v_i\}_{i=0}^{d}$ is a basis for W. $\square$