The Lie algebra $\mathcal{L}$

We will continue our discussion of K_n after an aside about Lie algebras.

Def 11 By a <u>Lie algebra</u> we mean a vector space L over $\mathbb{C}$ together with a map

$$[\,,\,] : L \times L \to L \qquad \text{"Lie bracket"}$$

such that

(i) $[\,,\,]$ is $\mathbb{C}$-linear in each argument

(ii) $[x, x] = 0 \quad \forall x \in L$

(iii) $\forall x, y, z \in L$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

"Jacobi identity"

Ex. 12 Let n = pos integer

$$L = \mathrm{Mat}_{n\times n}(\mathbb{C})$$

$$[x, y] = xy - yx \qquad \forall x, y \in L$$

ord matrix mult

then L with $[\,,\,]$ give a Lie algebra called $\mathfrak{gl}_n(\mathbb{C})$

pf (ex)

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Ex 13 Let n = pos integer

$$L = \{ x \in \mathrm{Mat}_{n \times n}(\mathbb{C}) \ / \ \mathrm{trace}(x) = 0 \}$$

$$[x,y] = xy - yx \qquad \forall x,y \in L$$

then L with $[\ ,\ ]$ is Lie algebra called $\mathfrak{sl}_n(\mathbb{C})$

pf (ex)

Ex 14 $\mathfrak{sl}_2(\mathbb{C})$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = h$$

pf (ex)

Ex 15 Let V denote a finite-diml vector space over $\mathbb{C}$

L = $\mathbb{C}$-vector space of all $\mathbb{C}$-lin transformations $V \to V$

$$[x,y] = xy - yx \qquad \forall x,y \in L$$

composition of functions

then L with $[\ ,\ ]$ gives a Lie algebra denoted $\mathfrak{gl}(V)$

pf (ex)

5/26/10

Ex 16 Let V denote a finite-diml vector space over $\mathbb{C}$

L = $\mathbb{C}$-vector space of all $\mathbb{C}$-lin trans $V \to V$ that have trace 0.

$[x,y] = xy - yx \qquad \forall x,y \in L$

then L with $[\,,\,]$ gives Lie algebra denoted $sl(V)$

pf (ex)

Def 17 Given Lie algebras L and L'.

By a <u>homomorphism</u> (resp. <u>isomorphism</u>) of Lie algebras from L to L' we mean a $\mathbb{C}$-linear transformation (resp. $\mathbb{C}$-lin bijection) $\sigma : L \to L'$ such that

$$\sigma([x,y]) = [\sigma(x), \sigma(y)] \qquad \forall x,y \in L$$

Lem 18 Let V denote a finite-diml vector space over $\mathbb{C}$

Let $\{v_i\}_{i=1}^n$ denote a basis for V

(i) the map

$$gl(V) \longrightarrow gl_n(\mathbb{C})$$

$$x \longrightarrow \text{matrix rep } x \text{ rel } \{v_i\}_{i=1}^n$$

is an iso of Lie algebras.

(ii) the map

$$sl(V) \longrightarrow sl_n(\mathbb{C})$$

$$x \longrightarrow \text{matrix rep } x \text{ rel } \{v_i\}_{i=1}^n$$

is an iso of Lie algebras.

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Back to DRGs — Return to notation of prev. lecture
we continue to discuss K_n

Following ex 12 we see the vector space $\mathrm{Mat}_X(\mathbb{C})$

together with

$$[u,v] = uv - vu$$

is a Lie algebra, denoted $gl_X(\mathbb{C})$

Def 19 Let $\mathcal{L}$ denote the Lie subalgebra of $gl_X(\mathbb{C})$

generated by A, A^*.

[this means L is the minimal subspace of $gl_X(\mathbb{C})$
that contains A, A^* and is closed under $[\,,\,]$]

Next goal: show

$$A,\ A^*,\ [A, A^*],\ \Phi$$

is a basis for $\mathcal{L}$. In particular $\dim \mathcal{L} = 4$

Obs since $\Phi \in Z(T)$, the following hold in $gl_X(\mathbb{C})$:

$$[\Phi, A] = 0, \qquad [\Phi, A^*] = 0$$

LEM 20 the following relations hold in $gl_X(\mathbb{C})$

(i) $$[A,[A,A^*]] = n(n-2)(A-\Phi) + n^2(A^*-\Phi)$$

(ii) $$[A^*,[A^*,A]] = n(n-2)(A^*-\Phi) + n^2(A-\Phi)$$

pf (i) Show each side equals

$$-2n^2 E_0 + n^3 E_0 E_0^* + n^3 E_0^* E_0$$

LHS:

$$[A,A^*] = [nE_0 - I,\ nE_0^* - I]$$
$$= n^2[E_0, E_0^*]$$

$$[A,[A,A^*]] = [nE_0 - I,\ [E_0,E_0^*]]\,n^2$$
$$= [E_0,[E_0,E_0^*]]\,n^3$$
$$= \Big(E_0(E_0E_0^* - E_0^*E_0) - (E_0E_0^* - E_0^*E_0)E_0\Big)n^3$$
$$= \Big(E_0E_0^* - 2\underbrace{E_0E_0^*E_0}_{n^{-1}E_0} + E_0^*E_0\Big)n^3$$
$$= -2n^2E_0 + n^3E_0E_0^* + n^3E_0^*E_0 \quad \checkmark$$

To eval RHS use

$$A = nE_0 - I \qquad A^* = nE_0^* - I$$

obs

$$-n(n-2) - n^2 = -2n(n-1)$$

By L9,
$$-2n(n-1)\Phi = -n^3(E_0 + E_0^* - E_0E_0^* - E_0^*E_0) + 2n(n-1)I$$

RHS calc

term	coeff	I	E_0	E_0^*	$E_0E_0^*$	$E_0^*E_0$
A	$n(n-2)$	-1	n			
A^*	n^2	-1		n		
Ψ	$-n^3$		1	1	-1	-1
Ψ	$2n(n-1)$	1				
		0	$-2n^2$	0	n^3	n^3

pf of (i) is complete.

pf of (ii) is similar. $\square$

LEM 21 A, A^* satisfy

(i) $$[A, [A, [A, A^*]]] = n^2 [A, A^*]$$

(ii) $$[A^*, [A^*, [A^*, A]]] = n^2 [A^*, A]$$

"Dolan Grady relations"

pf (i) In the equation of Lem 20 (i) take the Lie bracket of each side with A. Obs

$$[A, A] = 0$$

Also $$[A, \Phi] = A\Phi - \Phi A = 0$$

since Φ is central

$\square$

(ii) Similar to (i)

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LEM 22 We have

(i) $$\Phi = \frac{n(n-2)A + n^2 A^* - [A,[A,A^*]]}{2n(n-1)}$$

(ii) $$\Phi = \frac{n(n-2)A^* + n^2 A - [A^*,[A^*,A]]}{2n(n-1)}$$

(iii) $\Phi \in \mathcal{L}$

pf (i), (ii) In LEM 20 solve for Φ $\square$

(iii) follows by (i) or (ii) and def of $\mathcal{L}$.

LEM 23

(i) the elements $A, A^*, [A, A^*], \Phi$ form a basis for the vector space $\mathcal{L}$.

(ii) $\dim \mathcal{L} = 4$.

pf (i) Each of $A, A^*, [A, A^*]$ is in $\mathcal{L}$ by the def of $\mathcal{L}$.

$\Phi \in \mathcal{L}$ by LEM 22.

the elements $A, A^*, [A, A^*], \Phi$ are lin indep by Cor 8 (ii) and since A, A^*, AA^*, A^*A are lin indep

the Span$(A, A^*, [A, A^*], \Phi)$ is closed under $[\,,\,]$ by LEM 20 and $[\Phi, A] = 0$, $[\Phi, A^*] = 0$

Result follows. $\square$

(ii) Clear from (i)

LEM 24

$$\mathcal{L} = \{ y \in T \mid \mathrm{trace}(y) = 0 \}$$

pf Let $\mathcal{L}' = $ RHS

dim $T = 5$ and $I \in T$, $I \notin \mathcal{L}'$ so

$$\dim \mathcal{L}' \le 4.$$

Each of

$$A, \quad A^*, \quad [A, A^*], \quad \Phi$$

is in T and has trace 0. These form a basis for $\mathcal{L}$

so $\mathcal{L} \subseteq \mathcal{L}'$.

But $\dim \mathcal{L} = 4$ so $\mathcal{L} = \mathcal{L}'$. $\square$

The elements R, L, F of $\mathcal{L}$

LEM 25 The following is a basis for $\mathcal{L}$ that is orthog with respect to $\langle , \rangle$

$$
R = \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & \\ 1 & \\ \vdots & 0 \\ 1 & \end{array}\right)
\qquad
L = \left(\begin{array}{c|cccc} 0 & 1 & 1 & \cdots & 1 \\ \hline 0 & & & 0 & \end{array}\right)
$$

$$
F = \left(\begin{array}{c|cccc} 0 & & & 0 & \\ \hline & 0 & 1 & 1 & \cdots \\ & 1 & 0 & 1 & \cdots \\ 0 & 1 & 1 & 0 & \\ & \vdots & \vdots & & \ddots \end{array}\right)
\qquad
A^* = \left(\begin{array}{c|cccc} n-1 & & & 0 & \\ \hline & -1 & & & 0 \\ & & -1 & & \\ 0 & & & -1 & \\ & 0 & & & \ddots \\ & & & & -1 \end{array}\right)
$$

pf R, L, F, A^* are lin indep, contained in T, and have trace 0

So they form a basis for $\mathcal{L}$.

One checks R, L, F, A^* are mutually orthog rel $\langle , \rangle$ $\square$

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So far we have two bases for $\mathcal{L}$:

$$A, A^*, [A, A^*], \Phi$$

$$R, L, F, A^*$$

We now write each basis in terms of the other one.

LEM 26

(i) $A = R + L + F$

(ii) $[A, A^*] = nR - nL$

(iii) $\Phi = \frac{n}{2(n-1)} F + \frac{n-2}{2(n-1)} A^*$

pf (i) Inspection

(ii) We gave $[A, A^*]$ in Lec 1

(iii) Use ~~Cor~~ LEM 9 (iii) $\square$

LEM 27

(i) $$F = \frac{2-n}{n} A^* + \frac{2(n-1)}{n} \Phi$$

(ii) $$R = \frac{nA + (n-2)A^* + [A, A^*] + 2(1-n)\Phi}{2n}$$

(iii) $$L = \frac{nA + (n-2)A^* - [A, A^*] + 2(1-n)\Phi}{2n}$$

pf (i) Use LEM 26 (iii)

(ii), (iii) By LEM 26 (i), (ii)

$$R + L = A - F$$

$$R - L = [A, A^*] n^{-1}$$

So

$$2R = A - F + [A, A^*] n^{-1}$$

$$2L = A - F - [A, A^*] n^{-1}$$

Result follows. $\square$

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the actions of $\mathcal{L}$ on e_0V, e_1V

Recall standard basis $\hat{x}$, $\mathbb{1}-\hat{x}$ for e_0V

LEM 28 Relative the st. basis $\hat{x}$, $\mathbb{1}-\hat{x}$

$$A : \begin{pmatrix} 0 & n-1 \\ 1 & n-2 \end{pmatrix} \qquad A^* : \begin{pmatrix} n-1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[A, A^*] : \begin{pmatrix} 0 & n(1-n) \\ n & 0 \end{pmatrix} \qquad \Phi : \begin{pmatrix} \frac{n-2}{2} & 0 \\ 0 & \frac{n-2}{2} \end{pmatrix}$$

$$R : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad L : \begin{pmatrix} 0 & n-1 \\ 0 & 0 \end{pmatrix}$$

$$F : \begin{pmatrix} 0 & 0 \\ 0 & n-2 \end{pmatrix}$$

pf A, A^* ✓

Use $[A, A^*] = AA^* - A^*A$

We saw Φ acts on e_0V as $\frac{n-2}{2} I$

To get the matrix representations of R, L, F use LEM 25(ex). $\square$

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LEM 29 Given $y \in \mathcal{L}$ such that $y\, e_0 V = 0$.

Then $y = 0$. "$\mathcal{L}$-module $e_0 V$ is faithful"

pf Recall

$$R, L, F, A^*$$

is a basis for $\mathcal{L}$.

In Lem 28 the 2×2 matrices representing

R, L, F, A^* are lin indep.

Result follows. $\square$

thm 30

(i) the map

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & gl(e_0 V) \\ B & \longrightarrow & B|_{e_0 V} \end{array}$$

↖ restriction of B to $e_0 V$

is an isomorphism of Lie algebras.

(ii) the Lie algebra $\mathcal{L}$ is isomorphic to $gl_2(\mathbb{C})$

pf (i) Call the map σ.

By cnstr σ is homomorphism of Lie algebras.

show σ is bijection.

Recall $\dim \mathcal{L} = 4$, $\dim gl(e_0 V) = 4$ since $e_0 V$ has dim 2.

Suf to show σ is injective. This follows from LEM 29.

(ii) By LEM 18 (i) and since $\dim e_0 V = 2$ $\square$

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We now give an explicit Lie algebra iso

$$\mathcal{L} \to gl_2(\mathbb{C})$$

obs $gl_2(\mathbb{C})$ has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{I+h}{2}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{I-h}{2}$$

Consider the map

$$\sigma : \begin{array}{ccc} \mathcal{L} & \longrightarrow & gl_2(\mathbb{C}) \\ B & \longrightarrow & \text{matrix rep } B \text{ rel st. basis } \hat{x},\ \mathbb{1} - \hat{x} \end{array}$$

σ is Lie alg iso by L 18 (i) and P 30 (i)

LEM 31 With above notation

σ sends

$$\begin{aligned} R &\to f \\ L &\to (n-1)e \\ F &\to \frac{n-2}{2}(I-h) \\ A^* &\to \frac{n}{2}h + \frac{n-2}{2} I \end{aligned}$$

σ^{-1} sends

$$\begin{aligned} e &\to \frac{L}{n-1} \\ f &\to R \\ h &\to \frac{A^*-F}{n-1} \\ I &\to \frac{n}{(n-1)(n-2)} F + \frac{1}{n-1} A^* \end{aligned}$$

pf Use the 2x2 matrix representations in L 28 (ex)

We can now easily compute the Lie bracket $[\cdot,\cdot]$ on the basis R, L, F, A^*

LEM 32 We have

$[\cdot,\cdot]$	R	L	F	A^*
R	0	$F-A^*$	$(2-n)R$	nR
L	A^*-F	0	$(n-2)L$	$-nL$
F	$(n-2)R$	$(2-n)L$	0	0
A^*	$-nR$	nL	0	0

pf For notational convenience identify $\mathcal{L}$ with $gl_2(\mathbb{C})$ via the map σ in LEM 31.

Obs
$$
\begin{aligned}
[R,L] &= [f,(n-1)e] \\
&= (n-1)[f,e] \\
&= (1-n)[e,f] \\
&= (1-n)\,h \\
&= (1-n)\,\frac{A^*-F}{n-1} \\
&= F-A^*
\end{aligned}
$$

other entries similarly found. (ex) $\square$

We now consider the action of $\mathcal{L}$ on $e_i V$ 5/26/10

LEM 33 We have

el in $\mathcal{L}$	action on $e_i V$
A	-1
A^*	-1
$[A, A^*]$	0
Φ	-1
R	0
L	0
F	-1

pf We saw this already for A, A^*, Φ

$[A, A^*]$: clear

R, L, F : Use LEM 27 (ex) $\square$

The center of $\mathcal{L}$

Aside on general Lie algebras

Given any Lie algebra L define

$$Z(L) = \{ x \in L \mid [x,y] = 0 \quad \forall y \in L \}$$

By an <u>ideal</u> in L we mean a subspace $K \subseteq L$ such that

$$[K, L] \subseteq K$$

Let K denote an ideal of L. Certainly

$$[K, K] \subseteq K$$

so K is a Lie subalgebra of L

Obs $\quad Z(L)$ is an ideal of L $\quad$ (ex)

Ex 34 $\quad$ For $L = gl_n(\mathbb{C})$

$$Z(L) = \mathrm{Span}(I)$$

In particular

$$\dim Z(L) = 1.$$

pf ex.

Back to DRGs

we continue discussing K_n

LEM 35

$$T = \mathcal{L} + \mathbb{C} I \qquad \text{(dir sum)}$$

pf Recall

$\mathcal{L} \subseteq T$

$\dim \mathcal{L} = 4$

$\dim T = 5$

$I \in T, \quad I \notin \mathcal{L}$

Result follows. $\square$

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LEM 36

(i) $\quad Z(T) = Z(\mathcal{L}) + \mathbb{C} I$

(ii) $\quad Z(\mathcal{L}) = Z(T) \cap \mathcal{L}.$

pf (i) follows from LEM 35 and since $I \in Z(T)$

(ii) follows from (i) $\square$

5/26/10

thm 37

$$Z(\mathcal{L}) = \mathbb{C}\,\Phi$$

pf We saw

$$Z(\mathcal{L}) = Z(T) \cap \mathcal{L}$$

$$Z(T) = \mathrm{Span}(\Phi, I)$$

$$\Phi \in \mathcal{L}, \qquad I \notin \mathcal{L}.$$

Result follows. $\square$

Note Here is another pf of thm 37:

the center of $gl_2(\mathbb{C})$ is spanned by the 2×2 identity matrix I.

Consider the Lie algebra iso

$$\sigma : \mathcal{L} \to gl_2(\mathbb{C})$$

from LEM 31.

Using LEM 28

$$\sigma(\Phi) = \frac{n-2}{2} I$$

therefore Φ is a basis for $Z(\mathcal{L})$.