Lec 5 Friday Jan 31 1/31/14

1.5 Linear first order equations

We now consider diff equations of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$ "linear, 1st order"

$P(x)$, $Q(x)$ any functions of x

Solution method:

Multiply both sides by an "integrating factor"

$$e^{F(x)}$$ $F(x)$ to be determined

So

$$\underbrace{e^{F(x)}\frac{dy}{gx} + e^{F(x)}P(x)y}_{\text{compare}} = e^{F(x)}Q(x) \quad *$$

$$\frac{d}{dx}\left(e^{F(x)}y\right)$$

$$\frac{d}{dx}\left(e^{F(x)}y\right) \overset{\text{prod rule}}{=} e^{F(x)}\frac{dy}{dx} + e^{F(x)}F'(x)\,y$$

To get LHS of * choose $F(x)$ s.t.

$$F'(x) = P(x)$$

So

$$F(x) = \int P(x)\,dx$$

Now * becomes

$$\frac{d}{dx}\left(e^{F(x)}y\right) = e^{F(x)}Q(x)$$

So

$$e^{F(x)}y = \int e^{F(x)}Q(x)\,dx$$

So

$$y = e^{-F(x)}\int e^{F(x)}Q(x)\,dx$$

this gives the gen sol.

Ex Solve the problem

$$xy' + 5y = 7x^2 \qquad y(2) = 5$$

Sol Write

$$y' + \underbrace{\frac{5}{x}}_{P(x)} y = \underbrace{7x}_{Q(x)}$$

Find integrating factor $e^{F(x)}$ with $F(x) = \int P(x)\,dx$:

$$\int \frac{5}{x}\,dx = 5 \ln x$$

[any particular sol will do here]

$$e^{5\ln x} = \left(e^{\ln x}\right)^5 = x^5$$

$$\underbrace{x^5\left(y' + \frac{5}{x}y\right)}_{\frac{d}{dx}(x^5 y)} = \underbrace{x^5(7x)}_{7x^6}$$

$$x^5 y = \int 7x^6\,dx$$

$$= x^7 + C \qquad C = \text{const}$$

Solve for y

$$y = x^2 + Cx^{-5} \qquad \text{(gen sol)}$$

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Find C using $y(2) = 5$

$$5 = \underset{\overset{\shortparallel}{4}}{2^{2}} + C\,\underset{\overset{\shortparallel}{\frac{1}{32}}}{2^{-5}}$$

$$C = 32$$

$$y = x^{2} + 32x^{-5}$$

Ex Solve the problem

$$\frac{dy}{dx} + y = e^{x} \qquad y(0) = 1$$

Sol

$$\frac{dy}{dx} + \underbrace{1}_{P(x)} y = \underbrace{e^{x}}_{Q(x)}$$

Find integrating factor $e^{F(x)}$ $F(x) = \int P(x)\,dx$

$$\int 1\,dx = x$$

e^{x} = int factor

$$e^{x}\left(\frac{dy}{dx} + y\right) = e^{x}e^{x}$$

$$\frac{d}{dx}\left(e^{x}y\right) = e^{2x}$$

$$e^{x}y = \int e^{2x}\,dx$$

$$= \frac{e^{2x}}{2} + C$$

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$$y = e^{-x}\left(\frac{e^{2x}}{2} + C \right)$$

$$y = \frac{e^{x}}{2} + C e^{-x}$$ (gen sol)

To find C use $y(0)=1$:

$$1 = \frac{e^{0}}{2} + C e^{-0}$$

$$= \frac{1}{2} + C$$

$$C = \frac{1}{2}$$

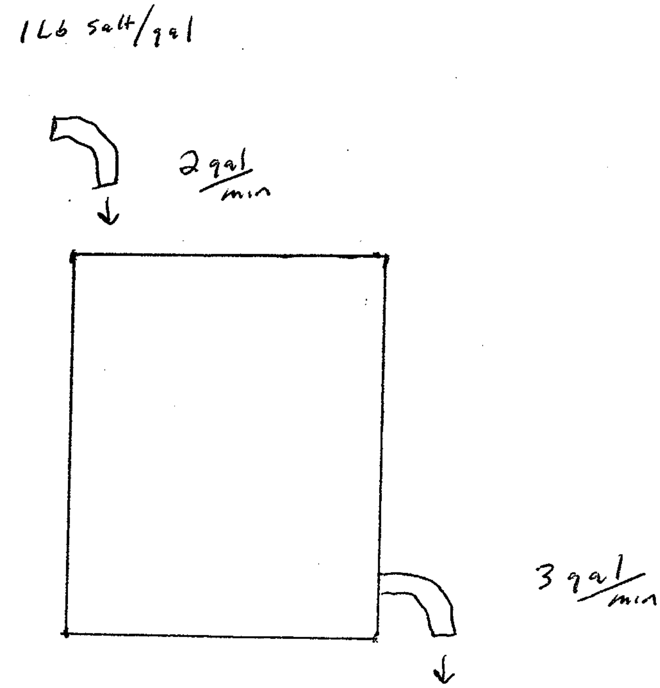
$$y = \frac{e^{x} + e^{-x}}{2}$$

Ex A tank initially contains 60 gal pure water

Salt water containing 1 Lb salt per gal enters the tank at 2 gal/min

The (perfectly mixed) solution leaves the tank at 3 gal/min.

Describe the amt of salt in tank as a function of time. When does this amount reach its maximum value?



At first tank holds 60 gal pure water

gal salt water in tank after t min is

$$60 - t$$

all gone after $t = 60$ min

Define

$$x(t) = \#\ \text{Lb salt in tank after } t \text{ min}$$

$$(0 \le t \le 60)$$

So $\quad x(0) = 0$

Find $x(t)$

Observe:

For each time t and for a short time interval Δt

$$\underset{\text{Lb salt}}{x(t+\Delta t)} = \underset{\text{Lb salt}}{x(t)} + \underset{\text{gal/min}}{2}\ \underset{\text{min}}{\Delta t}\ \underset{\text{Lb salt/gal}}{1} - \underset{\text{gal/min}}{3}\ \underset{\text{min}}{\Delta t}\ \underset{\text{Lb salt/gal}}{\frac{x(t)}{60-t}}$$

[in lim as $\Delta t \to 0$]

$$x'(t) = \lim \frac{x(t+\Delta t) - x(t)}{\Delta t} = 2 - 3\,\frac{x(t)}{60-t}$$

So

$$x'(t) + \frac{3}{60-t}\,x(t) = 2$$

Find integrating factor:

$$\int \frac{3}{60-t}\,dt = -3\ln(60-t)$$

$$e^{-3\ln(60-t)} = (60-t)^{-3}$$

int factor

$$(60-t)^{-3}\left(x'(t) + \frac{3}{60-t}\, x(t) \right) = 2(60-t)^{-3}$$

$$\parallel$$

$$\frac{d}{dt}\left((60-t)^{-3} x(t) \right)$$

$$(60-t)^{-3} x(t) = \int 2(60-t)^{-3}\,dt$$

$$= (60-t)^{-2} + C$$

$$x(t) = 60 - t + C(60-t)^{3} \quad \text{(gen sol)}$$

Find C using $x(0)=0$

$$0 = 60 + C\,60^{3}$$

$$C = -\frac{1}{60^{2}} = -\frac{1}{3600}$$

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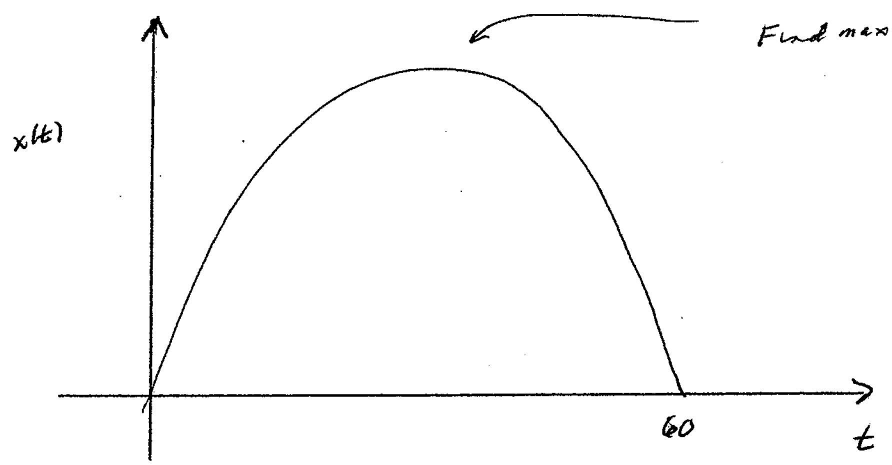
$$x(t) = 60 - t - \frac{(60-t)^3}{60^2}$$



Find t such that $x(t) = \max$

Here $x'(t) = 0$

$$x'(t) = -1 + \frac{3(60-t)^2}{60^2}$$

Require

$$1 = \frac{3(60-t)^2}{60^2}$$

$$\frac{60^2}{3} = (60-t)^2$$

$$\frac{60}{\sqrt{3}} = 60 - t$$

$$t = 60 - \frac{60}{\sqrt{3}}$$

This is when $x(t)$ is maximal

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Find $x\left(60 - \frac{60}{\sqrt{3}}\right)$

$$x\left(60-\frac{60}{\sqrt{3}}\right) = \frac{60}{\sqrt{3}} - \left(\frac{60}{\sqrt{3}}\right)^3 \frac{1}{60^2}$$

$$= \frac{60}{\sqrt{3}} - \frac{60}{3\sqrt{3}}$$

$$= \frac{60}{\sqrt{3}}\left(1 - \frac{1}{3}\right)$$

$$= \frac{60}{\sqrt{3}} \frac{2}{3}$$

$$= \frac{40}{\sqrt{3}}$$

Max value of $x(t)$ is

$$\frac{40}{\sqrt{3}} \simeq 23.09 \quad \text{Lb salt}$$

$\square$

Ex Solve the problem

$$y' + 2xy = x \qquad y(0) = -2$$

Sol Find integrating factor

$$\int 2x\,dx = x^2$$

$$e^{x^2} \quad \text{is integ factor}$$

$$e^{x^2}(y' + 2xy) = xe^{x^2}$$

$$\shortparallel$$

$$\frac{d}{dx}\left(e^{x^2}y\right)$$

$$e^{x^2}y = \int xe^{x^2}dx$$

$$= \frac{e^{x^2}}{2} + C$$

$$y = \frac{1}{2} + Ce^{-x^2}$$

Find C using $y(0) = -2$

$$-2 = \frac{1}{2} + C$$

$$C = -\frac{5}{2}$$

$$y = \frac{1 - 5e^{-x^2}}{2}$$

Ex Solve the problem

$$xy' + y = 3xy \qquad y(1) = 0$$

Sol Write

$$xy' + (1-3x)\,y = 0$$

$$y' + \underbrace{\frac{1-3x}{x}}_{P(x)}\, y = \underbrace{0}_{Q(x)}$$

Find integ factor.

$$\int \frac{1-3x}{x}\,dx = \int x^{-1}\,dx - \int 3\,dx$$

$$= \ln x - 3x$$

$$e^{\ln x - 3x} = e^{\ln x} e^{-3x}$$

$$= x e^{-3x}$$

$$x e^{-3x}\left(y' + \frac{1-3x}{x}\, y\right) = x e^{-3x}(0)$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\frac{d}{dx}\left(x e^{-3x} y\right) \qquad\qquad 0$$

$$x e^{-3x} y = \int 0\,dx$$

$$= C$$

$$y = C\frac{e^{3x}}{x}$$ gen sol.

Find C using $y(1)=0$

$$0 = C\frac{e^0}{1} = C$$

our ans is

$$y \equiv 0$$ $\square$