MANILA Lecture 4 Monday May 31

5/31/10

the equitable basis for $sl_2(\mathbb{C})$

Recall the Lie algebra $sl_2(\mathbb{C})$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Define

$$x = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \qquad y = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \qquad z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= 2e - h \qquad\qquad = -2f - h \qquad\qquad = h$$

Then x, y, z is a basis for $sl_2(\mathbb{C})$ and

$$[x, y] = 2x + 2y, \qquad [y, z] = 2y + 2z, \qquad [z, x] = 2z + 2x \qquad (*)$$

(ex)

Def 57 Let V denote a vector space over $\mathbb{C}$ with dim 2. Recall Lie algebra $sl(V)$

A basis x, y, z for $sl(V)$ is called <u>equitable</u> whenever it satisfies $(*)$.

We saw $sl(V)$ is iso $sl_2(\mathbb{C})$ and $sl_2(\mathbb{C})$ has an equit basis, so $sl(V)$ has an equit basis. We now construct many equit bases for $sl(V)$.

LEM 58 With ref to Def 57

pick 3 mutually distinct 1-dim'l subspaces of V, denoted V_1, V_2, V_3. Obs V is the direct sum of any two of V_1, V_2, V_3.

Define linear trans

$x: V \to V, \qquad y: V \to V, \qquad z: V \to V$

such that

$$(x+I)V_1 = 0, \qquad (x-I)V_2 = 0,$$

$$(y+I)V_2 = 0, \qquad (y-I)V_3 = 0,$$

$$(z+I)V_3 = 0, \qquad (z-I)V_1 = 0.$$

Then x, y, z is an equit basis for $sl(V)$.

pf Each of x, y, z has eigvals $1, -1$ so has trace 0.

Therefore $x, y, z \in sl(V)$.

Pick $0 \neq v_1 \in V_1$ $\qquad 0 \neq v_2 \in V_2$

So v_1, v_2 basis for V

Rel the basis v_1, v_2

$$x: \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{= -h} \qquad y: \underbrace{\begin{pmatrix} 1 & 0 \\ \alpha & -1 \end{pmatrix}}_{= h + \alpha f} \qquad 0 \neq \alpha \in \mathbb{C}$$

$$z: \underbrace{\begin{pmatrix} 1 & \beta \\ 0 & -1 \end{pmatrix}}_{= h + \beta e} \qquad 0 \neq \beta \in \mathbb{C}$$

Using these matrix reps

x, y, z are lin indep, hence a basis for $sl(V)$

Check $[x, y] = 2x + 2y$:

Use matrix reps:

$$[-h, h+\alpha f] \overset{?}{=} 2(-h) + 2(h+\alpha f)$$

$$\shortparallel$$

$$-\alpha [h, f]$$

$$\shortparallel$$

$$2\alpha f \quad \checkmark$$

The equations

$[y, z] = 2y + 2z$, $[z, x] = 2z + 2x$

are sim checked. $\square$

LEM 59 Given any Lie alg L over $\mathbb{C}$ of dim 3 5/31/10

Assume L has a basis x, y, z such that

$[x, y] = 2x+2y$ $\quad [y, z] = 2y+2z$ $\quad [z, x] = 2z+2x$

then L is iso $sl_2(\mathbb{C})$

pf Define an iso of vector spaces $L \to sl_2(\mathbb{C})$ such that

$x \longrightarrow 2e-h, \quad y \to -2f-h, \quad z \to h.$

This map respects the Lie bracket and is therefore an iso of Lie algebras. $\square$

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The tetrahedron Lie algebra

Let V denote a vector space over $\mathbb{C}$ with dim 2.

Let $\mathbb{I} = \{0,1,2,3\}$.

Let $\{V_i\}_{i \in \mathbb{I}}$ denote mut. distinct 1-dim'l subspaces of V

For distinct $i,j \in \mathbb{I}$

$$V = V_i + V_j \qquad (ds)$$

so $\exists$ unique lin trans

$$x_{ij} : V \to V$$

such that

$$(x_{ij} + I) V_i = 0, \qquad (x_{ij} - I) V_j = 0$$

x_{ij} has eigvals $1, -1$ so has trace 0.

Therefore $x_{ij} \in sl(V)$.

LEM 60 With above notation 5/31/10

(i) For dist $i, j \in \mathbb{I}$

$$x_{ij} + x_{ji} = 0$$

(ii) For mut dist $h, i, j \in \mathbb{I}$

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}$$

(iii) For mut dist $h, i, j, k \in \mathbb{I}$

$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4[x_{hi}, x_{jk}]$$

(Dolan-Grady)

In the above eqs the bracket means

$$[r, s] = rs - sr$$

pf (i) clear

(ii) same pf as LEM 58

(iii) Pick $0 \neq v_h \in V_h$ $\quad 0 \neq v_i \in V_i$

v_h, v_i basis for V

Rel this basis

$$x_{hi} : \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h \qquad x_{jk} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ah + be + cf$$

$a, b, c, d \in \mathbb{C}$

$a + d = 0$

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Check

$$\underbrace{\left[-h, \underbrace{\left[-h, \underbrace{\left[-h, ah + be + cf\right]}_{-2be+2cf}\right]}_{4be+4cf}\right]}_{-8be+8cf} \overset{?}{=} 4\underbrace{\left[-h, ah+be+cf\right]}_{-2be+2cf}$$

✓

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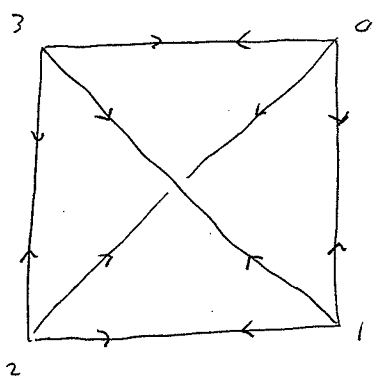
Def 61. Let $\boxtimes$ denote the Lie algebra / $\mathbb{C}$ defined by gens $\{ x_{ij} \mid i,j \in \mathbb{I},\ i \neq j \}$ and relations (i) – (iii) in Lem 57.

$\boxtimes$ is called the <u>tetrahedron</u> <u>algebra</u>

<u>Diagram</u>



Each directed arc represents a generator

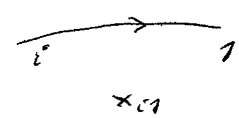
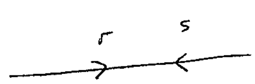
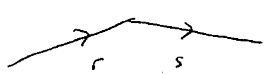
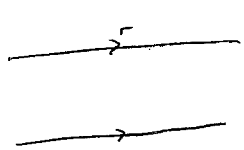


diagram	meaning
	$r + s = 0$
	$[r, s] = 2r + 2s$
	r, s satisfy Dolan-Grady rels

Back to DRGs

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Return to notation of prev lectures

Continue to discuss K_n

Recall $e_0 V$ has basis $\hat{x}, \mathbb{1}$.

Recall set $\mathbb{I} = \{0,1,2,3\}$.

We define 1-dim'l subspaces $\{V_i\}_{i \in \mathbb{I}}$ of $e_0 V$ as follows:

$$V_0 = \mathbb{C}\hat{x} \qquad V_1 = \mathbb{C}(\mathbb{1} - \hat{x})$$

$$V_2 = \mathbb{C}(n\hat{x} - \mathbb{1}) \qquad V_3 = \mathbb{C}\mathbb{1}$$

Obs $\{V_i\}_{i \in \mathbb{I}}$ are mut distinct.

By Lem 60 $\exists$ $\boxtimes$-module str on $e_0 V$ s.t.

$$(x_{ij} + I)V_i = 0, \qquad (x_{ij} - I)V_j = 0$$

for all dist $i,j \in \mathbb{I}$

Obs $\exists$ $\boxtimes$-module str on $e_1 V$ s.t.

$$x_{ij}\, e_1 V = 0$$

for all dist $i,j \in \mathbb{I}$

Since $V = e_0 V + e_1 V$ we now have a $\boxtimes$-module str on V

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LEM 62 The following hold on V

$$(*)$$

(i) $A - \Phi = \frac{n}{2} X_{23}$

(ii) $A^* - \Phi = \frac{n}{2} X_{10}$

(iii) $\tilde{H} = X_{30}$

pf (i) Rel the dual st. basis $\mathbb{1}, n\hat{x} - \mathbb{1}$

$$A - \Phi : \begin{pmatrix} \frac{n}{2} & 0 \\ 0 & -\frac{n}{2} \end{pmatrix}$$

$$X_{23} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So $(*)$ holds on $e_0 V$

Also $(*)$ holds on $e_i V$ since both sides 0 on $e_i V$.

Result follows. $\square$

(ii), (iii) Sim (ex)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\overset{\mathbb{C}\hat{x}}{0} & \overset{\mathbb{C}(\mathbb{1}-\hat{x})}{1} \arrow[l, "A^{x}-\Phi = \frac{n}{2} X_{10}"'] \arrow[d, dash] \arrow[dl, dash] \\
\underset{\mathbb{C}\mathbb{1}}{3} \arrow[u, "\tilde{H} = X_{30}"] & \underset{\mathbb{C}(n\hat{x}-\mathbb{1})}{2} \arrow[l, "A-\Phi = \frac{n}{2} X_{23}"] \arrow[ul]
\end{tikzcd}
$$

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LEM 63 Rel the split bases $\hat{x}$, $\mathbb{1}$

$$X_{01} : \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$X_{12} : \begin{pmatrix} \frac{n+1}{n-1} & \frac{2n}{n-1} \\ \frac{-2}{n-1} & \frac{1+n}{1-n} \end{pmatrix}$$

$$X_{23} : \begin{pmatrix} -1 & 0 \\ \frac{2}{n} & 1 \end{pmatrix}$$

$$X_{30} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X_{02} : \begin{pmatrix} -1 & -2n \\ 0 & 1 \end{pmatrix}$$

$$X_{13} : \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$$

pf

$$X_{01} \, \hat{x} = -\hat{x}$$

$$X_{01} \, \mathbb{1} = X_{01} \left(\mathbb{1} - \hat{x} \right) + X_{01} \, \hat{x}$$

$$= \mathbb{1} - \hat{x} \qquad - \hat{x}$$

$$= \mathbb{1} - 2\hat{x}$$

$$X_{12} \, \hat{x} = X_{12} \, \frac{n\hat{x} - \mathbb{1} + \mathbb{1} - x}{n-1}$$

$$= \frac{n\hat{x} - \mathbb{1} + \hat{x} - \mathbb{1}}{n-1}$$

$$= \frac{(n+1)\,\hat{x} - 2\,\mathbb{1}}{n-1}$$

$$X_{12} \, \mathbb{1} = X_{12} \, \frac{n\hat{x} - \mathbb{1} + n\left(\mathbb{1} - \hat{x}\right)}{n-1}$$

$$= \frac{n\hat{x} - \mathbb{1} - n\left(\mathbb{1} - \hat{x}\right)}{n-1}$$

$$= \frac{2n\,\hat{x} - (n+1)\,\mathbb{1}}{n-1}$$

$$X_{23}\ \hat{x} = X_{23}\ \frac{n\hat{x} - \mathbb{1} + \mathbb{1}}{n}$$

$$= \frac{\mathbb{1} - n\hat{x} + \mathbb{1}}{n}$$

$$= \frac{-n\hat{x} + 2\mathbb{1}}{n}$$

$$X_{23}\ \mathbb{1} = \mathbb{1}$$

$$X_{30}\ \hat{x} = \hat{x}$$

$$X_{30}\ \mathbb{1} = -\mathbb{1}$$

$$X_{02}\ \hat{x} = -\hat{x}$$

$$X_{02}\ \mathbb{1} = X_{02}\left(\mathbb{1} - n\hat{x} + n\hat{x}\right)$$

$$= \mathbb{1} - n\hat{x} - n\hat{x}$$

$$= -2n\hat{x} + \mathbb{1}$$

$$X_{13}\ \hat{x} = X_{13}\left(\hat{x} - \mathbb{1} + \mathbb{1}\right)$$

$$= \mathbb{1} - \hat{x} + \mathbb{1}$$

$$= -\hat{x} + 2\mathbb{1}$$

$$X_{13}\ \mathbb{1} = \mathbb{1}$$

$\square$

LEM 64 Rel the standard basis $\hat{x}$, $\mathbb{1}-\hat{x}$

$$X_{01}: \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X_{12}: \begin{pmatrix} 1 & 0 \\ \frac{2}{1-n} & -1 \end{pmatrix}$$

$$X_{23}: \begin{pmatrix} \frac{2-n}{n} & \frac{2(n-1)}{n} \\ \frac{2}{n} & \frac{n-2}{n} \end{pmatrix}$$

$$X_{30}: \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$

$$X_{02}: \begin{pmatrix} -1 & 2(1-n) \\ 0 & 1 \end{pmatrix}$$

$$X_{13}: \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

pf For each matrix B in L63 compute CBC^{-1} where

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

□

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LEM 65 the following hold on V:

$$(i) \quad n X_{01} - X_{02} + (1-n) X_{03} = 0$$

$$(ii) \quad n X_{10} - X_{13} + (1-n) X_{12} = 0$$

$$(iii) \quad n X_{23} - X_{20} + (1-n) X_{21} = 0$$

$$(iv) \quad n X_{32} - X_{31} + (1-n) X_{30} = 0$$

"Corner dependencies"

pf One checks these equations hold on $e_0 V$ using the matrix reps in L63.
These equations also hold on $e_1 V$ since each X_{ij} is 0 on $e_1 V$.
Result follows. $\square$

5/31/10

Our action of $\boxtimes$ on V induces a Lie algebra hom

$$Ev : \boxtimes \longrightarrow gl(V) \equiv gl_X(\mathbb{C})$$

LEM 66

(i) the image of $\boxtimes$ under Ev is $[\mathcal{L}, \mathcal{L}]$

(ii) the image of $\boxtimes$ under Ev is iso to $sl_2(\mathbb{C})$

pf (i) Denote the image by Im

$Im \supseteq [\mathcal{L}, \mathcal{L}]$:

Recall $[\mathcal{L}, \mathcal{L}]$ is gen by $A - \Phi$, $A^* - \Phi$.

By L62

$$A - \Phi = \frac{n}{2} Ev(x_{23}) \in Im$$

$$A^* - \Phi = \frac{n}{2} Ev(x_{10}) \in Im$$

$Im \subseteq [\mathcal{L}, \mathcal{L}]$: $[\mathcal{L}, \mathcal{L}]$ contains

$Ev(x_{23})$, $Ev(x_{10})$, $Ev(x_{30})$ by L62

Now using L65 we find $[\mathcal{L}, \mathcal{L}]$ contains

$Ev(x_{02})$, $Ev(x_{13})$, $Ev(x_{12})$.

Result follows.

(ii) Clear from (i) $\square$

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We now write each generator X_{ij} of $\boxtimes$ in terms of X_{01}, X_{23}.

LEM 67 The following hold in V:

$$X_{02} = \frac{n}{4}\left(2X_{01} - 2X_{23} + [X_{01}, X_{23}]\right)$$

$$X_{03} = \frac{n}{4(n-1)}\left(2X_{01} + 2X_{23} - [X_{01}, X_{23}]\right)$$

$$X_{12} = \frac{n}{4(n-1)}\left(-2X_{01} - 2X_{23} - [X_{01}, X_{23}]\right)$$

$$X_{13} = \frac{n}{4}\left(-2X_{01} + 2X_{23} + [X_{01}, X_{23}]\right)$$

pf To verify the first eq, use the matrix representation in L63.
To get the remaining three use L65 $\square$

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LEM 6.8 The following hold on V:

(i) $\frac{n}{n-1} X_{01} + X_{12} + \frac{n}{n-1} X_{23} + X_{30} = 0,$

"cycle dependencies"

(ii) $n X_{01} + X_{13} + n X_{32} + X_{20} = 0,$

(iii) $\frac{1}{1-n} X_{02} + X_{21} + \frac{1}{1-n} X_{13} + X_{30} = 0.$

pf (i) In L 6.5 add eq's (i), (iii)

(ii) - - - (ii), (iii)

(iii) - - - (i), (iii) $\square$

LEM 69 Relative the bases $\{\hat{y} \mid y \in X\}$ 5/31/10

$$X_{01}: \left(\begin{array}{c|ccc} -1 & & 0 & \\ \hline & \frac{1}{n-1} & \frac{1}{n-1} & \cdots \\ 0 & \frac{1}{n-1} & \frac{1}{n-1} & \\ & \vdots & & \ddots \end{array}\right) \qquad X_{12}: \left(\begin{array}{c|ccc} 1 & & 0 & \\ \hline \frac{2}{1-n} & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ \frac{2}{1-n} & \frac{1}{1-n} & \frac{1}{1-n} & \\ \vdots & \vdots & & \ddots \end{array}\right)$$

$$X_{23}: \left(\begin{array}{c|ccc} \frac{2-n}{n} & \frac{2}{n} & \frac{2}{n} & \cdots \\ \hline \frac{2}{n} & \frac{n-2}{n(n-1)} & \frac{n-2}{n(n-1)} & \cdots \\ \frac{2}{n} & \frac{n-2}{n(n-1)} & \frac{n-2}{n(n-1)} & \\ \vdots & \vdots & & \ddots \end{array}\right) \qquad X_{30}: \left(\begin{array}{c|ccc} 1 & \frac{2}{1-n} & \frac{2}{1-n} & \cdots \\ \hline & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ 0 & \frac{1}{1-n} & \frac{1}{1-n} & \\ & \vdots & & \ddots \end{array}\right)$$

$$X_{02}: \left(\begin{array}{c|ccc} -1 & -2 & -2 & \cdots \\ \hline & \frac{1}{n-1} & \frac{1}{n-1} & \cdots \\ 0 & \frac{1}{n-1} & \frac{1}{n-1} & \\ & \vdots & & \ddots \end{array}\right) \qquad X_{13}: \left(\begin{array}{c|ccc} 1 & & 0 & \\ \hline 2 & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ 2 & \frac{1}{1-n} & \frac{1}{1-n} & \\ \vdots & \vdots & & \ddots \end{array}\right)$$

pf Use Lem 64 as follows. 5/31/10

For each generator x_{ij} let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denote

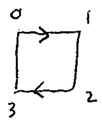
the matrix rep x_{ij} rel the standard basis $\hat{x}, \mathbb{1} - \hat{x}$

then the matrix rep x_{ij} rel $\{\hat{y} \mid y \in X\}$ is

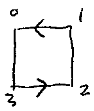
$$\left(\begin{array}{c|cccc} a & \frac{b}{n-1} & \frac{b}{n-1} & \cdots \\ \hline c & \frac{d}{n-1} & \frac{d}{n-1} & \\ c & \frac{d}{n-1} & \frac{d}{n-1} & \cdots \\ \vdots & \vdots & & \ddots \end{array}\right)$$

Indeed the entries are constant over each block by L43. $\square$

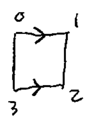
LEM A1 Rel the standard basis $\hat{x}$, $\mathbb{1}-\hat{x}$



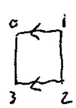
$$X_{01} + X_{23} = \begin{pmatrix} \frac{2(1-n)}{n} & \frac{2(n-1)}{n} \\ \frac{2}{n} & \frac{2(n-1)}{n} \end{pmatrix}$$



$$X_{10} + X_{32} = \begin{pmatrix} \frac{2(n-1)}{n} & \frac{2(1-n)}{n} \\ \frac{-2}{n} & \frac{2(1-n)}{n} \end{pmatrix}$$

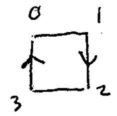


$$X_{01} + X_{32} = \begin{pmatrix} \frac{-2}{n} & \frac{2(1-n)}{n} \\ \frac{-2}{n} & \frac{2}{n} \end{pmatrix}$$

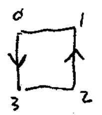


$$X_{10} + X_{23} = \begin{pmatrix} \frac{2}{n} & \frac{2(n-1)}{n} \\ \frac{2}{n} & \frac{-2}{n} \end{pmatrix}$$

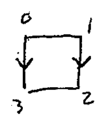
LEM A2 Rel the standard basis $\hat{x}$, $\mathbb{1} - \hat{x}$



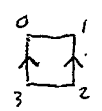
$$X_{12} + X_{30} = \begin{pmatrix} 2 & -2 \\ \frac{2}{1-n} & -2 \end{pmatrix}$$



$$X_{21} + X_{03} = \begin{pmatrix} -2 & 2 \\ \frac{2}{n-1} & 2 \end{pmatrix}$$

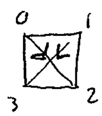


$$X_{03} + X_{12} = \begin{pmatrix} 0 & 2 \\ \frac{2}{1-n} & 0 \end{pmatrix}$$

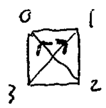


$$X_{30} + X_{21} = \begin{pmatrix} 0 & -2 \\ \frac{2}{n-1} & 0 \end{pmatrix}$$

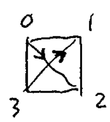
LEM A3 Rel the standard basis $\hat{x},\ \mathbb{1}-\hat{x}$



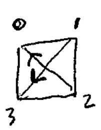
$$X_{02} + X_{13} = \begin{pmatrix} 0 & 2(1-n) \\ 2 & 0 \end{pmatrix}$$



$$X_{20} + X_{31} = \begin{pmatrix} 0 & 2(n-1) \\ -2 & 0 \end{pmatrix}$$



$$X_{02} + X_{31} = \begin{pmatrix} -2 & 2(1-n) \\ -2 & 2 \end{pmatrix}$$



$$X_{20} + X_{13} = \begin{pmatrix} 2 & 2(n-1) \\ 2 & -2 \end{pmatrix}$$

DEFA4 We define elements

$$\alpha,\ \beta,\ \gamma$$

in $[\mathcal{L},\mathcal{L}]$ as follows.

Rel st. basis $\hat{x},\ \mathbb{1}-\hat{x}$

$$\alpha = \frac{i}{\sqrt{n-1}}\begin{pmatrix} 0 & n-1 \\ -1 & 0 \end{pmatrix} \qquad i^2 = -1$$

$$\beta = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & n-1 \\ 1 & -1 \end{pmatrix}$$

$$\gamma = \sqrt{\frac{n-1}{n}}\begin{pmatrix} 1 & -1 \\ \frac{1}{1-n} & -1 \end{pmatrix}$$

(we take pos square roots)

LEM A5 On $e_0 V$ each of α, β, γ

has eigvals $1, -1$

pf In def A4 each matrix has $\det = -1$
and trace 0, so the eigvals are $1, -1$ $\square$

LEM A6 We have

$$[\alpha, \beta] = 2i\gamma \qquad i^2 = -1$$

$$[\beta, \gamma] = 2i\alpha$$

$$[\gamma, \alpha] = 2i\beta$$

pf Use the 2×2 matrix reps in Def A4 $\square$

We write the sums from LA1–LA3 in terms of α, β, γ

LEM A7 We have

	α	β	γ
$X_{01}+X_{23}$			$-2\sqrt{\frac{n-1}{n}}$
$X_{10}+X_{32}$			$2\sqrt{\frac{n-1}{n}}$
$X_{01}+X_{32}$		$\frac{-2}{\sqrt{n}}$	
$X_{10}+X_{23}$		$\frac{2}{\sqrt{n}}$	
$X_{12}+X_{30}$			$2\sqrt{\frac{n}{n-1}}$
$X_{21}+X_{03}$			$-2\sqrt{\frac{n}{n-1}}$
$X_{03}+X_{12}$	$\frac{-2i}{\sqrt{n-1}}$		
$X_{30}+X_{21}$	$\frac{2i}{\sqrt{n-1}}$		
$X_{02}+X_{13}$	$2i\sqrt{n-1}$		
$X_{20}+X_{31}$	$-2i\sqrt{n-1}$		
$X_{02}+X_{31}$		$-2\sqrt{n}$	
$X_{20}+X_{13}$		$2\sqrt{n}$	

LEM A8 We write the gens x_{ij} for $\boxtimes$ in terms of α, β, γ

	α	β	γ
X_{01}		$\frac{-1}{\sqrt{n}}$	$-\sqrt{\frac{n-1}{n}}$
X_{10}		$\frac{1}{\sqrt{n}}$	$\sqrt{\frac{n-1}{n}}$
X_{23}		$\frac{1}{\sqrt{n}}$	$-\sqrt{\frac{n-1}{n}}$
X_{32}		$\frac{-1}{\sqrt{n}}$	$\sqrt{\frac{n-1}{n}}$
X_{12}	$\frac{-i}{\sqrt{n-1}}$		$\sqrt{\frac{n}{n-1}}$
X_{21}	$\frac{i}{\sqrt{n-1}}$		$-\sqrt{\frac{n}{n-1}}$
X_{30}	$\frac{i}{\sqrt{n-1}}$		$\sqrt{\frac{n}{n-1}}$
X_{03}	$\frac{-i}{\sqrt{n-1}}$		$-\sqrt{\frac{n}{n-1}}$
X_{02}	$i\sqrt{n-1}$	$-\sqrt{n}$	
X_{20}	$-i\sqrt{n-1}$	$\sqrt{n}$	
X_{13}	$i\sqrt{n-1}$	$\sqrt{n}$	
X_{31}	$-i\sqrt{n-1}$	$-\sqrt{n}$	

here $i^2 = -1$