Lec 6 Monday Feb 3 2/3/14

2.1 Population Models

In this section we consider diff equations of the form

$$\frac{dx}{dt} = ax - bx^2$$

"Logistic equation" if $a,b>0$

"doomsday / extinction equation" if $a,b<0$

where a,b are constants

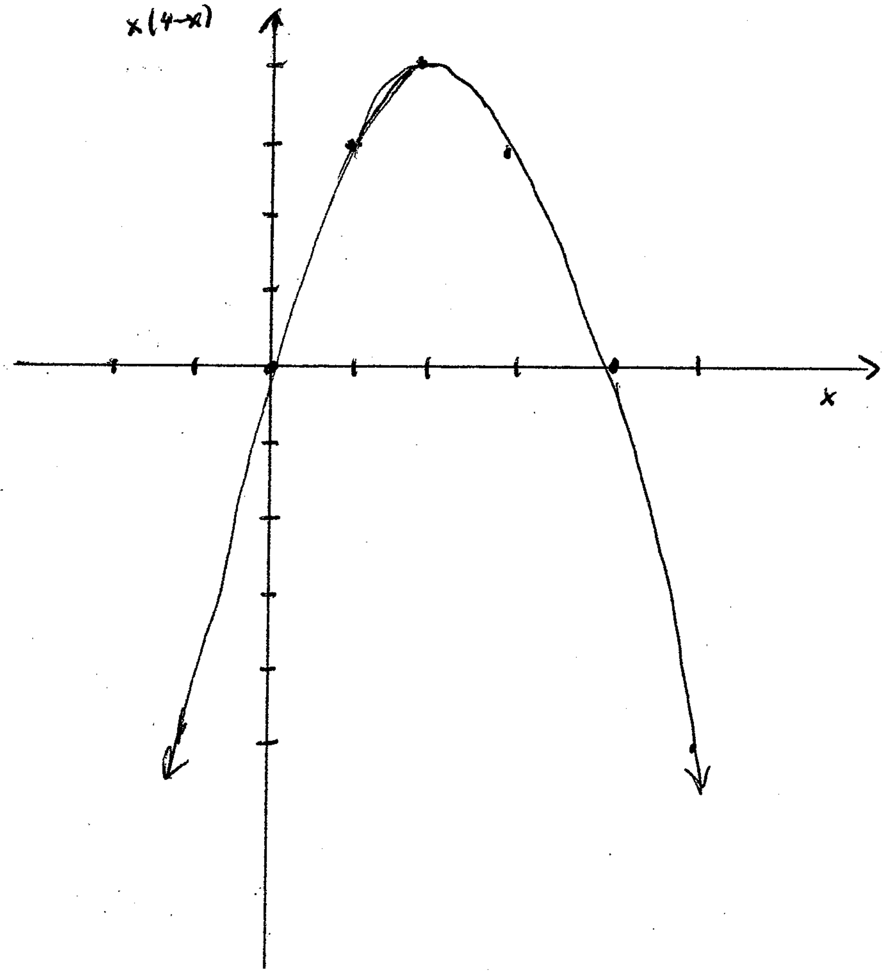
Ex Describe the solutions to

$$\frac{dx}{dt} = 4x - x^2 \qquad \text{(Logistic)}$$

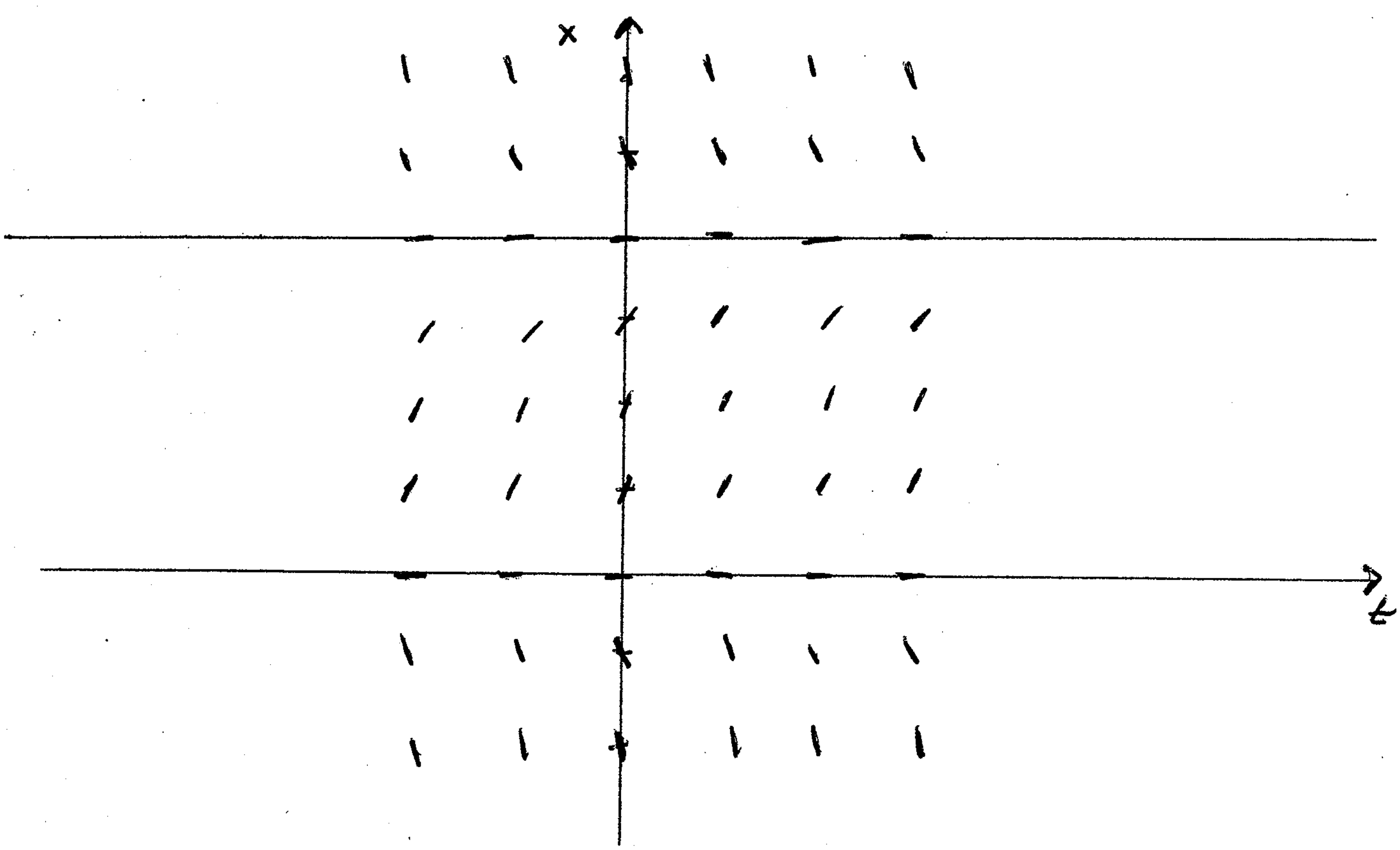
Sol First lets describe slope field

$$4x - x^2 = x(4-x)$$

x	$x(4-x)$
-2	-12
-1	-5
0	0
1	3
2	4
3	3
4	0
5	-5
6	-12



slope field

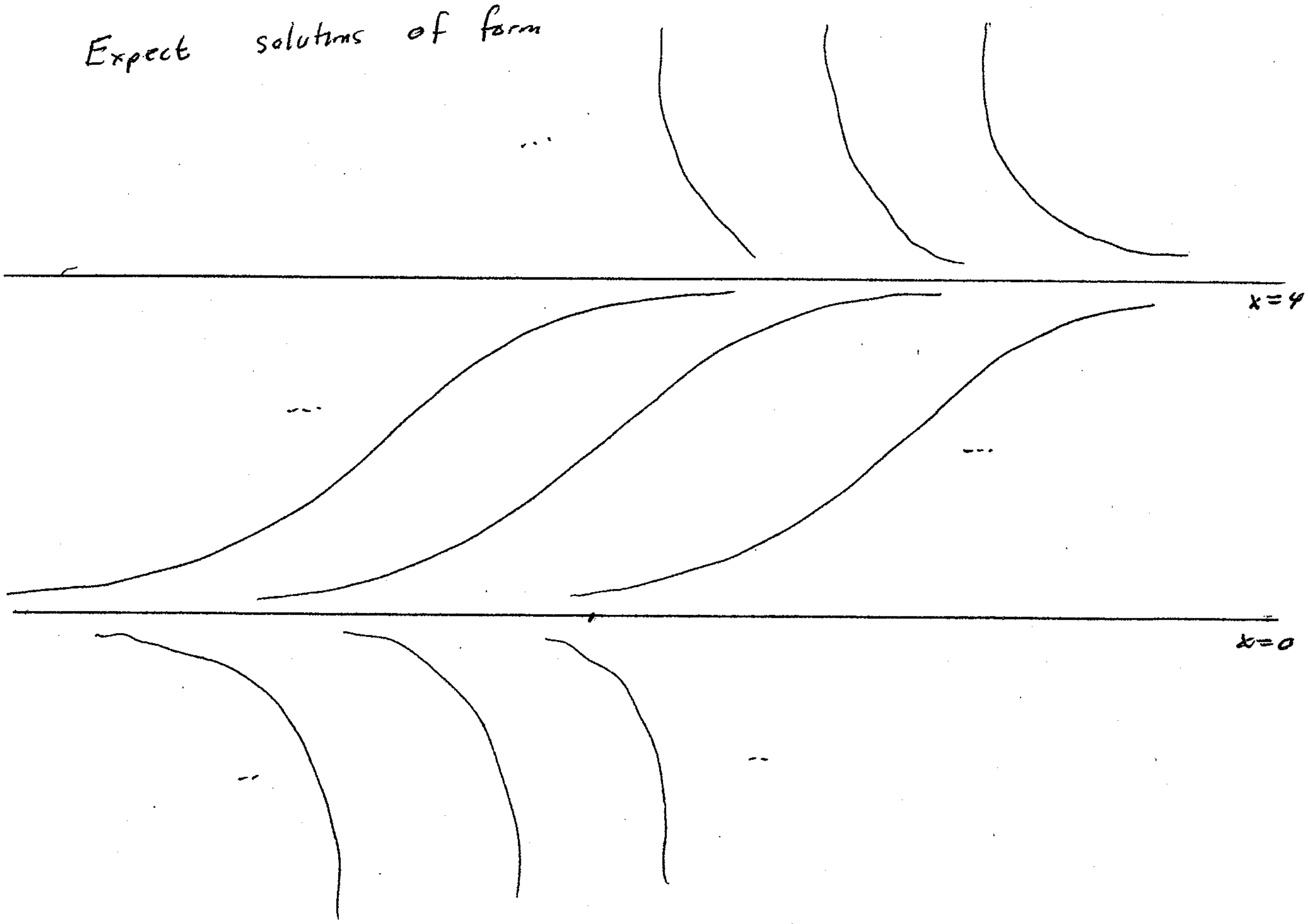


Constant functions

$x=0$, $x=4$

are both particular solutions.

Expect solutions of form



We now solve

$$\frac{dx}{dt} = x(4-x)$$

using separation of variables

$$\frac{dx}{x(4-x)} = dt$$

$$\int \frac{dx}{x(4-x)} = \int dt$$

$$= t + \text{const}$$

Find $\int \frac{dx}{x(4-x)}$

Use partial fraction decomposition:

Hunt for constants A, B such that

$$\frac{1}{x(4-x)} = \frac{A}{x} + \frac{B}{4-x}$$

$$\text{RHS} = \frac{A}{x}\frac{4-x}{4-x} + \frac{B}{4-x}\frac{x}{x}$$

$$= \frac{A(4-x) + Bx}{x(4-x)}$$

Require

$$1 = A(4-x) + Bx$$

Require

$$1 = 4A$$
$$0 = B - A$$

$$A = \frac{1}{4}, \qquad B = \frac{1}{4}$$

So

$$\frac{1}{x(4-x)} = \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right)$$

So

$$\int \frac{dx}{x(4-x)} = \frac{1}{4}\left(\ln|x| - \ln|4-x|\right) + \text{const}$$

$*$ becomes

$$\frac{1}{4}\left(\ln|x| - \ln|4-x|\right) = t + \text{const}$$

So

$$\ln|x| - \ln|4-x| = 4t + \text{const}$$

Exponentiate each side

Several cases $x < 0$, $0 \le x \le 4$, $x > 4$

all lead to

$$\frac{x}{4-x} = Ce^{4t} \qquad C = \text{const}$$

Case $C = 0$ gives $x = 0$

Assume $C \neq 0$

Solve for x:

$$x = (4-x)\,Ce^{4t}$$

$$x\left(Ce^{4t}+1\right) = 4Ce^{4t}$$

$$x = \frac{4Ce^{4t}}{Ce^{4t}+1}$$

$$x = \frac{4}{1 + C^{-1}e^{-4t}}$$

Write $x_0 = x(0)$

$$x_0 = \frac{4}{1+C^{-1}}$$

$$1 + C^{-1} = \frac{4}{x_0}$$

$$C^{-1} = \frac{4}{x_0} - 1 = \frac{4-x_0}{x_0}$$

$$x = \frac{4}{1 + \frac{4-x_0}{x_0} e^{-4t}}$$

$$x(t) = \frac{4x_0}{x_0 + (4-x_0)e^{-4t}}$$

x_0 free

(GSol)

Cases

$0 < x_0 < 4$

$x_0 > 4$

$x_0 < 4$

Case $0 < x_0 < 4$

$x_0 > 0, \quad 4 - x_0 > 0$

$$0 < \frac{4x_0}{x_0 + (4-x_0)e^{-4t}} < 4$$

As $t \longrightarrow \infty$

$e^{-4t} \longrightarrow 0$

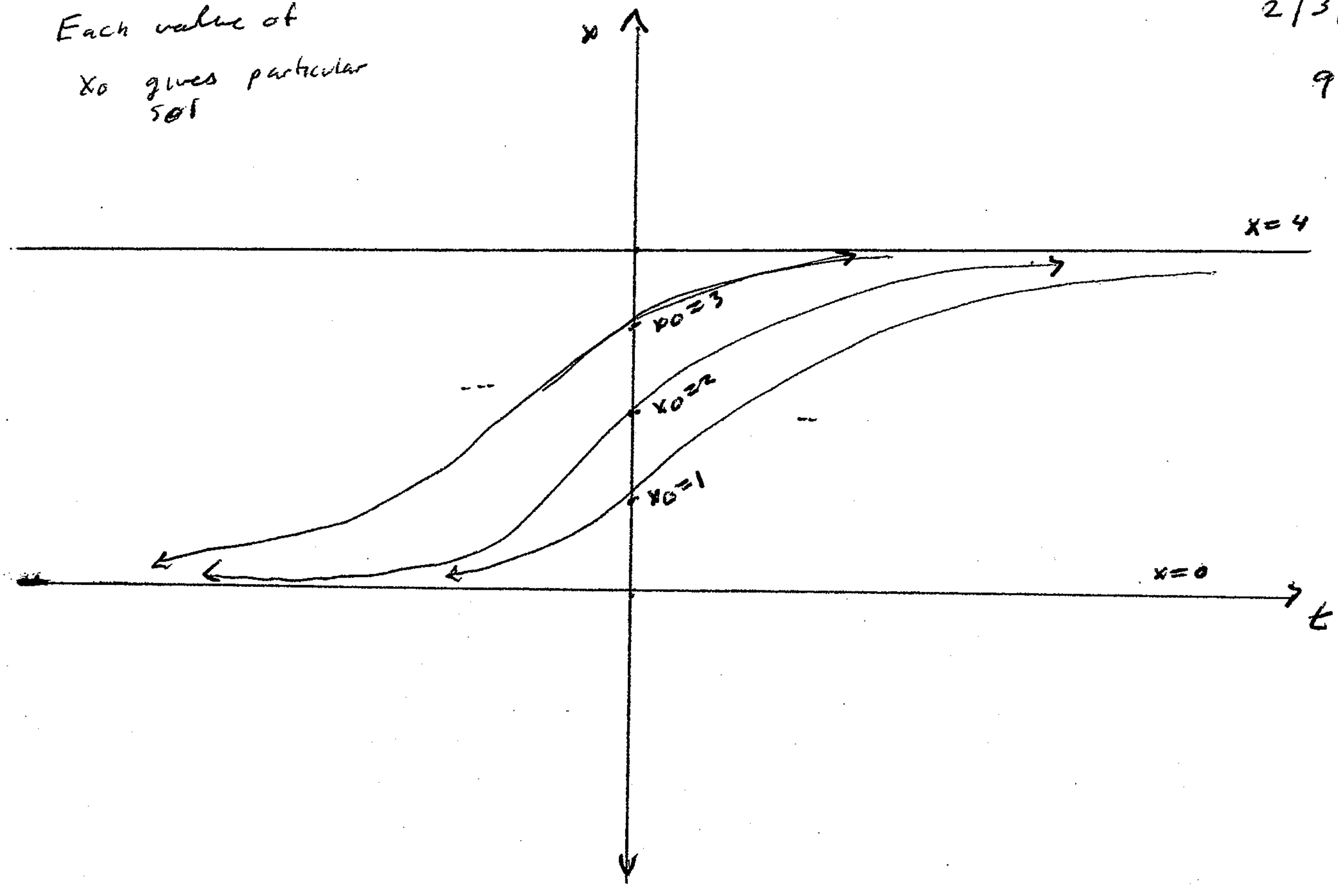
$$\frac{4x_0}{x_0 + (4-x_0)e^{-4t}} \longrightarrow 4$$

As $t \longrightarrow -\infty$

$e^{-4t} \longrightarrow \infty$

$$\frac{4x_0}{x_0 + (4-x_0)e^{-4t}} \longrightarrow 0$$

Each value of
x_0 gives particular sol



Cases $\quad x_0 > 4 \quad$ or $\quad x_0 < 0$

Here $\quad x_0, \quad x_0 - 4 \quad$ have same sign

$$\frac{x_0 - 4}{x_0} > 0$$

Recall

$$x(t) = \frac{4x_0}{x_0 + (4 - x_0)e^{-4t}}$$

Find t such that denom is 0

For this t,

$$x_0 + (4 - x_0)e^{-4t} = 0$$

$$e^{-4t} = \frac{x_0}{x_0 - 4}$$

$$e^{4t} = \frac{x_0 - 4}{x_0} > 0$$

$$4t = \ln \frac{x_0 - 4}{x_0}$$

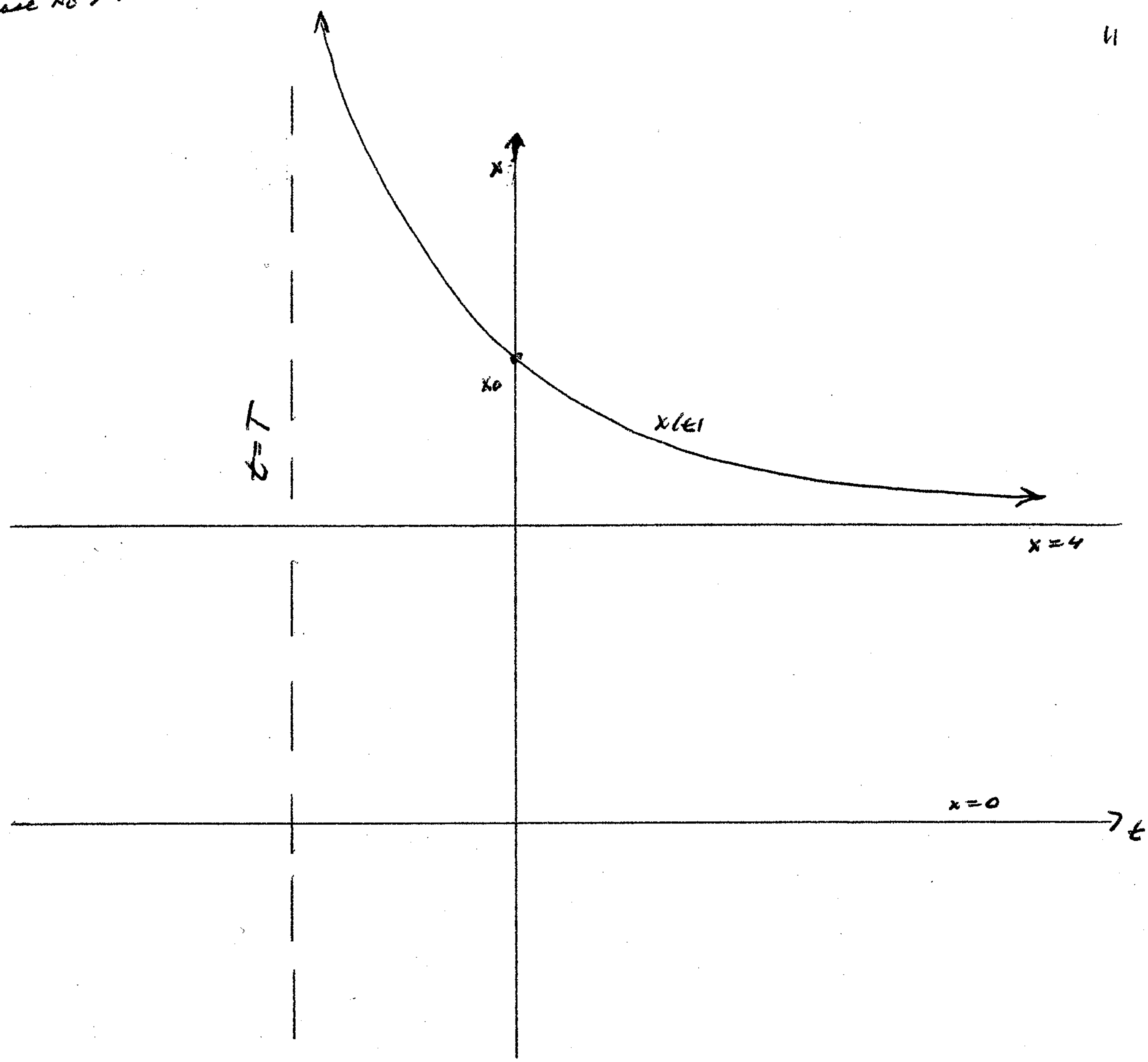
$$t = \underbrace{\frac{1}{4} \ln \frac{x_0 - 4}{x_0}}_{\text{call this } T}$$

One checks

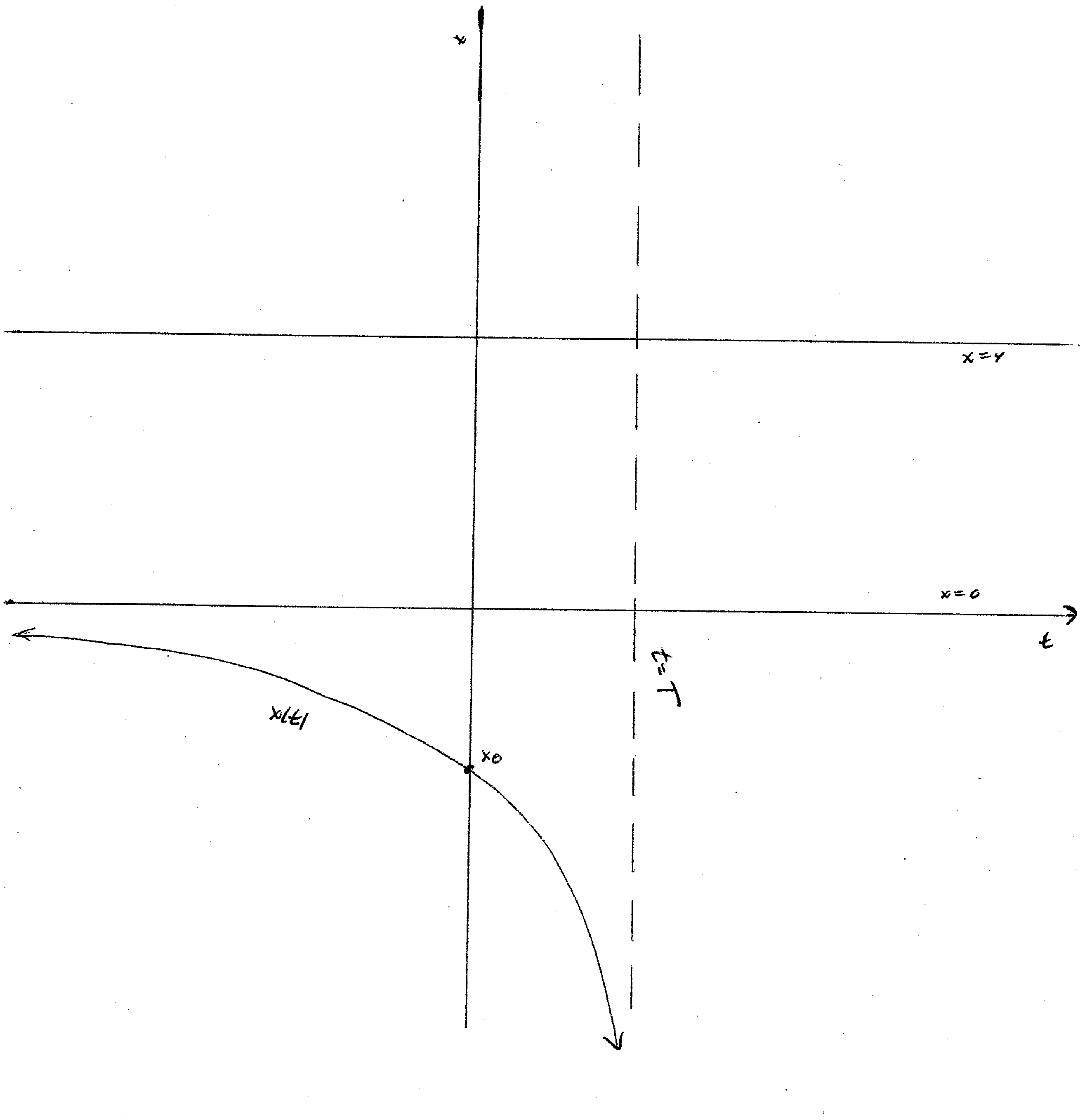
$$T < 0 \quad \text{if} \quad x_0 > 4$$

$$T > 0 \quad \text{if} \quad x_0 < 0$$

Case $x_0 > 4$



Case $x_0 < 0$



Note For the above example

Vieux

$x(t)$ = size of population at time t

(in millions)

$x_0 = x(0)$ is initial pop

For all $x_0 > 0$,

$$\lim_{t \to \infty} x(t) = 4$$

4 million = "Limiting population"

or "carrying capacity"

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Ex Describe sols to

$$\frac{dx}{dt} = -4x + x^2 \qquad (\text{doomsday/extinct}) \quad (\star)$$

Sol I use separation of vars as in prev example

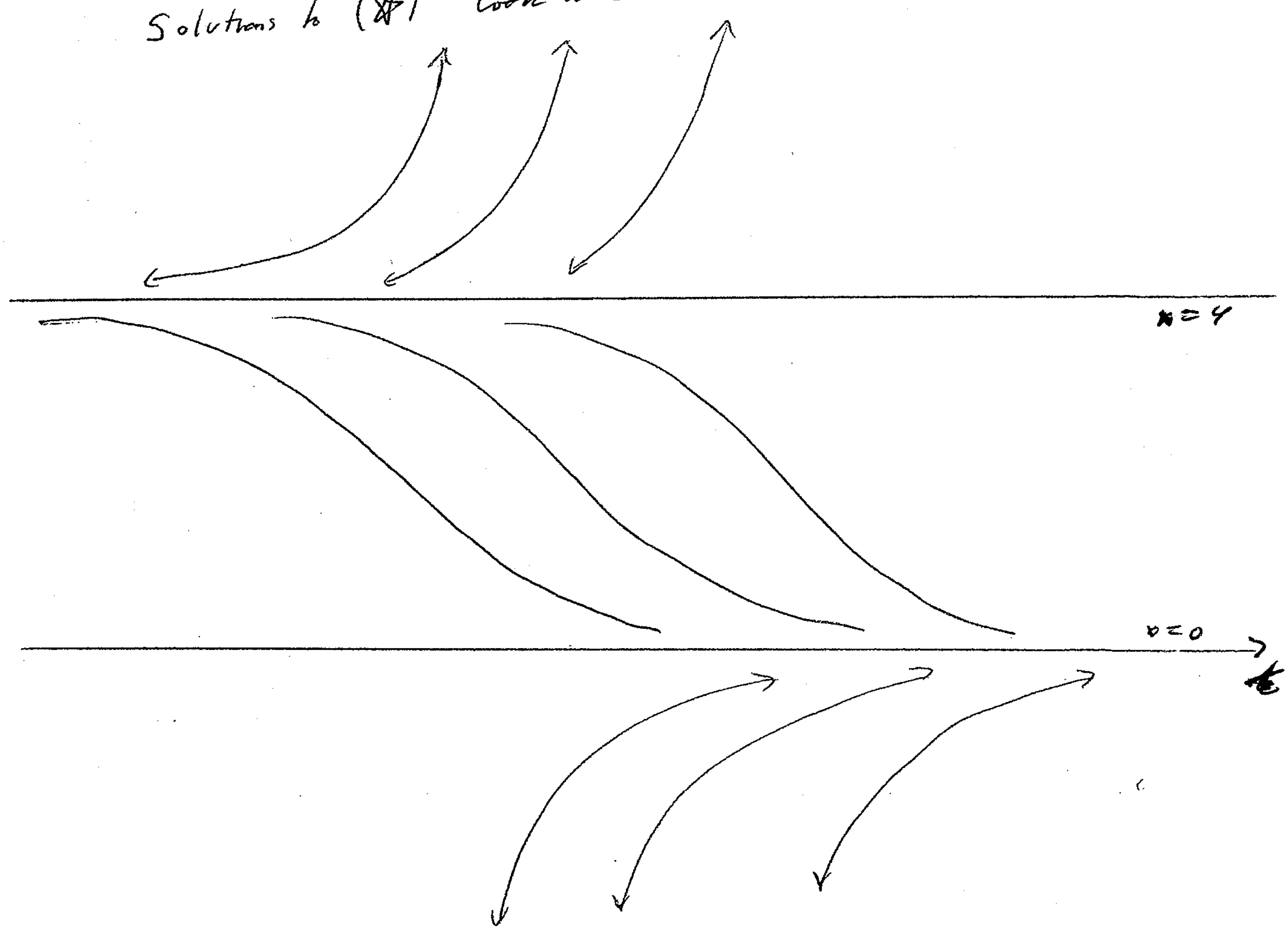
Sol II change vars $t \to -t$

Equation becomes

$$\frac{dx}{dt} = 4x - x^2$$

which we solved already

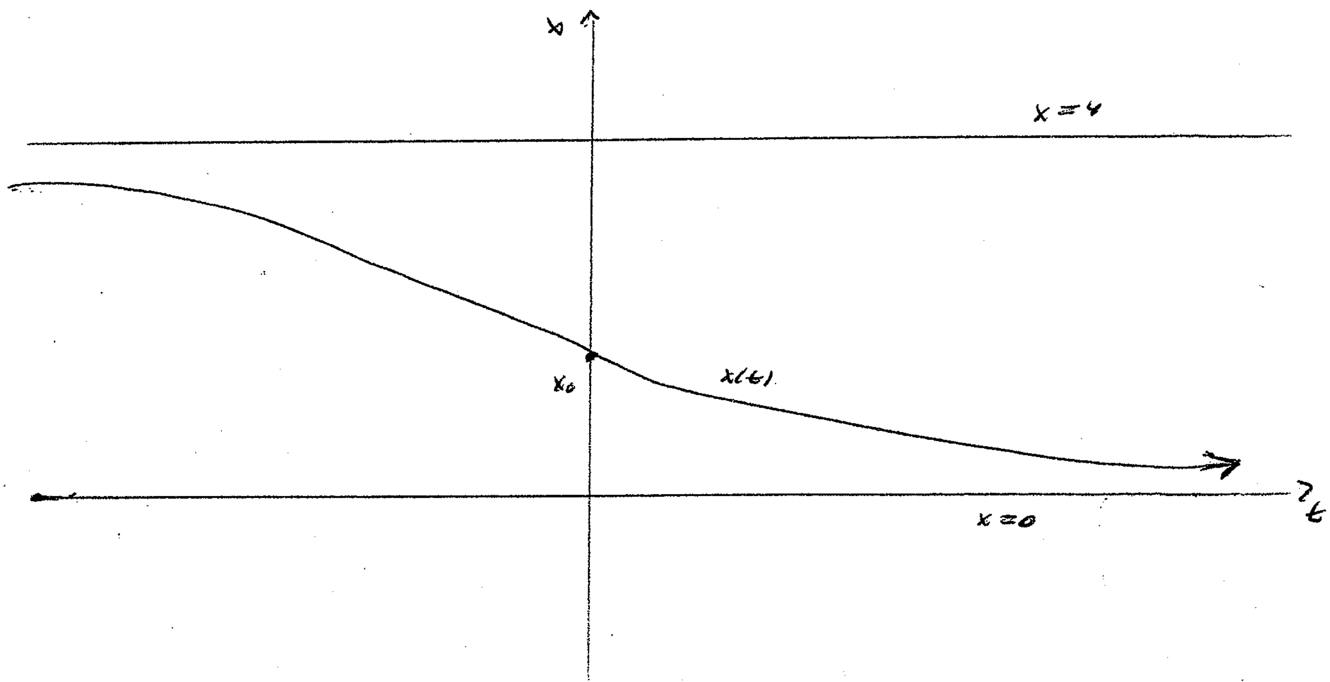
Solutions to $(\star)$ look like



the gen sol to (*) is

$$x(t) = \frac{4x_0}{x_0 + (4-x_0)e^{4t}} \qquad x_0 \text{ free}$$

Case $0 < x_0 < 4$



Case $x_0 < 0$ or $x_0 > 4$

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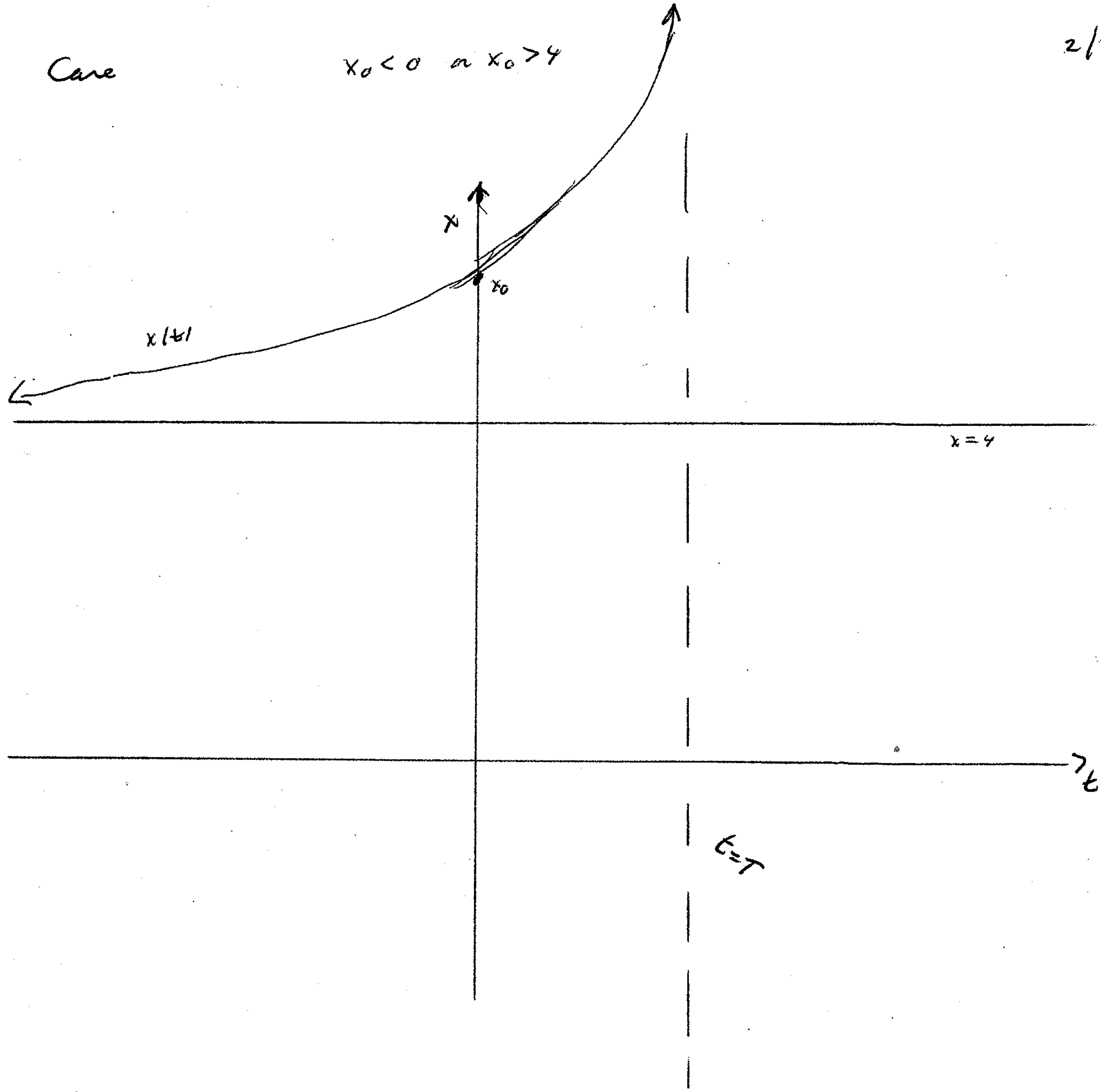
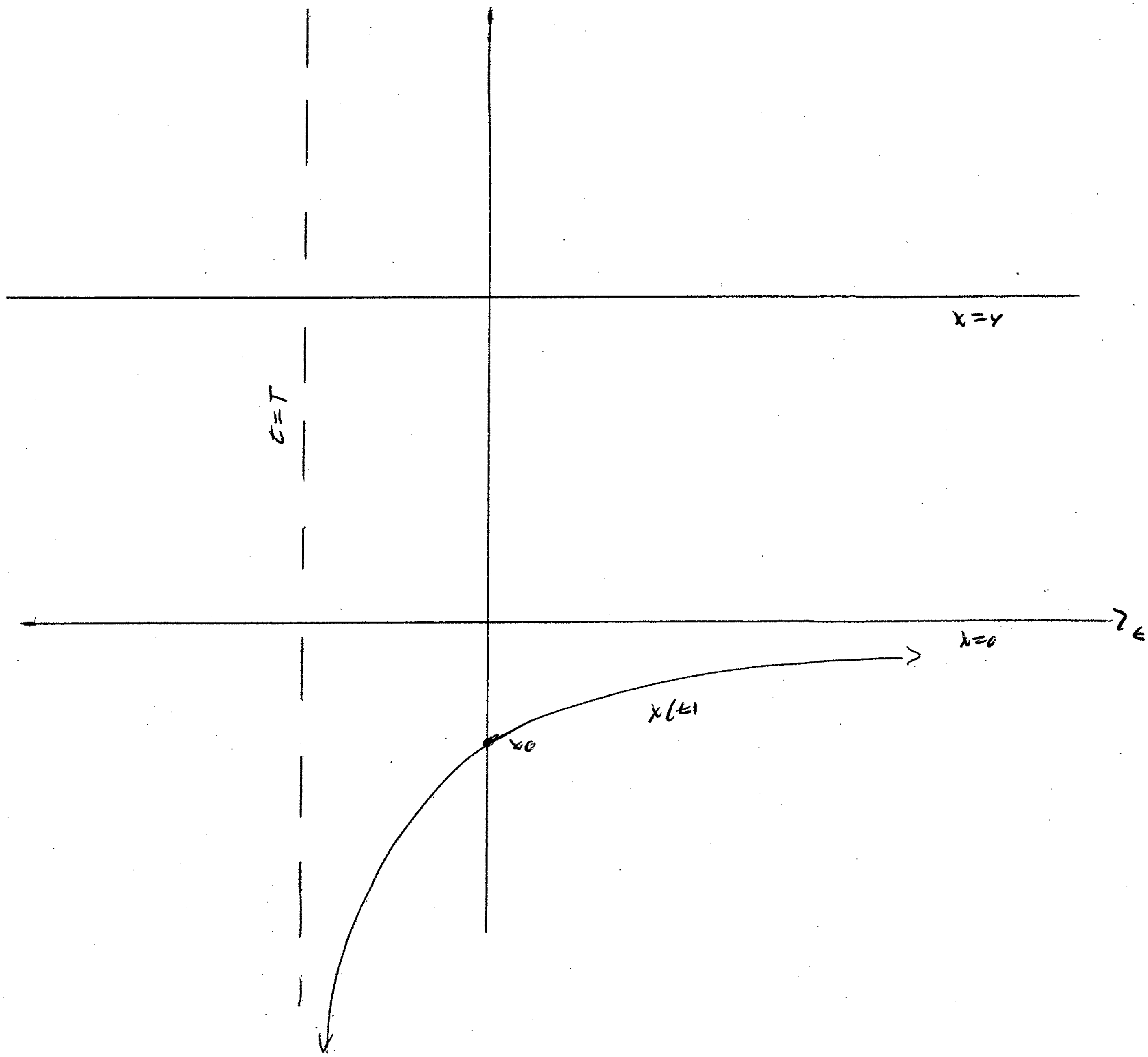


$$T = \frac{-1}{4} \ln \frac{x_0 - 4}{x_0}$$


t = T
x = y
t
x = 0
x(t)
x0

Note Again view

$x(t)$ = size of population at time t
(in millions)

$x_0 = x(0)$ is init pop

if $x_0 > 4$ then pop explosion at $t = T$
"doomsday"

if $0 < x_0 < 4$ then the pop shrinks to 0 as $t \to \infty$
"extinction"

4 million ($x_0 = 4$) is the "threshold population"