Math 542 **Midterm Exam, Spring 2016**
Prof: Paul Terwilliger

Your Name (please print) SOLUTIONS

NO CALCULATORS/ELECTRONIC DEVICES ALLOWED.

MAKE SURE YOUR CELL PHONE IS OFF.

Problem	Value	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. Let R and S denote nonzero rings. Prove that the ring $R \times S$ is not a field.

Solution. Pick nonzero $r \in R$ and nonzero $s \in S$. The elements $(r, 0)$ and $(0, s)$ of $R \times S$ are nonzero, and their product is zero. So $R \times S$ is not an Integral Domain. A field is an Integral Domain, so $R \times S$ is not a field.

2. Prove that the polynomial ring $\mathbb{Z}[x, y]$ is not a Euclidean domain.

Solution. The ring $R = \mathbb{Z}[x, y]$ is a UFD since $\mathbb{Z}$ is a UFD. The ideal $I = Rx + Ry$ of R consists of all the polynomials in R that have zero constant term. Therefore $I \neq R$. The ideal I is not principal, since x, y are irreducible with GCD equal to 1. The ring R is not a PID, and therefore not a Euclidean Domain.

3. Find all the ordered pairs r, s of positive integers such that $r^2 + s^2 = 999$.

Solution. Suppose that r and s are positive integers such that $r^2 + s^2 = 999$. Each of r^2, s^2 is equal to 0 or 1 (mod 4). So $r^2 + s^2$ is equal to 0 or 1 or 2 (mod 4). But 999 is equal to 3 (mod 4). Therefore r, s do not exist.

4. Let $\mathbb{F}$ denote a field and consider the polynomial ring $R = \mathbb{F}[x, y]$. Consider the ideals $I = R(x - y^2)$ and $J = R(x^2 - y^2)$ in R. Prove that the quotient rings R/I and R/J are not isomorphic.

Solution. Note that R is a UFD. The polynomial $x - y^2$ is irreducible in R, so R/I is an Integral Domain. The polynomial $x^2 - y^2 = (x + y)(x - y)$ is not irreducible in R, so R/J is not an Integral Domain. Therefore R/I and R/J are not isomorphic.

5. Let $\mathbb{F}$ denote a field. Let R denote the set of polynomials in $\mathbb{F}[x]$ that have x-coefficient zero. Note that R is a subring of $\mathbb{F}[x]$. Prove that R is not a UFD.

Solution. The units of R are the nonzero elements of $\mathbb{F}$. Note that $x^6 = (x^2)^3$ and $x^6 = (x^3)^2$. The elements x^2 and x^3 are irreducible in R, since no element in R has degree 1. The result follows.

6. Prove that the polynomial $x^3 + nx + 2$ is irreducible in $\mathbb{Z}[x]$, provided that $n \neq 1, -3, -5$.

Solution. Assume the given polynomial is reducible in $\mathbb{Z}[x]$. Then there exist integers a, b, c such that $x^3 + nx + 2 = (x^2 + ax + b)(x + c)$. Comparing the two sides we obtain $a + c = 0$ and $bc = 2$ and $n = ac + b$. The possible solutions for (a, b, c) are

$$(-2, 1, 2), \qquad (-1, 2, 1), \qquad (2, -1, -2), \qquad (1, -2, -1).$$

For these four possible solutions the corresponding values of n are $-3, 1, -5, -3$ respectively. These values of n are not allowed, so we have a contradiction.

7. For the $\mathbb{Z}$-modules $M = \mathbb{Z}/7\mathbb{Z}$ and $N = \mathbb{Z}/6\mathbb{Z}$, find all the elements in $\mathrm{Hom}_{\mathbb{Z}}(M, N)$.

Solution. $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ contains the zero homomorphism and nothing else. We now give the reason. For all $a \in M$ we have $7a = 0$. For all $b \in N$ we have $6b = 0$. For $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$ we have

$$\varphi(a) = 1\varphi(a) = 7\varphi(a) = \varphi(7a) = \varphi(0) = 0.$$

Therefore $\varphi = 0$.

8. Let $n = 1000$. Find the order of the group of units for the ring $\mathbb{Z}/n\mathbb{Z}$.

Solution. The answer is 400. The number of units in the ring $\mathbb{Z}/n\mathbb{Z}$ is equal to the number of positive integers up to n that are relatively prime to n. This is $\phi(n)$ where ϕ is the Euler function. We have $\phi(n) = \phi(10^3) = \phi(2^3 5^3) = \phi(2^3)\phi(5^3)$. Also $\phi(2^3) = 2^2 = 4$ and $\phi(5^3) = 5^3 - 5^2 = 100$. So $\phi(n) = 400$.

9. Let R denote a ring with $1 \neq 0$. Let M denote an R-module, and let N denote an R-submodule of M. Define $J = \{r \in R | ra = 0 \text{ for all } a \in N\}$. Prove that J is a 2-sided ideal in R.

Solution. One checks that J contains 0 and is closed under addition. So J is a subgroup of the abelian group R. For $r \in R$ and $s \in J$ we have $(rs)N = r(sN) = r(0) = 0$. Therefore $rs \in J$. We have $(sr)N = s(rN) \subseteq s(N) = 0$. Therefore $sr \in J$. We have shown that J is a 2-sided ideal in R.

10. Let R denote a commutative ring with $1 \neq 0$. Let F denote a free R-module with finite rank. Prove that the R-modules $\mathrm{Hom}_R(F, R)$ and F are isomorphic.

Solution. Denote the rank by t and view $F = R \times R \times \cdots \times R$ (t copies). For $1 \leq i \leq t$ let e_i denote the element of F that has 1 in coordinate i and 0 in all other coordinates. Consider the map $\theta : \mathrm{Hom}_R(F, R) \to F$ that sends $\varphi \mapsto (\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_t))$. We have $\theta(\varphi + \phi) = \theta(\varphi) + \theta(\phi)$ for all $\varphi, \phi \in \mathrm{Hom}_R(F, R)$. We have $\theta(r\varphi) = r\theta(\varphi)$ for all $r \in R$ and $\varphi \in \mathrm{Hom}_R(F, R)$. Therefore θ is an R-module homomorphism. We check that θ is injective. Given $\varphi \in \mathrm{Hom}_R(F, R)$ such that $\theta(\varphi) = 0$, we show that $\varphi = 0$. By construction $\varphi(e_i) = 0$ for $1 \leq i \leq t$. The elements $e_1, \ldots, e_t$ generate the R-module F, so $\varphi(x) = 0$ for all $x \in F$. Therefore $\varphi = 0$. We have shown that θ is injective. We check that θ is surjective. For $a \in F$ write $(a_1, a_2, \ldots, a_t)$. There exists $\varphi \in \mathrm{Hom}_R(F, R)$ such that $\varphi(e_i) = a_i$ for $1 \leq i \leq t$. By construction $\theta(\varphi) = a$. We have shown that θ is surjective. By the above comments θ is an R-module isomorphism.