MANILA Lecture 3 Monday May 31

Appendix 5/31/10

Consider problem 2 for K_n (expanded ver)

Consider

$$A A^* A^{-1} A^{*-1} \qquad A^* A A^{*-1} A^{-1}$$

$$A^* A^{-1} A^{*-1} A \qquad A^{-1} A^* A A^{*-1}$$

$$A^{-1} A^{*-1} A A^* \qquad A^{*-1} A^{-1} A^* A$$

$$A^{*-1} A A^* A^{-1} \qquad A A^{*-1} A^{-1} A^*$$

For each row the two entries are inverses.

In each col the four entries are similar

Find the eigenspaces / eigvalues.

Since $A, A^* \in T$ each of e_0V, e_1V

is left invariant by each of the above 8 matrices

<u>Action on e_1V :</u>

Each of A, A^* acts on e_1V as $-I$

So each of above 8 elements act on e_1V as I

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Action on e_0V:

Recall e_0V has basis $\hat{x}, \mathbb{1}$

Consider 4 vectors

$$\hat{x}, \quad \mathbb{1}-\hat{x}, \quad \mathbb{1}, \quad n\hat{x}-\mathbb{1}$$

Recall meaning

vector	eigenvector for	eigenvalue
$\hat{x}$	A^*	$n-1$
$\mathbb{1}-\hat{x}$	A^*	-1
$\mathbb{1}$	A	$n-1$
$n\hat{x}-\mathbb{1}$	A	-1

Also

$$A\hat{x} = \mathbb{1}-x \qquad A^{-1}(\mathbb{1}-\hat{x}) = \hat{x}$$

$$A^*\mathbb{1} = n\hat{x}-\mathbb{1} \qquad A^{*-1}(n\hat{x}-\mathbb{1}) = \mathbb{1}$$

Writing $W = e_0V$,

$$E_0^*W = \mathbb{C}\hat{x} \qquad E_1^*W = \mathbb{C}(\mathbb{1}-\hat{x})$$

$$E_0W = \mathbb{C}\mathbb{1} \qquad E_1W = \mathbb{C}(n\hat{x}-\mathbb{1})$$

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Find eigvectors for $AA^*A^{-1}A^{*-1}$ in EoV

$$\mathbb{1}-\hat{x} \xrightarrow{A^{*-1}} \mathbb{1}-\hat{x} \xrightarrow{A^{-1}} \hat{x} \xrightarrow{A^{*}} \hat{x} \xrightarrow{A} \mathbb{1}-x$$

coeff: -1 $\qquad$ $n-1$

$\mathbb{1}-\hat{x}$ is eigvector with eigval $1-n$

$$n\hat{x}-\mathbb{1} \xrightarrow{A^{*-1}} \mathbb{1} \xrightarrow{A^{-1}} \mathbb{1} \xrightarrow{A^{*}} n\hat{x}-\mathbb{1} \xrightarrow{A} n\hat{x}-\mathbb{1}$$

coeff: $\frac{1}{n-1}$ $\qquad$ -1

$n\hat{x}-\mathbb{1}$ is eigvector with eigval $\frac{1}{1-n}$

COR 57 $\quad$ For all 8 elements above the eigvals are

eigvalue	$1-n$	$\frac{1}{1-n}$	1
mult	1	1	$n-2$

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COR ~~58~~ For our 8 elements the eigenspaces are given below.

element	eigval $1-n$	eigval $\frac{1}{1-n}$	eigval 1
$A A^{*} A^{\neg} A^{*\neg}$	$\mathbb{C}(\mathbb{1}-\hat{x})$	$\mathbb{C}(n\hat{x}-\mathbb{1})$	$e_1 V$
$A^{*} A A^{*\neg} A^{\neg}$	$\mathbb{C}(n\hat{x}-\mathbb{1})$	$\mathbb{C}(\mathbb{1}-\hat{x})$	$e_1 V$
$A^{*} A^{\neg} A^{*\neg} A$	$\mathbb{C}\hat{x}$	$\mathbb{C}(n\hat{x}-\mathbb{1})$	$e_1 V$
$A^{\neg} A^{*} A A^{*\neg}$	$\mathbb{C}(n\hat{x}-\mathbb{1})$	$\mathbb{C}\hat{x}$	$e_1 V$
$A^{\neg} A^{*\neg} A A^{*}$	$\mathbb{C}\hat{x}$	$\mathbb{C}\mathbb{1}$	$e_1 V$
$A^{*\neg} A^{\neg} A^{*} A$	$\mathbb{C}\mathbb{1}$	$\mathbb{C}\hat{x}$	$e_1 V$
$A^{*\neg} A A^{*} A^{\neg}$	$\mathbb{C}(\mathbb{1}-\hat{x})$	$\mathbb{C}\mathbb{1}$	$e_1 V$
$A A^{*\neg} A^{\neg} A^{*}$	$\mathbb{C}\mathbb{1}$	$\mathbb{C}(\mathbb{1}-\hat{x})$	$e_1 V$

pf take for example

$$A^* A^{-1} A^{*-1} A = A^{-1}\left(A A^* A^{-1} A^{*-1}\right) A$$

For the eigval $1-n$ corresp eigenvector is

$$A^{-1}(\mathbb{1} - \hat{x}) = \hat{x}$$

For the eigval $\frac{1}{1-n}$ corresp eigvector is

$$A^{-1}(n\hat{x} - \mathbb{1}) = -(n\hat{x} - \mathbb{1})$$

Other entries similarly obtained (ex). $\square$

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$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{C}\hat{x} = E_0^*W \arrow[r, dash, "A^*"] \arrow[d, dash, "A^{-1}A^{*-1}AA^*"'] \arrow[dr, dash, "A^*A^{-1}A^{*-1}A" near start] & \mathbb{C}(\mathbb{1}-\hat{x}) = E_1^*W \arrow[d, dash, "AA^*A^{-1}A^{*-1}"] \\
\mathbb{C}\mathbb{1} = E_0W \arrow[r, dash, "A"'] \arrow[ur, dash, "A^{*-1}AA^*A^{-1}"' near start] & \mathbb{C}(n\hat{x}-\mathbb{1}) = E_1W
\end{tikzcd}$$

Each edge in tetrahedron represents an eigenspace decomp of W for the corresp operator.

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We have now solved problem 2 for K_n.

What about $\Gamma = H(D, n)$?

View $\Gamma = \underbrace{K_n \times K_n \times \cdots \times K_n}_{D}$ $\qquad V(\Gamma) = V(K_n)^{\otimes D}$

View $A_D = \overbrace{a \otimes a \otimes \cdots \otimes a}^{D}$ $\qquad a = A(K_n)$

$= a^{\otimes D}$

$A_D^* = (a^*)^{\otimes D}$ $\qquad a^* = A^*(K_n)$

obs
$$A_D A_D^* A_D^{-1} A_D^{*-1} = (a a^* a^{-1} a^{*-1})^{\otimes D}$$

To get eigenspaces/eigvals for $A_D A_D^* A_D^{-1} A_D^{*-1}$

use our results for $a a^* a^{-1} a^{*-1}$.

The main problem is to relate this to the split decomposition

for Γ.