Lec 4 Wednesday, Jan 29 1/29/14

1.4 Seperable equations + applications

We now consider diff equations of form

$$\frac{dy}{dx} = H(x,y) \qquad (*)$$

where the function $H(x,y)$ can be factored into

$$\left(\text{a function of } x\right) \times \left(\text{a function of } y\right)$$

"$*$ is seperable"

Write

$$H(x,y) = g(x)\, h(y)$$

Define

$$f(y) = \frac{1}{h(y)}$$

So

$$H(x,y) = \frac{g(x)}{f(y)}$$

(*) becomes

$$f(y)\ \frac{dy}{dx} = g(x) \qquad \text{“}f(y)\,dy = g(x)\,dx\text{”}$$

Define

$$F(y) = \int f(y)\,dy$$

$$G(x) = \int g(x)\,dx$$

then x, y are related by

$$F(y) = G(x) + C \qquad C = \text{const} \qquad (\star)$$

“implicit solution”

[solve $\star$ for y to get general solution]

Why it works:

show $F(y) - G(x)$ is constant

show $\frac{d}{dx}\left(F(y) - G(x) \right) = 0$

$$\frac{d}{dx} F(y) \overset{\text{ch rule}}{=} \frac{d}{dy} F(y)\ \frac{dy}{dx}$$

$$= f(y)\ \frac{dy}{dx}$$

$$= g(x)$$

$$= \frac{d}{dx} G(x) \qquad ✓$$

Ex Solve the problem

$$\frac{dy}{dx} = 6e^{2x-y} \qquad y(0) = 0$$

Sol seperate the variables

$$e^{2x-y} = e^{2x}e^{-y} = \frac{e^{2x}}{e^y}$$

so

$$e^y\,dy = 6e^{2x}\,dx$$

integrate both sides:

$$\int e^y\,dy = \int 6e^{2x}\,dx$$

$$\begin{array}{cc} \| & \| \\ e^y + \text{const} & 3e^{2x} + \text{const}' \end{array}$$

Implicit sol is

$$e^y = 3e^{2x} + C \qquad C = \text{const}$$

Find C using $y(0) = 0$:

$$e^0 = 3e^0 + C$$

$$1 = 3 + C$$

$$C = -2$$

$$e^y = 3e^{2x} - 2$$

so

$$y = \ln(3e^{2x} - 2)$$

$\square$

Ex Solve the problem

$$\frac{dy}{dx} = 2xy^2 + 3x^2y^2 \qquad y(1) = -1$$

Sol Write

$$\frac{dy}{dx} = (2x + 3x^2)y^2$$

$$\frac{dy}{y^2} = (2x + 3x^2)\,dx$$

$$\int \frac{dy}{y^2} = \int (2x + 3x^2)\,dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$y = \frac{-1}{x^2 + x^3 + C}$$

Find C using $y(1) = -1$:

$$-1 = \frac{-1}{1 + 1 + C}$$

$$C = -1$$

$$y = \frac{1}{1 - x^2 - x^3}$$

□

Ex Pick $0 \neq k \in \mathbb{R}$

Find gen sol for

$$\frac{dy}{dx} = ky$$

Sol

$$\frac{dy}{y} = k\,dx$$

$$\int \frac{dy}{y} = \int k\,dx$$

$$\ln|y| = kx + \text{Cnst}$$

$$y = Ce^{kx} \qquad C = \text{cnst}$$

y exhibits "exponential growth in x" if $k > 0$

" " "exponential decay in x" if $k < 0$

Ex At a bank, a savings account offers an interest rate of 3%
"compounded continuously"

You put in \$100

How does this amt grow over time?

Sol let $A(t)$ = # dollars in the act after t years

So $A(0) = 100$

write $r = .03$ "3 percent"

At any time t and for a short interval of time Δt

$$\underbrace{A(t+\Delta t)}_{} = \overbrace{A(t)}^{\text{"principal"}} + \overbrace{r\,A(t)\Delta t}^{\text{"interest"}}$$

[in limit as $\Delta t \to 0$]

Now $$A'(t) = \lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t} = rA$$

Solving $A'(t) = rA$

we get $A(t) = Ce^{rt}$ C const

Find C using $A(0) = 100$

$100 = Ce^0 = C$

$A(t) = 100e^{rt}$

$\square$

Ex A cake is removed from the oven at 210°F and left to cool at room temp 70° F.

After 30 min the cake is 140°F.

When will the cake be 100°F ?

Sol Define a function

$D(t) =$ the difference between the temp of the cake after t minutes and ambient temp 70

so

temp of cake after t min $= D(t) + 70$

From physics (Newton's Law of cooling)

$D'(t)$ is proportional to $D(t)$

so $D'(t) = k\,D(t)$ $k = \text{const}$

Gen sol is

$$D(t) = Ce^{kt}$$

Find C

$$D(0) = Ce^0 = C$$

$$\begin{gathered} D(0) \\ \shortparallel \\ 210 - 70 \\ \shortparallel \\ 140 \end{gathered}$$

$$D(t) = 140\, e^{kt}$$

Find k

$$D(30) = 140\, e^{k30}$$

$$\shortparallel$$

$$140 - 70$$

$$\shortparallel$$

$$70$$

$$\frac{1}{2} = e^{k30}$$

$$-\ln 2 = \ln \tfrac{1}{2} = 30k$$

$$k = -\frac{\ln 2}{30}$$

Find t s.t.

$$D(t) = 100 - 70$$

Require

$$30 = 140\, e^{kt}$$

$$\ln\left(\frac{3}{14}\right) = kt$$

$$t = \frac{1}{k} \ln\left(\frac{3}{14}\right)$$

$$\simeq 66.67 \text{ min}$$

About 66 min + 40 sec

$\square$

Ex A tank is shaped like a vertical cylinder

It initially contains water to the depth of 9 feet. A bottom plug is removed and water starts to flow out. After 1 hour the water depth is 4 feet. How long does it take to drain tank?

$t=0$ $t=1$ (hrs)

Sol Define

$y(t)$ = depth of water in tank after t hours

so $y(0) = 9$, $y(1) = 4$

Define

$V(t)$ = volume of water in tank after t hours

$V(t) = y(t)\ \pi r^2$ r = radius of tank

From physics (Torricelli's law)

water leaves the tank at a rate proportional to the square root of the depth above the hole

So $$\frac{dV(t)}{dt} = -k\sqrt{y(t)} \qquad k \text{ const}$$

obs

$$\frac{dV(t)}{dt} = \pi r^2 \frac{dy}{dt}$$

So

$$\pi r^2 \frac{dy}{dt} = -k\sqrt{y} \qquad (\text{seperable})$$

Find y

$$\frac{dy}{\sqrt{y}} = \frac{-k}{\pi r^2}\, dt$$

$$\int \frac{dy}{\sqrt{y}} = \int \frac{-k}{\pi r^2}\, dt$$

$$2\sqrt{y} = \frac{-k}{\pi r^2} t + \text{Const}$$

$$\sqrt{y} = \frac{-k}{2\pi r^2} t + C$$

To find C use $y(0) = 9$

$$3 = \sqrt{9} = 0 + C$$

$$\sqrt{y} = \frac{-k}{2\pi r^2} t + 3$$

to find k use $y(1) = 4$

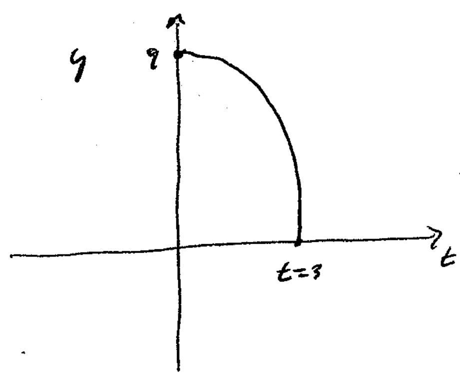
$$\underset{\overset{\shortparallel}{2}}{\sqrt{4}} = \frac{-k}{2\pi r^2} 1 + 3$$

$$k = 2\pi r^2$$

Now

$$\sqrt{y} = 3 - t$$

$$y(t) = (3-t)^2$$



$y(t) = 0$ when $t = 3$

water is gone after 3 hours $\square$

Ex Find all sols to

$$\frac{dy}{dx} = y^2 \qquad *$$

Sol: Sep vars

$$\frac{dy}{y^2} = dx$$

$$\int \frac{dy}{y^2} = \int dx$$

$$-\frac{1}{y} = x + C$$

$$y = \frac{-1}{x+C} \qquad C \text{ const}$$

But we can see directly that

$$y = 0$$

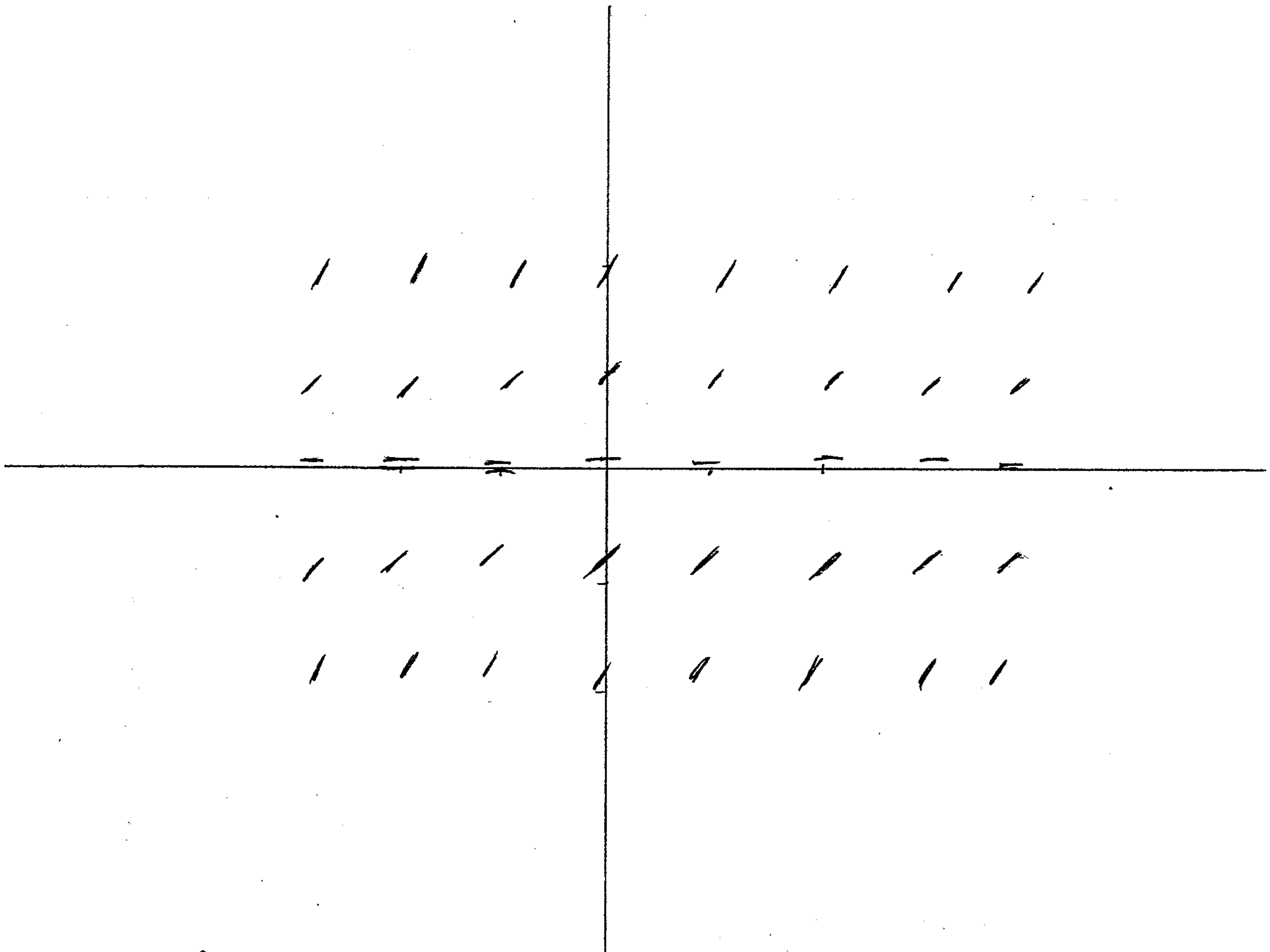
is a sol to $*$

View this as case "$C = \infty$"

Gen sol is

$$y = \frac{-1}{x+C} \qquad C \in \mathbb{R} \cup \infty$$

the slope field is



Each curve is partic sol