Lec 9 Monday Feb 10 2/10/14

3.1 Linear systems

We now shift topics from differential equations to linear algebra. Later we will apply linear algebra to differential equations.

Ex Find the solution set for

$$\left.\begin{aligned} 3x + 2y &= 9 \\ x - y &= 8 \end{aligned}\right\} \quad *$$

Here x, y are the "unknowns" or "variables"

A <u>solution</u> to $*$ is an ordered pair (x, y) of real numbers that makes each equation true.

The <u>solution set</u> is the set of all solutions.

The equations $*$ are <u>linear</u> in the variables x, y. Such an equation has the form

$$ax + by = c \qquad a, b, c \in \mathbb{R}$$

terms such as

$$x^2, \quad \sqrt{x}, \quad xy, \quad \frac{1}{x}$$

$$\tan x, \quad e^x \quad \text{etc}$$

or forbidden

Solve *:

obs

$$x = y + 8$$

So

$$3(y+8) + 2y = 9$$

$$5y + 24 = 9$$

$$5y = -15$$

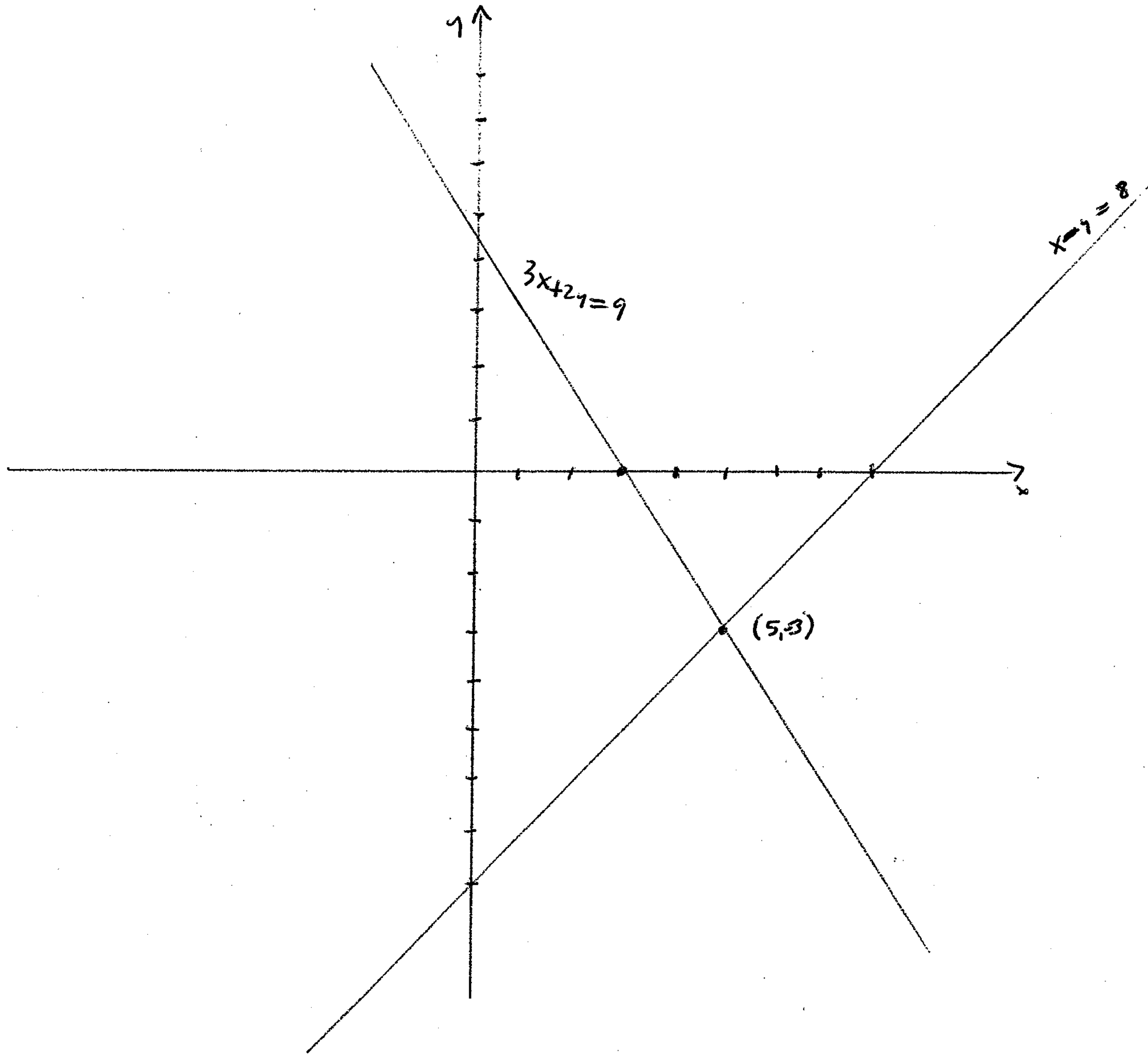
$$y = -3$$

$$x = -3 + 8$$

$$x = 5$$

$x = 5, \ y = -3$ is unique solution.

Geometric interp of *



Solution set for $3x+2y=9$ is a line

Solution set for $x-y=8$ is a line

Solution set for * is where the lines intersect.

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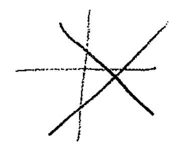
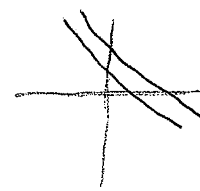
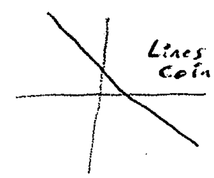
Given a system of 2 linear equations in x, y. There are 3 possibilities for the solution set

Case	example	graph
unique sol	$3x+2y=9$ $x-y=8$	
No sol	$x+y=1$ $2x+2y=3$	
∞ many sols	$x+y=1$ $2x+2y=2$	

Ex Consider the linear system in the variables x, y, z:

$$\left.\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}\right\} \quad *$$

Find the solution set.

<u>Strategy</u> We employ three types of moves to transform $*$ into a simpler linear system that has the same solution set.

(i) For some equation multiply each side by the same non 0 constant

(ii) Interchange two equations

(iii) Add a constant multiple of some equation to another equation

"elementary operations"

Using the elementary operations we try to put * in the form

$$\_\,x + \_\,y + \_\,z = \_$$
$$\_\,y + \_\,z = \_$$
$$\_\,z = \_$$

"triangular form"

We then "backsolve" to find z, then y, then x.

I interchange eqs one and two:

$$x + 3y + 2z = 2$$
$$2x + 7y + 3z = 11$$
$$3x + 7y + 9z = -12$$

II Replace eq 2 by eq 2 − 2 eq 1:

$$x + 3y + 2z = 2$$
$$y - z = 7$$
$$3x + 7y + 9z = -12$$

III Replace eq 3 by eq 3 − 3 eq 1:

$$x + 3y + 2z = 2$$
$$y - z = 7$$
$$-2y + 3z = -18$$

III Replace eq 3 by eq 3 + 2 eq 2:

$$x + 3y + 2z = 2$$
$$y - z = 7 \qquad \text{(triangular form)}$$
$$z = -4$$

Backsolve

$$z = -4$$

$$\begin{aligned} y &= 7 + z \\ &= 7 - 4 \\ &= 3 \end{aligned}$$

$$\begin{aligned} x &= 2 - 3y - 2z \\ &= 2 - 9 + 8 \\ &= 1 \end{aligned}$$

Unique sol to * is

$x = 1, \quad y = 3, \quad z = -4$

Double Check:

$$2 \cdot 1 + 7 \cdot 3 + 3(-4) = 11 \checkmark$$

$$1 \cdot 1 + 3 \cdot 3 + 2(-4) = 2 \checkmark$$

$$3 \cdot 1 + 7 \cdot 3 + 9(-4) = -12 \checkmark$$

Ex Find the solution set for the linear system

$$\left.\begin{aligned} x - 3y + 2z &= 6 \\ x + 4y - z &= 4 \\ 5x + 6y + z &= 20 \end{aligned}\right\} \ast$$

Sol Apply elem ops to put * in triangular form:

I Replace eq 2 by eq 2 - eq 1:

$$\begin{aligned} x - 3y + 2z &= 6 \\ 7y - 3z &= -2 \\ 5x + 6y + z &= 20 \end{aligned}$$

II Replace eq 3 by eq 3 - 5 eq 1:

$$\begin{aligned} x - 3y + 2z &= 6 \\ 7y - 3z &= -2 \\ 21y - 9z &= -10 \end{aligned}$$

III Replace eq 3 by eq 3 − 3 eq 2:

$$\begin{aligned} x - 3y + 2z &= 6 \\ 7y - 3z &= -2 \\ 0 &= -4 \end{aligned}$$

(tr form)

the Last equation shows

No sol (ie Solution set is empty)

Ex Find the solution set for the linear system

$$\left.\begin{aligned} x + y - z &= 5 \\ 3x + y + 3z &= 11 \\ 4x + y + 5z &= 14 \end{aligned}\right\} *$$

Sol Apply elem ops to put $*$ in triang form

I Replace eq 2 by eq 2 $- 3$ eq 1:

$$\begin{aligned} x + y - z &= 5 \\ -2y + 6z &= -4 \\ 4x + y + 5z &= 14 \end{aligned}$$

II Replace eq 3 by eq 3 $- 4$ eq 1:

$$\begin{aligned} x + y - z &= 5 \\ -2y + 6z &= -4 \\ -3y + 9z &= -6 \end{aligned}$$

$\underline{III}$ Replace eq 2 by $-\frac{1}{2}$ eq 2:

$$x + y - z = 5$$
$$y - 3z = 2$$
$$-3y + 9z = -6$$

$\underline{IV}$ Replace eq 3 by eq 3 + 3 eq 2:

$$x + y - z = 5$$
$$y - 3z = 2$$
$$0 = 0$$

(triangular form)

Backsolve:

no constraint on z, so write

$$z = t \qquad \text{free}$$

$$y = 2 + 3z$$
$$= 2 + 3t$$

$$x = 5 - y + z$$
$$= 5 - (2 + 3t) + t$$
$$= 3 - 2t$$

Sol set is

$x = 3-2t, \quad y = 2+3t, \quad z = t \quad t$ free

Double check:

$1(3-2t) + 1(2+3t) - t = 5$ ✓

$3(3-2t) + 1(2+3t) + 3t = 11$ ✓

$4(3-2t) + 1(2+3t) + 5t = 14$ ✓

Application of Linear systems to diff eqs

Consider function

$$y = e^{5x}$$

Obs

$$y' = 5e^{5x}$$
$$= 5y$$

So

$$y'' = 5y' = 25y$$

So $y = e^{5x}$ is a particular sol to

$$y'' - 25y = 0$$

Consider The function

$$y = e^{-5x}$$

Here

$$y' = -5e^{-5x}$$
$$= -5y$$

So

$$y'' = -5y' = 25y$$

So $y = e^{-5x}$ is another particular sol to

$$y'' - 25y = 0$$

As we will see, the gen solution to

$$y'' - 25y = 0$$

is

$$y = Ae^{5x} + Be^{-5x} \qquad A, B \text{ constants.}$$

Ex Solve the initial value problem

$$y'' - 25y = 0,$$

$$y(0) = 10, \qquad y'(0) = 20$$

Sol

$$y = Ae^{5x} + Be^{-5x} \qquad A, B = \text{cnst}$$

Find A, B.

obs

$$y' = 5Ae^{5x} - 5Be^{-5x}$$

Require

$$10 = y(0) = A + B$$

$$20 = y'(0) = 5A - 5B$$

Solve the linear system in variables A, B:

$$A + B = 10$$
$$5A - 5B = 20$$

$A = 7, \quad B = 3$ is unique sol

so

$$y = 7e^{5x} + 3e^{-5x}$$

$\square$