Lec 7 Wednesday Feb 5 2/5/14

2.2 Equilibrium Solutions + Stability

Consider a diff. equation of form

$$\frac{dx}{dt} = f(x) \qquad f(x) \text{ is any polynomial in } x$$

For each solution $x(t)$ we consider

$$\lim_{t\to\infty} x(t) \quad \text{and} \quad \lim_{t\to -\infty} x(t)$$

Ex Describe the sols to

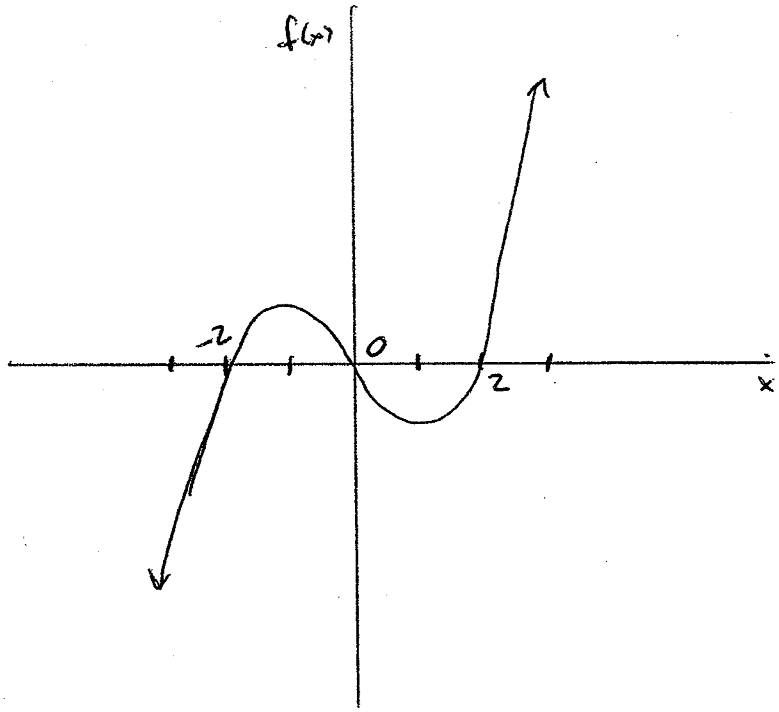
$$\frac{dx}{dt} = x^3 - 4x$$

Sol graph the function

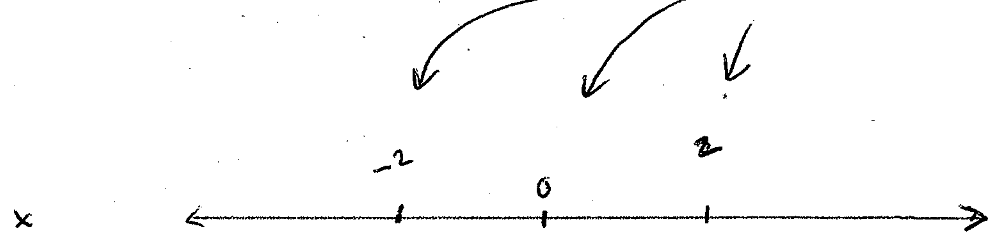
$$f(x) = x^3 - 4x$$

$$f(x) = x(x^2-4)$$

$$= x(x-2)(x+2)$$

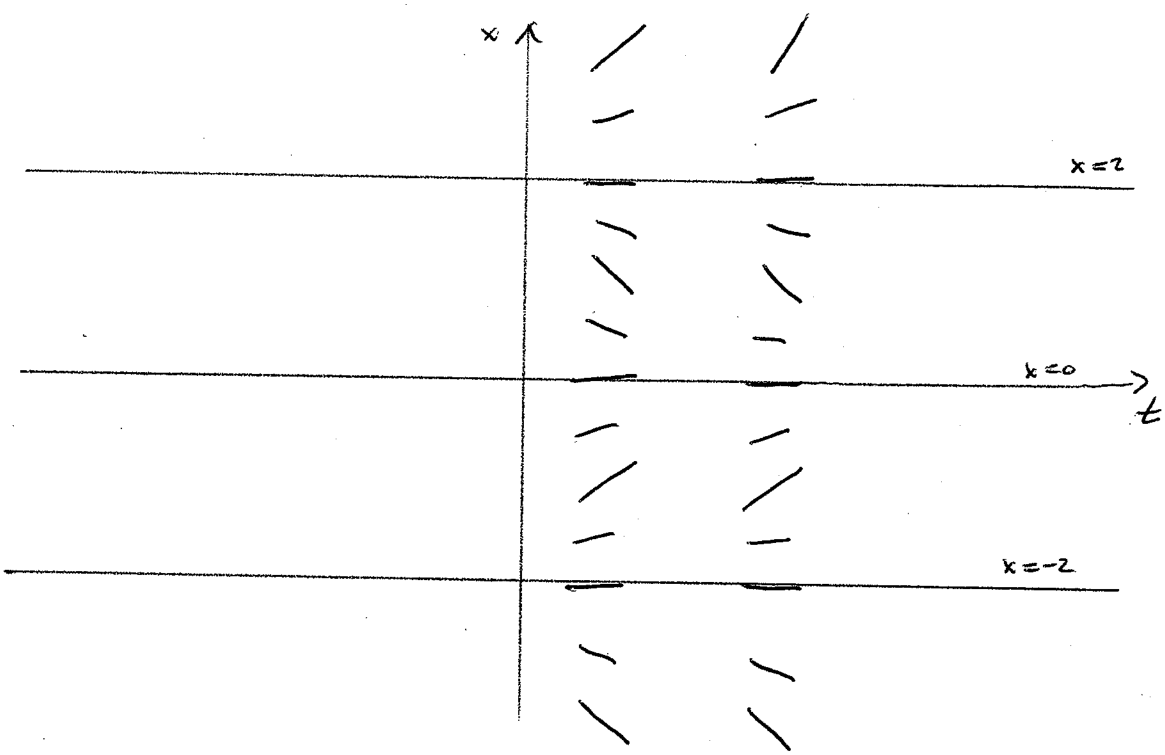


"critical points"



$\frac{dx}{dt}$ neg pos neg pos

sketch slope field:



Constant functions

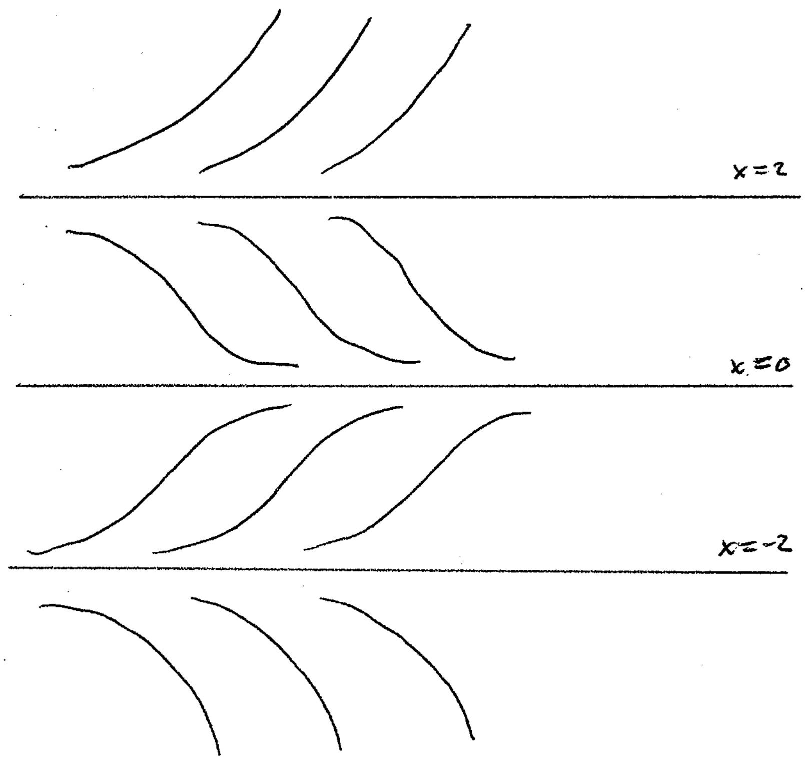
$x = -2$, $x = 0$, $x = 2$

are particular sols

"equilibrium solutions"

Remaining sols look like:

As $t \to \infty$ solutions $x(t)$ tend towards



∞

"unstable"

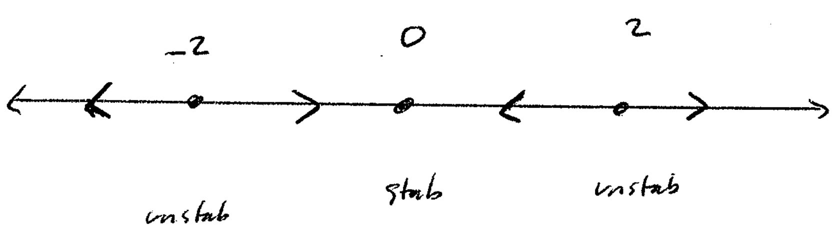
0

"stable"

0

"unstable"

$-\infty$



"phase diagram"

An equilibrium sol $x = a$ is <u>stable</u> whenever for nearby sols $x(t)$

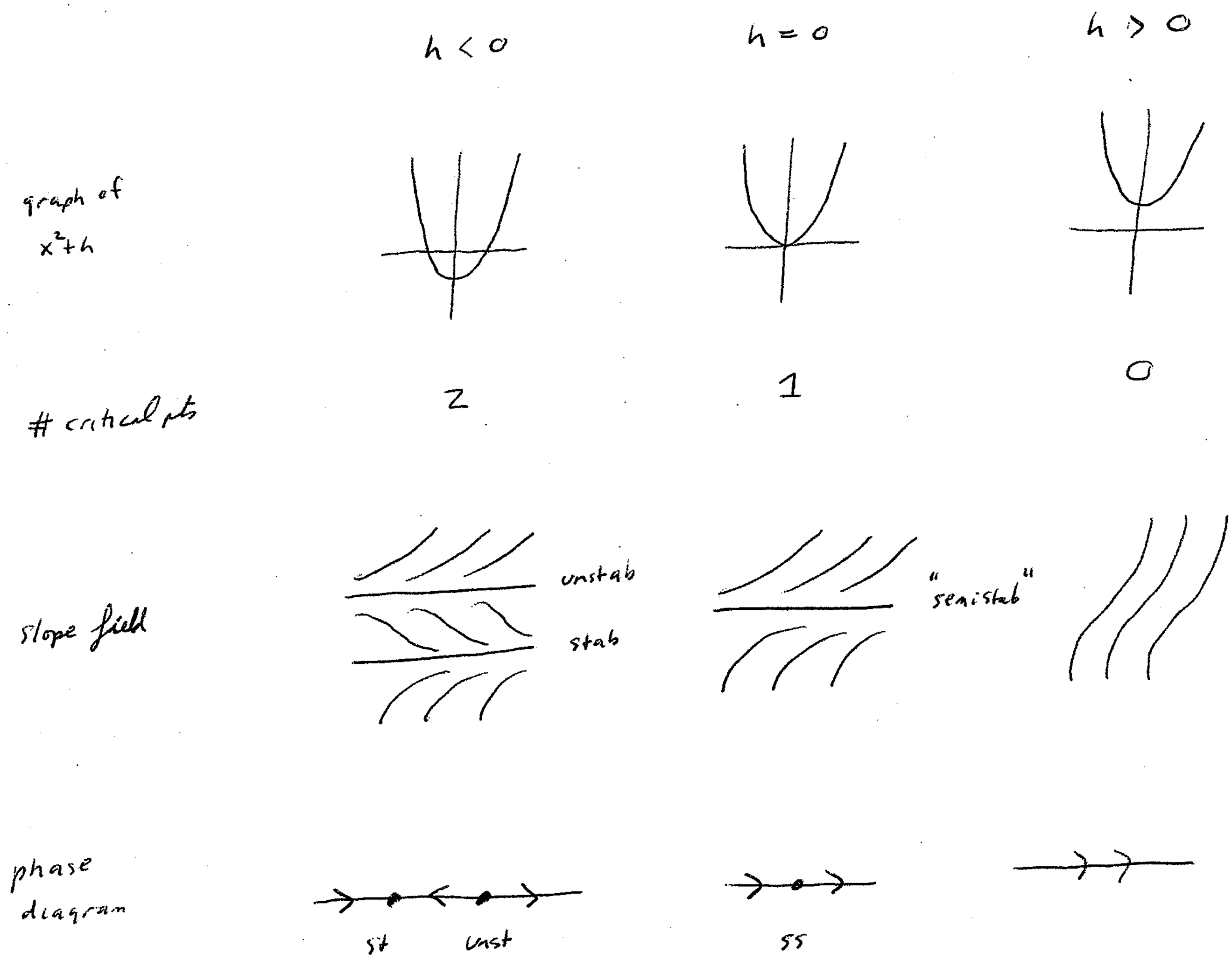
$$\lim_{t \to \infty} x(t) = a$$

Ex For the diff eq

$$\frac{dx}{dt} = x^2 + h \qquad h = \text{const}$$

How do solutions depend on h?

Sol Compare



$h = 0$ is called a <u>bifurcation point</u> for the parameter h. This where qualitative nature of solutions change.

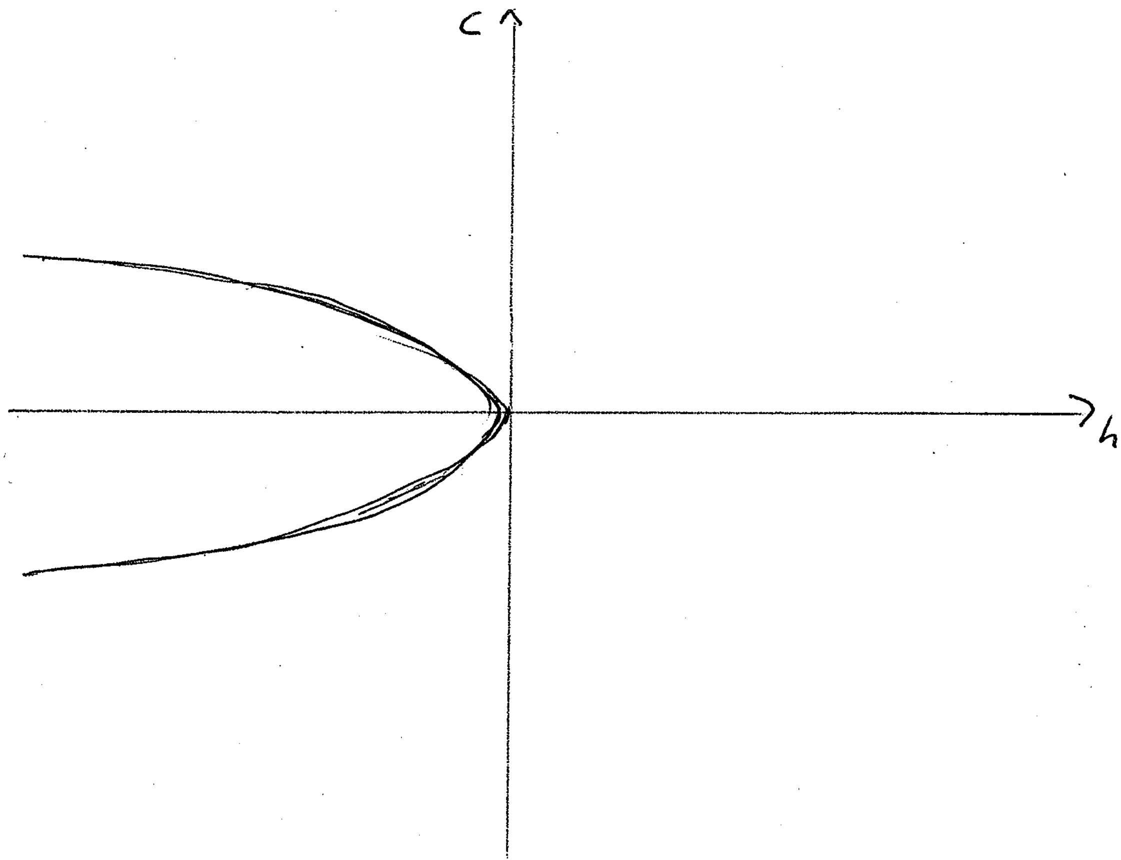
Ex. Draw a graph that shows all the points (h, c) where $h \in \mathbb{R}$ and c is a critical point for the equation

$$\frac{dx}{dt} = x^2 + h$$

Sol. For each h the critical pts are the solutions to

$$x^2 + h = 0$$

h	crit pts c
-4	$2, -2$
-3	$\sqrt{3}, -\sqrt{3}$
-2	$\sqrt{2}, -\sqrt{2}$
-1	$1, -1$
0	0
1	none
2	$\vdots$
3	



" bifurcation diagram "

is graph of

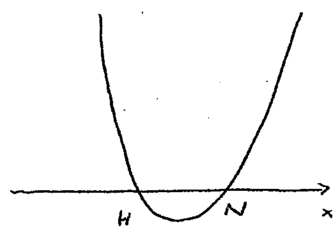
$$c^2 + h = 0$$

Ex Find gen sol to

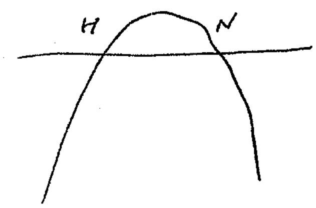
$$\frac{dx}{dt} = k(x-H)(x-N) \qquad k, H, N \in \mathbb{R}$$
$$k \neq 0, \quad H < N$$

Sol

sketch graph of $f(x) = k(x-H)(x-N)$

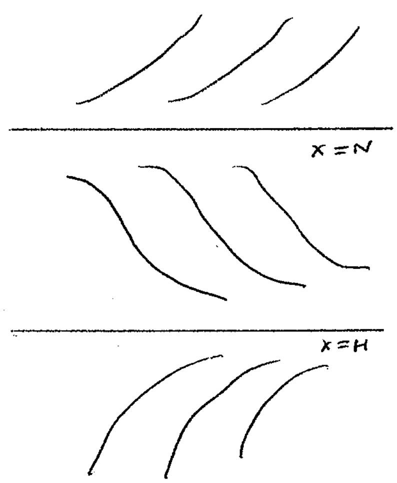


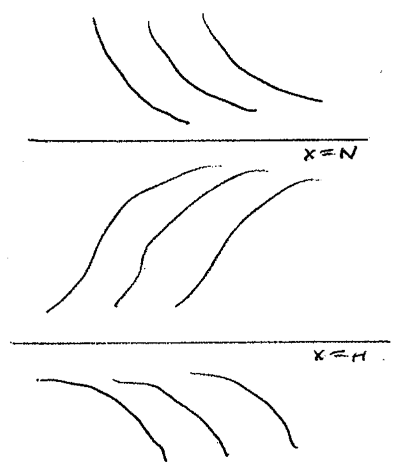
$k > 0$



$k < 0$

slope field





Solve using separation of variables

$$\frac{dx}{dt} = k(x-H)(x-N)$$

$$\frac{dx}{(x-H)(x-N)} = k\,dt$$

$$\int \frac{dx}{(x-H)(x-N)} = \int k\,dt \qquad *$$

$$= kt + \text{const}$$

Using partial fraction decomp

$$\frac{1}{(x-H)(x-N)} = \frac{1}{N-H}\left(\frac{1}{x-N} - \frac{1}{x-H}\right)$$

$*$ gives

$$\frac{1}{N-H}\left(\ln|x-N| - \ln|x-H|\right) = kt + \text{const}$$

$$\ln|x-N| - \ln|x-H| = (N-H)kt + \text{const}$$

exponentiate :

$$\frac{x-N}{x-H} = C e^{(N-H)kt} \qquad C = \text{const} \qquad **$$

Write $\qquad x_0 = x(0)$

Set $t=0$ to get

$$\frac{x_0 - N}{x_0 - H} = C$$

Solve $**$ for x:

$$x - N = (x-H) C e^{(N-H)kt}$$

$$x\left(1 - C e^{(N-H)kt}\right) = N - H C e^{(N-H)kt}$$

$$x = \frac{N - H C e^{(N-H)kt}}{1 - C e^{(N-H)kt}}$$

Eliminate C:

$$X = \frac{N - H \dfrac{X_0 - N}{X_0 - H} e^{k(N-H) \cdot t}}{1 - \dfrac{X_0 - N}{X_0 - H} e^{k(N-H) \cdot t}}$$

$$x(t) = \frac{N(X_0 - H) + H(N - X_0) e^{k(N-H) \cdot t}}{X_0 - H + (N - X_0) e^{k(N-H) \cdot t}}$$

x_0 free

(Gen Sol)

Other forms

Ex Find gen sol to

$$\frac{dx}{dt} = k(x-H)(N-x)$$

$k, H, N \in \mathbb{R}$

$k \neq 0, \ H < N$

Sol this is prev example with k replaced by $-k$

Gen Sol is

$$x(t) = \frac{N(x_0 - H) + H(N - x_0)\, e^{-k(N-H)t}}{x_0 - H + (N - x_0)\, e^{-k(N-H)t}}$$

x_0 free

Ex Find gen sol to Logistic eq

$$\frac{dx}{dt} = k x (M - x) \qquad k, M > 0$$

Sol this prev example with $H = 0$
and $N = M$

Gen sol is

$$x(t) = \frac{M x_0}{x_0 + (M - x_0) e^{-kMt}}$$

x_0 free

Ex Find gen solution to doomsday/extinction equation

$$\frac{dx}{dt} = k\,x\,(x-M) \qquad k,\ M > 0$$

Sol this is the eq

$$\frac{dx}{dt} = k\,(x-H)(x-N)$$

with $H = 0, \quad N = M$

Gen sol is

$$x(t) = \frac{M x_0}{x_0 + (M - x_0)\,e^{kMt}}$$

x_0 free

□