Lec 8 Friday Feb 7

2.4 Numerical Approximation: Euler's method

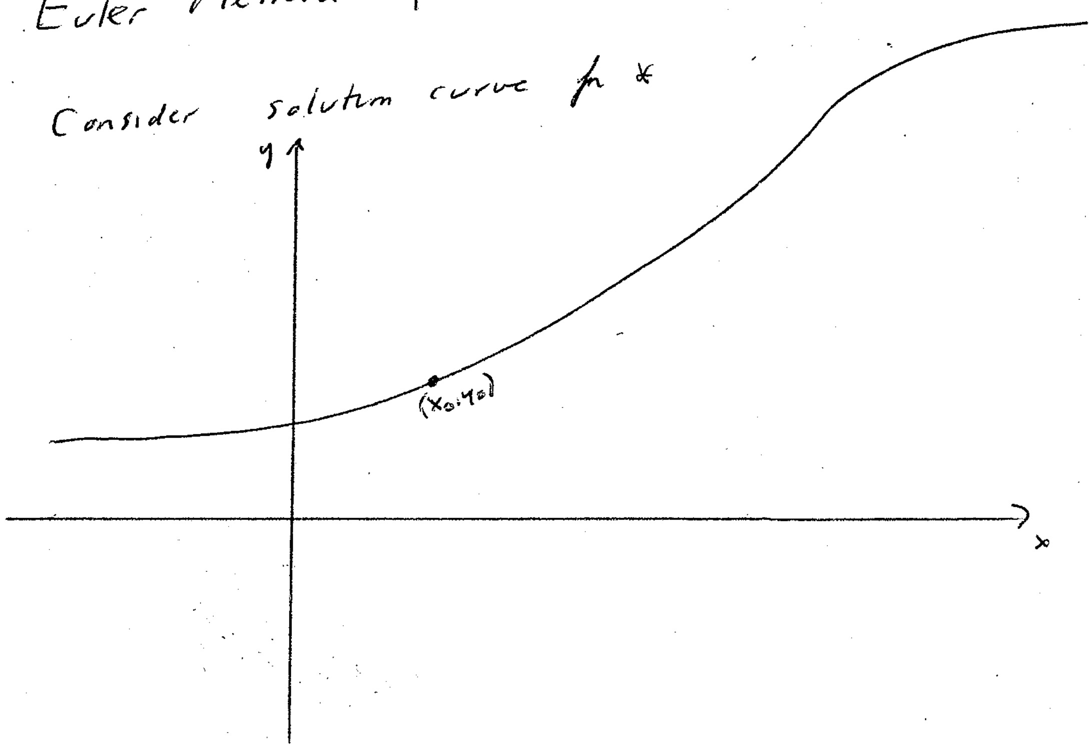
Consider a diff equation of form

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0 \qquad *$$

For some functions $f(x,y)$ we can't solve for y in closed form. In this case we approximate y using Euler method

Euler Method (Motivation)

Consider solution curve for $*$

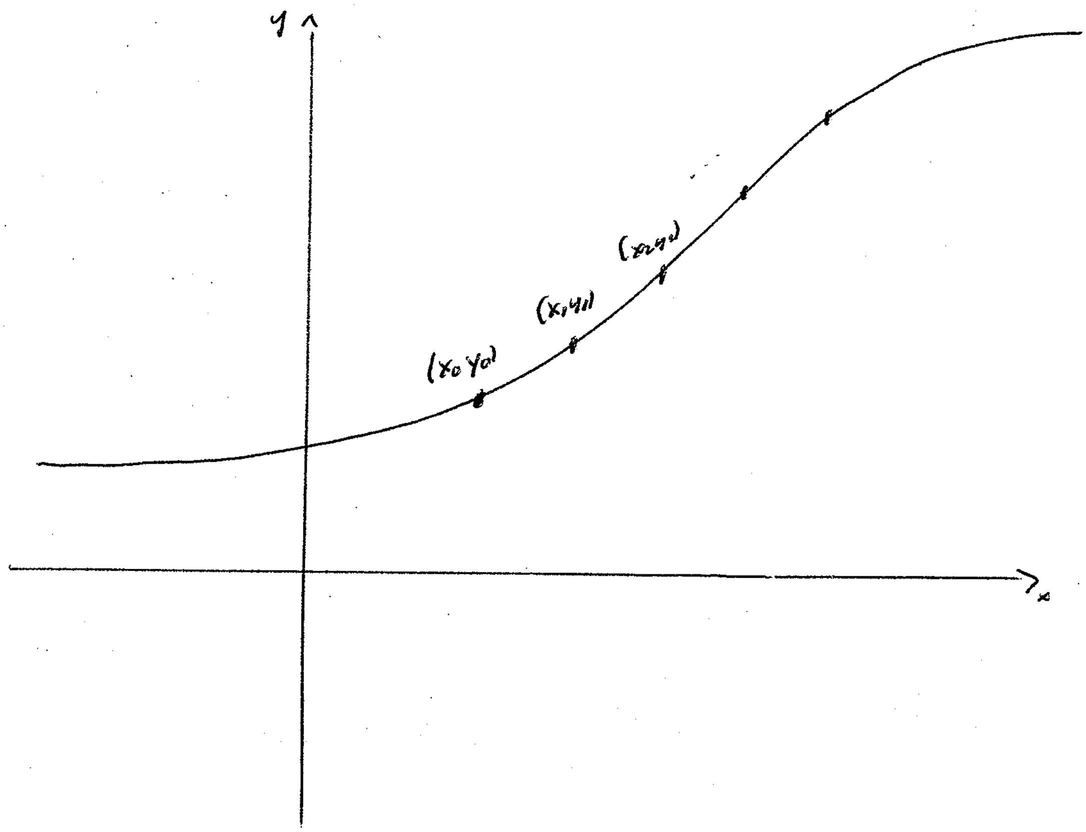


Pick a small positive "step size" h

Define
$$x_n = nh + x_0 \qquad n = 0, 1, 2 \ldots$$

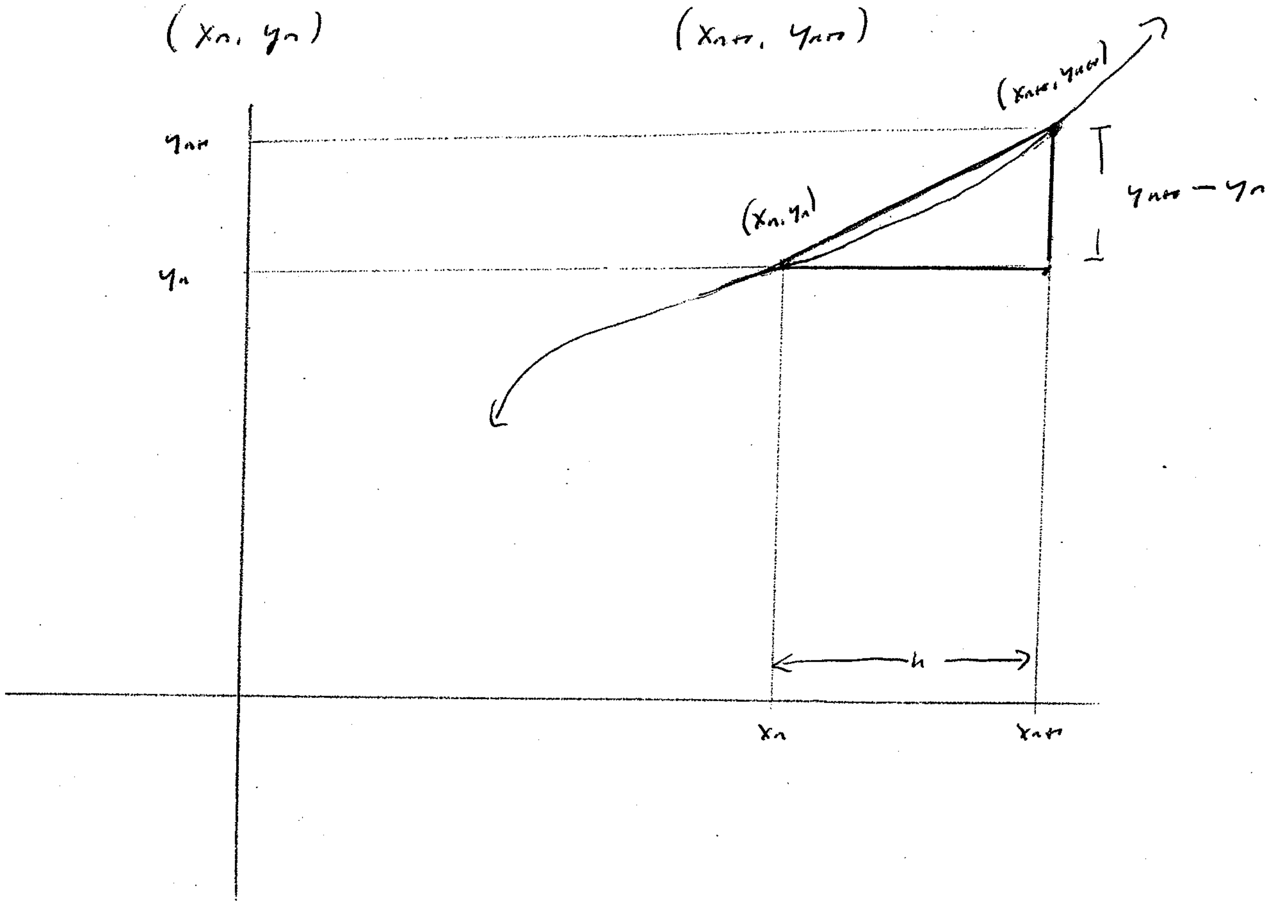
So
$$x_{n+1} - x_n = h \qquad n = 0, 1, 2 \ldots$$

Define
$$y_n = g(x_n) \qquad n = 0, 1, 2 \ldots$$



For $n = 0, 1, 2, \ldots$ compare pts

(x_n, y_n) (x_{n+1}, y_{n+1})



$$\frac{y_{n+1} - y_n}{h} = \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \text{slope of triangle}$$

$$\cong \frac{dy}{dx} \text{ (at } (x_n, y_n))$$

$$= f(x_n, y_n)$$

Solving for y_{n+1} we find

$$y_{n+1} = y_n + h f(x_n, y_n) \qquad n = 0, 1, 2, \ldots \quad ☆$$

Euler Method: Using ☆ we recursively find y_n for all $n = 0, 1, 2, \ldots$

Each y_n is approximately $y(x_n)$

Ex Use Euler Method to approx

the solution to

$$\frac{dy}{dx} = \underset{f(x,y)}{\underset{\shortparallel}{y}} \qquad y(0) = 1$$

[Of course exact sol is $y = e^x$]

Sol Pick small pos step size h

(In gen the smaller the h, the better the approx)

Define $x_n = nh$ $\qquad n = 0, 1, 2, \ldots$

Recursively define $y_0, y_1, y_2, \ldots$ such that

$$y_0 = 1$$

and

$$y_{n+1} = y_n + h\underbrace{f(x_n, y_n)}_{y_n} \qquad n = 0, 1, 2, \ldots$$

$$y_{n+1} = (1+h)\, y_n \qquad n = 0, 1, 2, \ldots$$

n	y_n
0	1
1	$1+h$
2	$(1+h)^2$
3	$(1+h)^3$
$\vdots$	$\vdots$

$$y_n = (1+h)^n \qquad n = 0, 1, 2, \ldots$$

By construction

$$e^{x_n} \overset{\text{approx}}{\approx} y_n \qquad n = 0, 1, 2, \ldots$$

So

$$e^{hn} \approx (1+h)^n \qquad n = 0, 1, 2, \ldots$$

Ex Estimate e

Sol. Pick large integer N

Pick step size $h = \frac{1}{N}$

So $hN = 1$

$$e = e^1$$
$$= e^{hN}$$
$$\simeq (1+h)^N$$
$$= \left(1 + \frac{1}{N}\right)^N$$

As N grows.

$$\left(1 + \frac{1}{N}\right)^N$$

becomes a better and better approx for e.

In other words

$$e = \lim_{N \to \infty} \left(1 + \frac{1}{N}\right)^N$$

Similarly

$$e^x = \lim_{N \to \infty} \left(1 + \frac{x}{N}\right)^N$$

Ex Use Euler method to approx the solution to

$$\frac{dy}{dx} = \underbrace{x - y}_{f(x,y)} \qquad y(0) = 1$$

Sol Pick small pos step size h

Define $x_n = nh$ $\qquad n = 0, 1, 2, \ldots$

Recursively def $y_0, y_1, \ldots$ by

$$y_0 = 1$$

and

$$y_{n+1} = y_n + h \underbrace{f(x_n, y_n)}_{= x_n - y_n} \qquad n = 0, 1, 2, \ldots$$

So

$$y_{n+1} = y_n + h(hn - y_n) \qquad n = 0, 1, 2, \ldots$$

$$y_{n+1} = (1-h) y_n + h^2 n \qquad n = 0, 1, 2, \ldots$$

Computing $y_0, y_1, y_2 \ldots$

one checks by induction that

$$y_n = 2(1-h)^n + hn - 1 \qquad n = 0, 1, 2, \ldots$$

this y_n is approximately $y(x_n)$

Ex For the equation

$$\frac{dy}{dx} = x - y \qquad y(0) = 2$$

estimate $y(3)$

Sol Pick large integer N

Pick step size $h = \frac{3}{N}$

Define

$$x_n = nh \qquad n = 0, 1, \ldots, N$$

Obs

$$x_N = Nh = N\frac{3}{N} = 3$$

So $y(3) = y(x_N) \approx y_N = 2(1-h)^N + hN - 1$

$$= 2\left(1 - \frac{3}{N}\right)^N + \frac{3}{N}N - 1$$

$$= 2\left(1 - \frac{3}{N}\right)^N + 2$$

our estimate for $y(3)$

Expect

$$y(3) = \lim_{N\to\infty}\left(2\left(1-\frac{3}{N}\right)^N + 2\right)$$

$$= 2e^{-3} + 2$$

For completeness we now solve in closed form

$$\frac{dy}{dx} = x - y \qquad y(0) = 1$$

Obs

$$\frac{dy}{dx} + y = x$$

Has form

$$\frac{dy}{dx} + \underset{\substack{\shortparallel \\ 1}}{P(x)}\, y = \underset{\substack{\shortparallel \\ x}}{Q(x)}$$

$$\int 1\, dx = x$$

$$e^x\left(\frac{dy}{dx} + y\right) = e^x x$$

$$\frac{d}{dx}\left(e^x y\right) = e^x x$$

$$e^x y = \int e^x x\, dx$$

$$= (x-1)e^x + C$$

Find C using $y(0)=1$

$$e^0 1 = (0-1)e^0 + C$$

$$1 = -1 + C$$

$$C = 2$$

$$e^x y = 2 + (x-1)e^x$$

$$y = 2e^{-x} + x-1$$

$\square$

(compare this with Euler approx)

Return to $y = e^x$

Ex We put \$100 in a bank act paying an interest rate r, compounded continuously. [so $r = .06$ for 6% interest]

Define
$$A(t) = \# \text{ dollars in act after } t \text{ years}$$

We saw earlier
$$A(t) = 100\, e^{rt}$$

Lets estimate $A(t)$ by assuming that the compounding intervals are finite.

Compounding interval	amt in acct after t years
yearly	$100(1+r)^t$
$\frac{1}{2}$ year	$100(1+\frac{r}{2})^{2t}$
quarterly	$100(1+\frac{r}{4})^{4t}$
$\frac{1}{N}$ year	$100(1+\frac{r}{N})^{Nt}$
$\vdots$	$\vdots$
continuous	$100 \lim_{N\to\infty}(1+\frac{r}{N})^{Nt}$

$$= 100\Big(\underbrace{\lim_{N\to\infty}(1+\tfrac{r}{N})^{N}}_{e^r}\Big)^t$$

$$= 100\, e^{rt}$$

Finite compounding gives "Euler estimate" for e^{rt}

$\square$