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The Lie algebra $[L, L]$

Aside on general Lie algebras

Given any Lie algebra L define

$$[L,L] = \mathrm{Span}\{ [x,y] \mid x,y \in L \}$$

then $[L,L]$ is an ideal of L (ex)

Suppose $L = gl(V)$ V = finite dim'l vector space over $\mathbb{C}$

then $[L,L] = sl(V)$ (ex)

Suppose $L = gl_n(\mathbb{C})$ $n \in \mathbb{Z}$, $n > 0$

then $[L,L] = sl_n(\mathbb{C})$ ex

Given Lie algebras L, L'

Given a Lie algebra hom $\sigma : L \to L'$

then the kernel

$$\ker(\sigma) = \{ x \in L \mid \sigma(x) = 0 \}$$

is an ideal of L (ex)

The image

$$\mathrm{Im}(\sigma) = \{ \sigma(x) \mid x \in L \}$$

is a Lie subalgebra of L' but not an ideal of L' in general (ex)

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Back to DRG's

Return to notation of prev. lecture

We continue to discuss K_n

Recall $\mathcal{L}$ iso $gl_2(\mathbb{C})$

so $[\mathcal{L},\mathcal{L}]$ iso $sl_2(\mathbb{C})$

In particular

$$\dim [\mathcal{L},\mathcal{L}] = 3.$$

LEM 38 The following is a basis for $[\mathcal{L},\mathcal{L}]$

$$A - \Phi, \quad A^* - \Phi, \quad [A, A^*] \qquad (*)$$

pf $[\mathcal{L},\mathcal{L}]$ contains $[A, A^*]$

By LEM 20 $[\mathcal{L},\mathcal{L}]$ contains both

$$n(n-2)(A - \Phi) + n^2 (A^* - \Phi)$$

$$n^2 (A - \Phi) + n(n-2)(A^* - \Phi)$$

So $[\mathcal{L},\mathcal{L}]$ contains both

$$A - \Phi, \quad A^* - \Phi$$

The elements $(*)$ are lin indep by Lem 23 (i)

Result follows since $\dim [\mathcal{L},\mathcal{L}] = 3$ $\square$

LEM 39 We have

$$
A - \Phi = \left(\begin{array}{c|cccc}
\frac{2-n}{2} & 1 & 1 & \cdots & 1 \\
\hline
1 & \frac{n-2}{2(n-1)} & \frac{n-2}{2(n-1)} & \cdots & \\
1 & \frac{n-2}{2(n-1)} & \frac{n-2}{2(n-1)} & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \\
1 & & & &
\end{array}\right)
$$

$$
A^{b} - \Phi = \left(\begin{array}{c|ccc}
\frac{n}{2} & & 0 & \\
\hline
 & \frac{n}{2(1-n)} & \frac{n}{2(1-n)} & \cdots \\
0 & \frac{n}{2(1-n)} & \frac{n}{2(1-n)} & \cdots \\
 & \vdots & \vdots & \ddots
\end{array}\right)
$$

pf Use Lem 9 (iii) $\square$

Lem 40

$$\mathcal{L} = [\mathcal{L}, \mathcal{L}] \quad + \quad Z(\mathcal{L})$$ (direct sum of ideals)

pf Recall

$\dim \mathcal{L} = 4$

$\dim [\mathcal{L}, \mathcal{L}] = 3$

$Z(\mathcal{L}) = \mathbb{C} \; \Phi$

Suf to show

$$\Phi \notin [\mathcal{L}, \mathcal{L}]$$

This follows from L 38 and since

$A, A^*, [A, A^*], \Phi$

are lin indep. $\square$

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Here is a characterization of $[\mathcal{L}, \mathcal{L}]$

Obs $e_1 V$ is a module for $\mathcal{L}$.

Action of $\mathcal{L}$ on $e_1 V$ induces a Lie algebra hom

$$\mathcal{L} \longrightarrow gl(e_1 V) \qquad (*)$$
$$y \to y|_{e_1 V}$$

LEM 41 With the above notation

$[\mathcal{L}, \mathcal{L}]$ is the kernel of the map $(*)$.

pf Recall $\mathcal{L}$ has basis

$$A, A^*, [A, A^*], \Phi$$

By LEM 33 the kernel of $(*)$ has basis

$$A - \Phi, \quad A^* - \Phi, \quad [A, A^*]$$

these vectors form a basis for $[\mathcal{L}, \mathcal{L}]$ $\square$

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LEM 42 The following are equal

(i) $[\mathcal{L}, \mathcal{L}]$

(ii) $[T, T]$

(iii) $\{ y \in T \,/\, tr(y/_W) = 0 \quad \forall$ irred Tmodules $W \}$

pf (i) $\subseteq$ (ii) ✓

(ii) $\subseteq$ (iii): Given $y, z \in T$ Given irred Tmodule W

$$tr\, [y,z]/_W = tr\left(y/_W\, z/_W - z/_W\, y/_W\right)$$

$$= tr\, y/_W\, z/_W - tr\, z/_W\, y/_W$$

$$= 0$$

(iii) $\subseteq$ (i): Given $y \in T$ that has trace 0 on all irred T-modules

Decompose V into direct sum of irred T-modules

y has trace 0 on each Tmodule in this sum

trace of y on V is sum of these traces, and is therefore 0

Now by Lem 24 $\quad y \in \mathcal{L}$.

Let W denote a nonprimary irred Tmodule

$y/_W$ has trace 0 and $\dim W = 1$ so $yW = 0$

$e_1 V$ is direct sum of nonprimary irred Tmodules so

$$y\, e_1 V = 0$$

Now $y \in [\mathcal{L}, \mathcal{L}]$ by Lem 41 $\square$

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We now give the action of $[\mathcal{L}, \mathcal{L}]$ on $e \circ V$

Relative the standard basis $\hat{x},\ \mathbb{1} - \hat{x}$

$$A - \Phi : \begin{pmatrix} \frac{2-n}{2} & n-1 \\ 1 & \frac{n-2}{2} \end{pmatrix}$$

$$A^* - \Phi : \begin{pmatrix} \frac{n}{2} & 0 \\ 0 & \frac{-n}{2} \end{pmatrix}$$

$$[A, A^*] : \begin{pmatrix} 0 & n(1-n) \\ n & 0 \end{pmatrix}$$

relative the split basis $\hat{x},\ \mathbb{1}$,

$$A - \Phi : \begin{pmatrix} \frac{-n}{2} & 0 \\ 1 & \frac{n}{2} \end{pmatrix}$$

$$A^* - \Phi : \begin{pmatrix} \frac{n}{2} & n \\ 0 & \frac{-n}{2} \end{pmatrix}$$

$$[A, A^*] : \begin{pmatrix} -n & -n^2 \\ n & n \end{pmatrix}$$

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Relative the dual standard basis $\mathbb{1}, n\hat{x} - \mathbb{1}$

$$A - \Phi \;:\; \begin{pmatrix} \frac{n}{2} & 0 \\ 0 & -\frac{n}{2} \end{pmatrix}$$

$$A^* - \Phi \;:\; \begin{pmatrix} \frac{2-n}{2} & n-1 \\ 1 & \frac{n-2}{2} \end{pmatrix}$$

$$[A, A^*] \;:\; \begin{pmatrix} 0 & n(n-1) \\ -n & 0 \end{pmatrix}$$

(ex)

Note $[\mathcal{L}, \mathcal{L}]$ is generated by

$$A - \Phi, \quad A^* - \Phi$$

pf $[\mathcal{L}, \mathcal{L}]$ has a basis

$$A - \Phi, \quad A^* - \Phi, \quad [A, A^*]$$

and

$$[A, A^*] = [A - \Phi, A^* - \Phi]$$

since Φ central.

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An orthogonal basis for $[\mathcal{L},\mathcal{L}]$

LEM 43 The following is a basis for $[\mathcal{L},\mathcal{L}]$ that is orthogonal wrt $\langle\, ,\rangle$:

$$R = \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & \\ 1 & \\ \vdots & 0 \\ 1 & \end{array}\right) \qquad L = \left(\begin{array}{c|cccc} 0 & 1 & 1 & \cdots & 1 \\ \hline 0 & & & 0 & \end{array}\right)$$

$$A^{*} - F = \left(\begin{array}{c|cccc} n-1 & & & 0 & \\ \hline & -1 & -1 & \cdots & \\ 0 & -1 & -1 & \cdots & \\ & \vdots & \vdots & \ddots & \end{array}\right)$$

pf By LEM 32 each of $R, L, A^{*}-F$ is in $[\mathcal{L},\mathcal{L}]$. These are lin indep and $\dim[\mathcal{L},\mathcal{L}] = 3$ so they form a basis for $[\mathcal{L},\mathcal{L}]$. One checks this basis is orthogonal wrt $\langle\, ,\rangle$. $\square$

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So far we have two bases for $[\mathcal{L}, \mathcal{L}]$:

$$A - \Phi, \quad A^* - \Phi, \quad [A, A^*]$$

$$R, \quad L, \quad A^* - F$$

We now write each basis in terms of the other one.

LEM 44

(i) $\quad A - \Phi = R + L + \frac{2-n}{2(n-1)}(A^* - F)$

(ii) $\quad A^* - \Phi = \frac{n}{2(n-1)}(A^* - F)$

(iii) $\quad [A, A^*] = nR - nL$

pf use Lem 26

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LEM 45

(i) $$R = \frac{n(A - \Phi) + (n-2)(A^* - \Phi) + [A, A^*]}{2n}$$

(ii) $$L = \frac{n(A - \Phi) + (n-2)(A^* - \Phi) - [A, A^*]}{2n}$$

(iii) $$A^* - F = \frac{2(n-1)}{n}(A^* - \Phi)$$

pf Use Lem 27

LEM 46 Rel standard basis $\hat{x}$, $\mathbb{1}-\hat{x}$

$$R: \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad L: \begin{pmatrix} 0 & n-1 \\ 0 & 0 \end{pmatrix}$$

$$A^* - F: \begin{pmatrix} n-1 & 0 \\ 0 & 1-n \end{pmatrix}$$

pf (ex)

Recall Lie algebra iso from LEM 31

$$\sigma: \mathcal{L} \to gl_2(\mathbb{C})$$

Restriction of σ to $[\mathcal{L}, \mathcal{L}]$ gives Lie algebra iso

$$[\mathcal{L}, \mathcal{L}] \to sl_2(\mathbb{C}) \qquad \text{*}$$

LEM 47 the iso * sends

$$R \to f$$
$$L \to (n-1)e$$
$$A^* - F \to (n-1)h$$

the inverse of * sends

$$f \to R$$
$$e \to \frac{L}{n-1}$$
$$h \to \frac{A^*-F}{n-1}$$

pf Use Lem 46 on LEM 31 $\square$

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LEM 47 We have

$[\,,\,]$	R	L	A^*-F
R	0	$F-A^*$	$2(n-1)R$
L	A^*-F	0	$2(1-n)L$
A^*-F	$2(1-n)R$	$2(n-1)L$	0

pf use L32 & L47. (ex) $\square$

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LEM 48

(i) $[\mathcal{L}, \mathcal{L}]$ is generated by R, L

(ii) $[R, [R, L]] = -2(n-1)R$

(iii) $[L, [L, R]] = -2(n-1)L$

pf (i) $A^* - F = [L, R]$ and

$R, L, A^* - F$ is a basis for $[\mathcal{L}, \mathcal{L}]$

(ii), (iii) Use Lem 47 (ex) $\square$

The basis $\tilde{R}, \tilde{L}, \tilde{H}$ for $[\mathcal{L}, \mathcal{L}]$

We now consider a basis for $[\mathcal{L}, \mathcal{L}]$ that is nice relative the split basis

Def 49 Put

$$\tilde{H} = \frac{nA - nA^* - [A, A^*]}{n(1-n)}$$

$$\tilde{R} = \frac{2-n}{2(1-n)}(A - I) + \frac{n}{2(n-1)}(A^* - I) + \frac{[A, A^*]}{2(n-1)}$$

$$\tilde{L} = \frac{2-n}{2(1-n)}(A^* - I) + \frac{n}{2(n-1)}(A - I) - \frac{[A, A^*]}{2(n-1)}$$

We obs $\tilde{H}, \tilde{R}, \tilde{L}$ are in $[\mathcal{L}, \mathcal{L}]$

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Recall the split basis $\hat{x}, \mathbb{1}$ for $e_0 V$

LEM 50 Relative the split basis $\hat{x}, \mathbb{1}$

$$\tilde{H} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{R} : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\tilde{L} : \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$$

$\square$

pf Use data below LEM 42 (ex)

LEM 51 The following is a basis for $[\mathcal{L},\mathcal{L}]$:

$$\tilde{H},\quad \tilde{R},\quad \tilde{L}$$

pf These elements are contained in $[\mathcal{L},\mathcal{L}]$ and lin indep by LEM 50. $\square$

LEM 52 We have

$$\tilde{H} = \left(\begin{array}{c|cccc} 1 & \frac{2}{1-n} & \frac{2}{1-n} & \cdots \\ \hline & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ 0 & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ & \vdots & \vdots & \ddots \end{array}\right)$$

$$\tilde{R} = \left(\begin{array}{c|cccc} 1 & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ \hline 1 & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ 1 & \frac{1}{1-n} & \frac{1}{1-n} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & & & \end{array}\right)$$

$$\tilde{L} = \left(\begin{array}{c|cccc} 0 & \frac{n}{n-1} & \frac{n}{n-1} & \cdots \\ \hline 0 & & 0 & \end{array}\right)$$

LEM 5.3 We have

$[\,,\,]$	$\tilde{R}$	$\tilde{L}$	$\tilde{H}$
$\tilde{R}$	0	$-n\tilde{H}$	$2\tilde{R}$
$\tilde{L}$	$n\tilde{H}$	0	$-2\tilde{L}$
$\tilde{H}$	$-2\tilde{R}$	$2\tilde{L}$	0

pf Use LSO ex. $\square$

LEM 54 We have

(i) $$A - \Phi = \tilde{R} - \frac{n}{2}\tilde{H}$$

(ii) $$A^* - \Phi = \tilde{L} + \frac{n}{2}\tilde{H}$$

(iii) $$[A, A^*] = n(\tilde{R} - \tilde{L} - \tilde{H})$$

pf Compare L50 with data below L42. (ex) $\square$

We now compare bases

$R, \; L, \; A^* - F$

$\tilde{R}, \; \tilde{L}, \; \tilde{H}$

LEM 55 We have

(i) $$R = -\frac{1}{n}\tilde{L} - \tilde{H} + \tilde{R}$$

(ii) $$L = \frac{n-1}{n}\tilde{L}$$

(iii) $$A^* - F = \frac{2(n-1)}{n}\tilde{L} + (n-1)\tilde{H}$$

pf use lem 45 and lem 54. (ex) $\square$

LEM 56 We have

(i) $$\tilde{R} = R + \frac{1}{1-n}L + \frac{A^* - F}{n-1}$$

(ii) $$\tilde{L} = \frac{n}{n-1}L$$

(iii) $$\tilde{H} = \frac{-2}{n-1}L + \frac{A^* - F}{n-1}$$

pf Backsolve using Lem 55. $\square$