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Lecture 1 Monday May 24

We will discuss Q-polynomial distance-regular graphs (DRG)

and their relation to tridiagonal pairs (TD pair)

Our focus is on examples such as Hamming graphs.

Recall Hamming graph $H(D, n)$ $(n \geq 2)$:

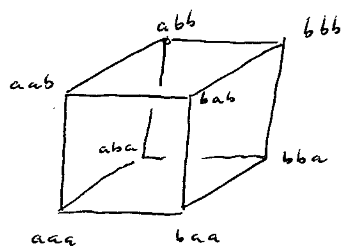
Fix a set S with $|S| = n$

Vertex set X = set of D-tuples of elements in S

Edge set $R = \{ xy \mid x, y \text{ differ in exactly 1 coord} \}$

Call $H(D, 2)$ the D-cube or hypercube

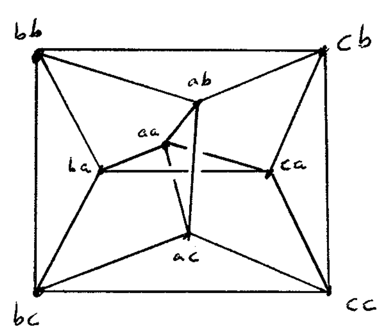
$H(3,2)$



$S = \{a, b\}$

$H(1, n)$ is complete graph K_n

$H(2,3)$



$S = \{a, b, c\}$

$H(D,n)$ is DRG with intersection numbers

(ex) $c_i = i$ $\quad b_i = (D-i)(n-1)$ $\quad 0 \le i \le D$

Adjacency matrix A has eigenvalues

$$\theta_i = (n-1)(D-i) - i$$

(ex)

$$= (n-1)D - ni \qquad 0 \le i \le D$$

θ_i has mult

(ex) $$m_i = \binom{D}{i}(n-1)^i \qquad 0 \le i \le D$$

Let

$$E_i = \text{primitive idempotent for } \theta_i \qquad 0 \le i \le D$$

Recall

$$E_i E_j = \delta_{ij} E_i \qquad 0 \le i,j \le D$$

$$I = \sum_{i=0}^{D} E_i$$

$$A = \sum_{i=0}^{D} \theta_i E_i$$

The subconstituent algebra T

For convenience we assume $n \geq 3$ from now on.

Until further notice fix $x \in X$

For $0 \leq i \leq D$ define a diagonal matrix $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad y \in X$$

$(\partial =$ path-length distance$)$

E_i^* is <u>ith dual idempotent rel x</u>

Obs

$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \leq i,j \leq D)$$

$$I = \sum_{i=0}^{D} E_i^*$$

Recall dual adjacency matrix

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$$

$\theta_i^* = \theta_i$ since $H(D,n)$ is self-dual

Subconstituent algebra $T = T(x)$ is subalgebra of $\mathrm{Mat}_X(\mathbb{C})$

gen by A, A^*

Helpful to start with case $D=1$ ie K_n

Recall basic facts:

$E_0 = n^{-1} J$ $\qquad$ J = all 1's matrix in $Mat_X(\mathbb{C})$

$$E_1 = I - E_0 = I - n^{-1} J \qquad A = J - I = (n-1)E_0 - E_1$$

A has all diag entries 0 so

$$tr(A) = 0$$

$$E_0^* = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \qquad E_1^* = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{matrix} \end{array}\right)$$

$$A^* = (n-1) E_0^* - E_1^*$$

$$= \left(\begin{array}{c|c} n-1 & 0 \\ \hline 0 & \begin{matrix} -1 & & 0 \\ & -1 & \\ 0 & & \ddots \; -1 \end{matrix} \end{array}\right)$$

Obs

$$tr(A^*) = 0$$

We have

$$n E_0^* E_0 E_0^* = E_0^*, \qquad n E_0 E_0^* E_0 = E_0 \qquad (ex)$$

Each of the following is a basis for T:

(i) $I, \; E_0, \; E_0^*, \; E_0 E_0^*, \; E_0^* E_0$ $\qquad (ex)$

(ii) $I, \; A, \; A^*, \; AA^*, \; A^*A$

Recall a Hermitean inner product on $\mathrm{Mat}_X(\mathbb{C})$:

$$\langle u, v \rangle = \mathrm{tr}(u \bar{v}^t) \qquad \forall u, v \in \mathrm{Mat}_X(\mathbb{C})$$

The following is a basis for T that is orthog rel $\langle , \rangle$:

$$E_0^* = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & O \end{array}\right) \qquad E_1^* = \left(\begin{array}{c|c} 0 & O \\ \hline O & \begin{matrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{matrix} \end{array}\right)$$

$$E_0^* J E_1^* = \left(\begin{array}{c|c} 0 & 1 \; 1 \cdots 1 \\ \hline 0 & O \end{array}\right) \qquad E_1^* J E_0^* = \left(\begin{array}{c|c} 0 & O \\ \hline \begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix} & O \end{array}\right)$$

$$E_1^* J E_1^* - E_1^* = \left(\begin{array}{c|c} 0 & O \\ \hline O & \begin{matrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \\ \vdots & \vdots & & 0 \end{matrix} \end{array}\right)$$

For Later use we mention

$$AA^* - A^*A = \left(\begin{array}{c|cccc} 0 & -n & -n & \cdots & -n \\ \hline n & & & & \\ n & & & & \\ \vdots & & & 0 & \\ n & & & & \end{array}\right)$$

$$E_0 = \frac{A + I}{n} \qquad E_0^* = \frac{A^* + I}{n}$$

The center of T

Recall: an element $u \in T$ called central whenever $uv = vu \quad \forall v \in T$

Define

$$Z(T) = \{u \in T \mid u \text{ central}\} \qquad \text{"center of } T\text{"}$$

$Z(T)$ is subalgebra of T

We now describe $Z(T)$

Define

$$e_0 = \left(\begin{array}{c|cccc} 1 & & & 0 & \\ \hline & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \\ 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \\ & \vdots & \vdots & \ddots & \end{array}\right)$$

Obs

$$\bar{e}_0 = e_0, \qquad e_0^t = e_0$$

$$e_0 = \frac{n}{n-1}\left(E_0 + E_0^* - E_0 E_0^* - E_0^* E_0\right) \qquad \text{(ex)}$$

$$e_0 \in T$$

$$e_0 E_0^* = E_0^* e_0 = E_0^*$$

$$e_0 E_0 = E_0 e_0 = E_0$$

so

$$e_0 \in Z(T)$$

Define $e_1 = 1 - e_0$ so

$$e_0 + e_1 = 1, \qquad e_1 \in Z(T)$$

Obs

$$e_0^2 = e_0 \qquad e_1^2 = e_1 \qquad \text{(ex)}$$

$$e_0 e_1 = 0 \qquad e_0 e_1 = 0$$

" e_0, e_1 are orthog idempotents "

Obs

$\langle\ \rangle$	e_0	e_1
e_0	2	0
e_1	0	$n-2$

(ex)

the $\mathbb{C}$-alg T is iso

$$Mat_{2\times 2}(\mathbb{C}) \oplus \mathbb{C}$$

An iso $\sigma : Mat_{2\times 2}(\mathbb{C}) \oplus \mathbb{C} \to T$ is

y	$\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&0\\1&0\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&0\\0&1\end{pmatrix}, 0\right\}$	$\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}, 1\right\}$
$\sigma(y)$	E_0^*	$\frac{E_0^* J E_1^*}{n-1}$	$\frac{E_1^* J E_0^*}{n-1}$	$\frac{E_1^* J E_1^*}{n-1}$	e_1

Image of $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right\}$ under σ is e_0

" " " $Mat_{2\times 2}(\mathbb{C}) \oplus 0$ " " " $e_0 T$

" " " $0 \oplus \mathbb{C}$ " " " $e_1 T$

Thm 1 The matrices e_0, e_1 form a basis for $Z(T)$

pf We saw $e_0, e_1 \in Z(T)$

e_0, e_1 lin indep since they are orthog idempotents

show $\dim Z(T) = 2$

Suf to show center of

$$Mat_{2\times 2}(\mathbb{C}) \oplus \mathbb{C} \qquad *$$

has dim 2.

Center of $Mat_{2\times 2}(\mathbb{C})$ is spanned by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ hence dim 1

Center of $\mathbb{C}$ is $\mathbb{C}$. dim 1

Center of $*$ has dim 2 $\square$

T-modules

Recall standard module $V = \mathbb{C}^X$ (col vectors)

Bil form on V:

$$\langle u, v \rangle = u^t \bar{v} \qquad u, v \in V$$

"Hermitean dot product"

Obs

$$\langle Bu, v \rangle = \langle u, \bar{B}^t v \rangle \qquad \forall B \in Mat_X(\mathbb{C}) \quad \forall u, v \in V$$

$\forall y \in X$ define

$$\hat{y} = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow y \text{ coord}$$

$\{ \hat{y} \mid y \in X \}$ is orthonormal basis for V

Obs

$$V = e_0 V + e_1 V \qquad \text{(orthog dir sum)}$$

$e_0 V$ is T-module since $\forall B \in T$

$$B e_0 V = e_0 B V \subseteq e_0 V$$

Sim $e_1 V$ is T-module.

It will turn out $e_0 V$ is irred but $e_1 V$ is not in gen.

obs

e_0 acts on $e_0 V$ as 1

e_0 ... $e_1 V$ as 0

e_1 ... $e_0 V$ as 0

e_1 ... $e_1 V$ as 1

Describe action of T on $e_0 V$

def

$$\mathbb{1} = \sum_{y \in X} \hat{y}$$ "all 1's vector in V"

LEM 2 Each of the following is a basis for $e_0 V$:

(i) $\hat{x}, \ \mathbb{1} - \hat{x}$ "standard basis"

(ii) $\hat{x}, \ \mathbb{1}$ "split basis"

(iii) $\mathbb{1}, \ n\hat{x} - \mathbb{1}$ "dual standard basis"

pf (i) $e_0 V = \mathrm{Span}\{ e_0 \hat{y} \ / \ y \in X \}$

$$e_0 \hat{x} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{1}{n-1} \end{array}\right) \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$$

$$= \hat{x}$$

For $y \neq x$

$$e_0 \hat{y} = \left(\begin{array}{c|ccc} 1 & & 0 & \\ \hline & \ddots & & ⋰ \\ 0 & & \frac{1}{n-1} & \\ & ⋰ & & \ddots \end{array}\right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

$$= \frac{1}{n-1} \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \frac{\mathbb{1} - \hat{x}}{n-1}$$

So

$$e_0 V = \operatorname{Span}\{\hat{x}, \mathbb{1} - \hat{x}\}$$

Obs $\hat{x}, \mathbb{1} - x$ lin indep hence basis ✓

(ii), (iii) clear from (i) $\square$

We recognize $e_0 V$ is the <u>primary</u> <u>module</u> for T

Relative the st. basis $\hat{x}, \mathbb{1} - \hat{x}$

$A : \begin{pmatrix} 0 & n-1 \\ 1 & n-2 \end{pmatrix}$ $\qquad A^* : \begin{pmatrix} n-1 & 0 \\ 0 & -1 \end{pmatrix}$

$E_0 : \begin{pmatrix} \frac{1}{n} & \frac{n-1}{n} \\ \frac{1}{n} & \frac{n-1}{n} \end{pmatrix}$ $\qquad E_0^* : \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$E_1 : \begin{pmatrix} \frac{n-1}{n} & \frac{1-n}{n} \\ \frac{-1}{n} & \frac{1}{n} \end{pmatrix}$ $\qquad E_1^* : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

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Rel split basis $\hat{x}$, $\mathbb{1}$,

A: $\begin{pmatrix} -1 & 0 \\ 1 & n-1 \end{pmatrix}$ A^*: $\begin{pmatrix} n-1 & n \\ 0 & -1 \end{pmatrix}$

E_0: $\begin{pmatrix} 0 & 0 \\ \frac{1}{n} & 1 \end{pmatrix}$ E_0^*: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

E_1: $\begin{pmatrix} 1 & 0 \\ -\frac{1}{n} & 0 \end{pmatrix}$ E_1^*: $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$

Rel dual st. basis $\mathbb{1}$, $n\hat{x} - \mathbb{1}$

A: $\begin{pmatrix} n-1 & 0 \\ 0 & -1 \end{pmatrix}$ A^*: $\begin{pmatrix} 0 & n-1 \\ 1 & n-2 \end{pmatrix}$

E_0: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ E_0^*: $\begin{pmatrix} \frac{1}{n} & \frac{n-1}{n} \\ \frac{1}{n} & \frac{n-1}{n} \end{pmatrix}$

E_1: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ E_1^*: $\begin{pmatrix} \frac{n-1}{n} & \frac{1-n}{n} \\ \frac{-1}{n} & \frac{1}{n} \end{pmatrix}$

Describe action of T on $e_1 V$

LEM 3 Both

$$(A+I)\, e_1 V = 0 \qquad\qquad (A^*+I)\, e_1 V = 0$$

Pf $A+I = n E_0$ and

$$E_0 e_1 = E_0 (1-e_d)$$
$$= E_0 - E_0$$
$$= 0$$

So $(A+I)\, e_1 V = 0$

Case of A^* is sim. $\square$

COR 4 Each 1-dml subspace of $e_1 V$ is an irred T-module. On this module each of $A+I$, A^*+I is 0

Pf by LEM 3 $\square$

LEM 5 We have

$$e_1 V = \mathrm{Span}\left\{ \hat{y} - \hat{z} \;/\; y, z \in X, \quad x, y, z \text{ mut dist} \right\}$$

Pf View $e_1 V = (e_0 V)^\perp$ $\dim e_1 V = n-2$

$\forall\, y, z \in X$ s.t x, y, z m. dist

$$\langle \hat{y} - \hat{z}, \hat{x} \rangle = 0$$

$$\langle \hat{y} - \hat{z}, \mathbb{1} \rangle = 0$$

so $\hat{y} - \hat{z} \in e_1 V$

Fix $y \in X$ $y \neq x$

$\exists$ $n-2$ verts $z \in X$ s.t x, y, z mut dist.

Obs

$$\{ \hat{y} - \hat{z} \;/\; z \neq x,\; z \neq y \}$$ lin indep

hence span $e_1 V$ $\square$

In summary we have

elt in T	action on e_1V
A	-1
E_0	0
E_1	1
A^*	-1
E_0^*	0
E_1^*	1
e_0	0
e_1	1

The central element Φ

LEM 6 Let W denote an irred T-module for K_n

Then the trace of A, A^* on W is given below

W	primary	non primary
trace of A on W	$n-2$	-1
trace of A^* on W	$n-2$	-1

pf (ex) $\square$

Def 7 Let

$$\Phi = \frac{n-2}{2} e_0 - e_1$$

$$= \frac{n}{2} e_0 - I$$

Obs $tr(\Phi) = \frac{n}{2} \cdot 2 - n = 0$

Φ acts on $e_0 V$ as $\frac{n-2}{2} I$

Φ ... $e_1 V$ as $-I$

So

trace of Φ on primary T-module is $n-2$

- - - nonprimary ... -1

COR 8 Let W denote an irred T-module for K_n

Then each of $A - \Phi$, $A^* - \Phi$ have trace 0 on W.

LEM 9

(i)
$$\Phi = \frac{n^2}{2(n-1)}\left(E_0 + E_0^* - E_0 E_0^* - E_0^* E_0\right) - I$$

(ii)
$$\Phi = \frac{(n-2)(A + A^*) - AA^* - A^*A}{2(n-1)}$$

(iii)
$$\Phi = \left(\begin{array}{c|cccc} \frac{n-2}{2} & & & 0 & \\ \hline & \frac{2-n}{2(n-1)} & \frac{n}{2(n-1)} & \cdots & \\ 0 & \frac{n}{2(n-1)} & \frac{2-n}{2(n-1)} & & \\ & \vdots & & \ddots & \end{array}\right)$$

pf (i) Use def of $\Phi = \frac{n}{2} e_0 - I$ and def of e_0

(ii) In (i) expand using $E_0 = \frac{A+I}{n}$ $\quad E_0^* = \frac{A^*+I}{n}$

(iii) Use $\Phi = \frac{n}{2} e_0 - I$ $\square$

LEM 10 the elements Φ, I form a basis for $Z(T)$.

pf Φ, I are lin indep els of $Z(T)$ $\square$

We remark that T has a basis

$$I,\ A,\ A^*,\ [A, A^*],\ \Phi.$$